Letter to the Editor

On a linear perturbation of the Laguerre operator

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Abstract

In Koekoek and Koekoek (On a differential equation for Koornwinder’s generalized Laguerre polynomials, Proc. Amer. Math. Soc. 112 (1991) 1045–1054) a differential equation is introduced, which is of finite order if \( \alpha \) is a nonnegative integer and is of infinite order for other values of \( \alpha > -1 \), and can be interpreted as a linear perturbation of the differential operator for Laguerre polynomials, having orthogonal polynomials, generalizations of Laguerre polynomials (Laguerre type polynomials), as eigenfunctions. In this letter we give two new representations of this operator (with the perturbation operator in factorized form) in the finite order case and we mention some properties of this operator.

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1. Introduction

In [11] Koornwinder introduced the polynomials \( \{L_{n}^{\alpha,M}(x)\}_{n=0}^{\infty} \), defined by

\[
L_{n}^{\alpha,M}(x) = L_{n}^{(\alpha)}(x) + M Q_{n}^{(\alpha)}(x)
\]

with \( Q_{0}^{(\alpha)}(x) = 0 \) and for \( n \in \{1,2,3,\ldots\} \)

\[
Q_{n}^{(\alpha)}(x) = \binom{n+\alpha}{n-1} L_{n}^{(\alpha)}(x) + \binom{n+\alpha}{n} D L_{n}^{(\alpha)}(x)
\]

\[
= - \binom{n+\alpha}{n} \frac{x}{\alpha+1} L_{n-1}^{(\alpha+2)}(x),
\]

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\( \mathbf{D} = \frac{d}{dx} \), which are orthogonal with respect to the inner product

\[
(f, g) = \frac{1}{\Gamma(x+1)} \int_0^\infty f(x)g(x)x^xe^{-x}dx + Mf(0)g(0), \quad M \geq 0, \quad x > -1.
\]

They are usually called Laguerre type polynomials, which are studied in great detail in [9]. For \( x \in \{0, 1, 2\} \) they have been considered earlier (see [12–14], where differential equations were found for these polynomials). In [7] (see also [10]) Koekoek and Koekoek discovered a differential operator having the Laguerre type polynomials \( \{L_n^\mu(x)\}_{n=0}^\infty \) \( (x > -1) \) as eigenfunctions, generalizing the earlier results.

Since for the classical Laguerre polynomials \( \{L_n^\mu(x)\}_{n=0}^\infty \) it is known that

\[
L_n^{(x+1)}L_n^{(x)}(x) = nL_n^{(x+2)}(x)
\]

with

\[
L_n^{(x)} := -xD^2 - (x + 1 - x)D,
\]

an operator of the form

\[
L_n^{(x)} + MA_n^{(x)}
\]

is considered, where

\[
A_n^{(x)} := \sum_{i=1}^\infty a_i(x; \alpha)D^i,
\]

and numbers \( \{\alpha_n^{(x)}\}_{n=0}^\infty \) such that

\[
[L_n^{(x)} + MA_n^{(x)}]L_n^{(x)}(x) = [n + MA_n^{(x)}]L_n^{(x)}(x)
\]

for \( n \in \{0, 1, 2, \ldots\} \). It is easy to derive that \( \alpha_0^{(x)} = 0 \) and that

\[
L_n^{(x)}D_n^{(x)}(x) - nD_n^{(x)}(x) = D^2L_n^{(x)}(x) - DL_n^{(x)}(x) = L_{n-1}^{(x+2)}(x).
\]

By substituting Eq. (1.1) into Eq. (1.7) and by equating the coefficients of \( M \) and \( M^2 \) on both sides, it is derived that

\[
A_n^{(x)}L_n^{(x)}(x) = \alpha_n^{(x)}L_n^{(x)}(x) - \left( \frac{n + x}{n} \right) L_{n-1}^{(x+2)}(x),
\]

and

\[
A_n^{(x)}Q_n^{(x)}(x) = \alpha_n^{(x)}Q_n^{(x)}(x),
\]

for all real \( x \) and \( n \in \{1, 2, 3, \ldots\} \). Koekoek and Koekoek succeeded in showing that these two systems of equations for the unknown constants \( \{\alpha_n^{(x)}\}_{n=1}^\infty \) and the unknown functions \( \{a_i(x; \alpha)\}_{i=1}^\infty \), have a unique solution given by

\[
\alpha_n^{(x)} = \left( \frac{n + x + 1}{n - 1} \right), \quad n \in \{1, 2, 3, \ldots\},
\]

\[
a_i(x; \alpha) = \left( \frac{1}{i!} \sum_{j=1}^i (-1)^{i+j} \left( \frac{x + 1}{j - 1} \right) \left( \frac{x + 2}{i - j} \right) (x + 3)^{-j} \right), \quad i \in \{1, 2, 3, \ldots\}.
\]

From Eq. (1.12) it follows at once that the differential operator \( A_n^{(x)} \) is of order \( 2x + 4 \) if \( x \) is a nonnegative integer and of infinite order if \( x \) is not a nonnegative integer. A more direct approach to this result is given in [1] and the connection to spectral theory of differential operators is discussed in [5].
2. Factorizations of $A^{(x)}$

We study the case that the operator $A^{(x)}$ is of finite order. Therefore in the rest of this letter we will assume that $x$ is a nonnegative integer.

2.1. The first factorization

We can write

$$A_n^{(x)} = \frac{1}{(x+2)!} \prod_{j=0}^{x+1} (n+j),$$

and since by Eqs. (1.2), (1.4) and (1.8) we have

$$\left( L^{(x)} + \frac{x+1}{x} I \right) Q_n^{(x)}(x) = nQ_n^{(x)}(x),$$

it follows from Eq. (1.10) that for all $n \in \mathbb{N}$

$$A^{(x)} Q_n^{(x)}(x) = \left[ \frac{1}{(x+2)!} \prod_{j=0}^{x+1} \left( L^{(x)} + \left( \frac{x+1}{x} + j \right) I \right) \right] Q_n^{(x)}(x),$$

where $I$ denotes the identity operator. The polynomials $\{Q_n^{(x)}(x)\}_{n=1}^{\infty}$ are a basis for the space of all polynomials divisible by $x$. In order to conclude that

$$A^{(x)} = \frac{1}{(x+2)!} \prod_{j=0}^{x+1} \left( L^{(x)} + \left( \frac{x+1}{x} + j \right) I \right),$$

it is sufficient to prove the following

**Lemma 2.1.** The operator $\prod_{j=0}^{x+1} \left( L^{(x)} + \left( \frac{x+1}{x} + j \right) I \right)$ annihilates constants.

**Proof.** It is easy to verify that for all $j \in \{0, 1, 2, \ldots\}$

$$\left[ L^{(x)} + \left( \frac{x+1}{x} + j \right) I \right] \frac{1}{x^j} = \frac{(j+1)(-j+x+1)}{x^{j+1}}.$$

By applying this consecutively for $j = 0, 1, \ldots, x+1$ we obtain the desired result. □

From Eq. (2.1) it follows immediately that $A^{(x)}$ is of order $2x + 4$ and that

$$a_{2x+4}(x; x) = \frac{(-x)^{x+2}}{(x+2)!},$$

which also can be seen from Eq. (1.12).

2.2. The second factorization

Another factorization of the operator $A^{(x)}$ can be obtained from Eq. (1.9). If we remember Eq. (1.8) it is easy to construct a linear differential operator which has the same working on
all the Laguerre polynomials (for \( n = 0 \) this is trivial) as the operator \( A^{(\xi)} \) and therefore coincides with \( A^{(\xi)} \). In fact

\[
A^{(\xi)} = \frac{1}{(\alpha + 2)!} \prod_{j=0}^{2+\xi}(L^{(\xi)} + jI) - \frac{1}{\alpha!}(D^2 - D) \prod_{j=1}^{\xi}(L^{(\xi)} + jI)
\]

\[
= \frac{1}{(\alpha + 2)!}[(L^{(\xi)} + (\alpha + 1)I)L^{(\xi)} - (\alpha + 2)(\alpha + 1)(D^2 - D)] \prod_{j=1}^{\xi}(L^{(\xi)} + jI).
\]

The fourth order operator \( [L^{(\xi)} + (\alpha + 1)I]L^{(\xi)} - (\alpha + 2)(\alpha + 1)(D^2 - D) \) can be factorized. After some calculations we find a second factorization of \( A^{(\xi)} \):

\[
A^{(\xi)} = \frac{x}{(\alpha + 2)!}[xD^2 + (2\alpha + 4 - x)D - (\alpha + 2)I][D^2 - D] \prod_{j=1}^{\xi}(L^{(\xi)} + jI). \tag{2.2}
\]

In the case that \( \alpha = 0 \), we interpret \( \prod_{j=1}^{\xi}(L^{(\xi)} + jI) = I \).

### 3. Properties of the operator \( A^{(\xi)} \)

- The operator \( A^{(\xi)} \) maps polynomials of degree \( n(n \geq 1) \) onto the subspace of polynomials of degree \( n \) that have no constant term. By Eq. (1.10) the polynomials \( \{Q^{(\xi)}_n(x)\}_{n=1}^{\infty} \) are eigenfunctions of this operator.
- The coefficients \( \{a_i(x; \alpha)\}_{i=1}^{2\alpha+4} \) have the interesting property that

\[
\sum_{i=1}^{2\alpha+4} a_i(x; \alpha) = 0. \tag{3.1}
\]

This was proved in [8]. Another proof, which also holds in a more general situation, is given in [3]. There it appeared that the real reason for this property is that \( e^x \) is an eigenfunction of \( L^{(\xi)} \). Yet another proof can be given by using formula (2.2). In fact

\[
\prod_{j=1}^{\xi}(L^{(\xi)} + jI)e^x = \prod_{j=1}^{\xi}(x + 1 + j)e^x,
\]

showing that

\[
(D^2 - D) \prod_{j=1}^{\xi}(L^{(\xi)} + jI)e^x = 0.
\]

It follows that \( A^{(\xi)}e^x = 0 \), which implies Eq. (3.1).
- The operator \( L^{(\xi)} + MA^{(\xi)} \) is self-adjoint with respect to the inner product (1.3) for all \( f, g \in C^{2\alpha+4} \), such that (1.3) exists. Since the polynomials are dense in \( C^{2\alpha+4} \) and the polynomials \( \{L^{\frac{\alpha}{M}}_n(x)\}_{n=0}^{\infty} \) are a basis for the polynomials, it suffices to show that for all \( n, m \in \{0, 1, 2, \ldots\} \)

\[
((L^{(\xi)} + MA^{(\xi)})L^{\frac{\alpha}{M}}_n(x), L^{\frac{\alpha}{M}}_m(x)) = (L^{\frac{\alpha}{M}}_n(x), (L^{(\xi)} + MA^{(\xi)})L^{\frac{\alpha}{M}}_m(x)).
\]

For \( n = m \) this is obvious and for \( n \neq m \)

\[
((L^{(\xi)} + MA^{(\xi)})L^{\frac{\alpha}{M}}_n(x), L^{\frac{\alpha}{M}}_m(x)) = (n + M\alpha^{(\xi)})(L^{\frac{\alpha}{M}}_n(x), L^{\frac{\alpha}{M}}_m(x)) = 0
\]
and
\[
(L^a_n M(x), (L^a(x) + MA(x)) L^a_n M(x)) = (m + Mz^a_n)(L^a_n M(x), L^a_n M(x)) = 0
\]
by the orthogonality. With respect to the Laguerre inner product
\[
\langle f, g \rangle = \frac{1}{T} \int_0^\infty f(x)g(x)x^e e^{-x} \, dx
\]
this is still true for all polynomials which have no constant term, which is rather obvious, since there the two inner products coincide, but
\[
\langle (L^a(x) + MA(x)) L^a_n M(x), 1 \rangle = (n + Mz^a_n)\langle L^a_n M(x), 1 \rangle
\]

\[
= (n + Mz^a_n)\langle L^a_n(x) + MQ^a_n(x), 1 \rangle
\]

\[
= -\left(\frac{n + \alpha}{n}\right) \frac{(n + Mz^a_n)M}{\alpha + 1} \langle xL^a_{n-1}(x), 1 \rangle
\]

\[
= -\frac{(n + \alpha)!}{n!} (n + Mz^a_n)M
\]

\[
\langle L^a_n M(x), (L^a(x) + MA(x)) 1 \rangle = 0.
\]
This shows that the statement in [6], that the differential operator \( L^a(x) + MA(x) \) is symmetrizable with symmetry factor \( x^e e^{-x} \) is not completely correct.

- Generalizations of the operator \( A(x) \) to cases, where the inner product contains derivatives (Sobolev type Laguerre polynomials), are derived in [2,4,8].

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References