Global superconvergence in combinations of Ritz–Galerkin-FEM for singularity problems

Zi-Cai Li\textsuperscript{a,},*, Qun Lin\textsuperscript{b}, Ning-Ning Yan\textsuperscript{b}

\textsuperscript{a}Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, ROC 80424, Taiwan
\textsuperscript{b}Institute of Systems Science, Academic Sinica, Beijing 100080, People’s Republic of China

Received 9 February 1998; received in revised form 21 December 1998

Abstract

This paper combines the piecewise bilinear elements with the singular functions to seek the corner singular solution of elliptic boundary value problems. The \textit{global} superconvergence rates $O(h^{2.5})$ can be achieved by means of the techniques of Lin and Yan (The Construction and Analysis of High Efficient FEM, Hobei University Publishing, Hobei, 1996) for different coupling strategies, such as the nonconforming constraints, the penalty integrals, and the penalty plus hybrid integrals, where $\delta (>0)$ is an arbitrarily small number, and $h$ is the maximal boundary length of quasiuniform rectangles used. A little effort in computation is paid to conduct a posteriori interpolation of the numerical solutions, $u_h$, only on the subregion used in finite element methods. This paper also explores an equivalence of superconvergence between this paper and Z.C. Li, Internat. J. Numer. Methods Eng. 39 (1996) 1839–1857 and J. Comput. Appl. Math. 81 (1997) 1–17. © 1999 Elsevier Science B.V. All rights reserved.

MSC: 65N10; 65N30

Keywords: Elliptic equation; Singularity problem; Superconvergence; Combined method; Coupling technique; Finite-element method; The Ritz–Galerkin method; Penalty method; Hybrid method

1. Introduction

In this paper, the \textit{global} superconvergence rates of gradient of the error on the entire solution domain are established by combinations of the Ritz–Galerkin and finite-element methods (simply written as RGM–FEMs). There exist many reports on superconvergence at specific points, see [3,13–15,17] and in particular in the monograph of Wahlbin [16]. The traditional superconvergence in

\textsuperscript{\#} This work was supported by the research grants from the National Science Council of Taiwan under Grants No. NSC86-2125-M110-009.

* Corresponding author.

E-mail address: zcli@math.nsysu.edu.tw (Z.-C. Li)

0377-0427/99/$-see front matter © 1999 Elsevier Science B.V. All rights reserved.

PII: S0377-0427(99)00079-5
[3,13–17] is devoted to specific nodal solutions; this paper is devoted to the global superconvergence in the entire subdomains for RGM–FEM, based on Lin’s techniques [10–12] (also see [2]).

For solving singularity problems, optimal convergence $O(h)$ of RGM–FEM is reported in [9]; superconvergence $O(h^{2-\delta})$, $0 < \delta \ll 1$, of combinations of the Ritz–Galerkin and finite difference methods (RGM–FDM) is proven in [6] for average nodal derivatives, where $h$ is the maximal boundary length of finite elements or the maximal meshspacing of difference grids. Note that this kind of superconvergence is also equivalent to the global superconvergence in this paper.

Let the solution domain $S$ be divided into the singular and regular subdomains $S_1$ and $S_2$, respectively. Suppose that the regular domain $S_1$ can be partitioned into quasiuniform rectangles, and that the bilinear elements are chosen as in the finite-element method in $S_1$. Five different combinations are discussed in this paper, to match the FEM with the Ritz–Galerkin method using singular solutions that fit the solution singularity best. Under the assumption, $u \in H^3(S_1)$, the global superconvergence rates can be achieved as

$$
\|u_t - \tilde{u}_h\|_{1,S_1} + \|u_L - \tilde{u}_L\|_{1,S_1} = O(h^{2-\delta}),$
$$
\|u - \Pi^2_{2h}\tilde{u}_h\|_{1,S_1} + \|u_L - \tilde{u}_L\|_{1,S_2} = O(h^{2-\delta}), \quad 0 < \delta \ll 1,
$$

where $u_t$, $\tilde{u}_h$, and $\Pi^2_{2h}\tilde{u}_h$ are the solution interpolant, the numerical solution and a posteriori interpolant of $\tilde{u}_h$, respectively. $\| \cdot \|_{1,S_1}$ is the Sobolev norm in space $H^1(S)$. Here $u_t$ and $\tilde{u}_L$ are the true and approximate expansions with $L + 1$ singular functions, respectively. Five combinations are discussed in this paper together to provide a deep view of algorithm nature, and their comparisons.

Below, we first describe five combinations of the RGM–FEMs in the next section, and then derive the global convergence rates in Sections 3 and 4 for different combinations. In the last section, we give some comparisons and report some numerical experiments.

2. The combinations of RGM–FEM

Consider the Poisson equation with the Dirichlet boundary condition

$$
- \Delta u = - \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f(x,y), \quad (x,y) \text{ in } S, \quad (2.1)
$$
$$
u \big|_\Gamma = 0, \quad (x,y) \text{ on } \Gamma, \quad (2.2)
$$

where $S$ is a polygonal domain, $\Gamma$ the exterior boundary $\partial S$ of $S$, and $f$ smooth enough. Let the solution domain $S$ be divided by a piecewise straight line $I_0$ into two subdomains $S_1$ and $S_2$. The Ritz–Galerkin method is used in $S_2$, where there may exist a singular point, and the finite-difference method is used in $S_1$. For simplicity, the subdomain $S_1$ is again split into small quasiuniform rectangles $\Box_{ij}$ only, where $\Box_{ij} = \{ (x,y), x_i \leq x \leq x_{i+1}, y_j \leq y \leq y_{j+1} \}$. This confines the subdomain $S_1$ to be rectangles or the “L” shape, and those consisting of rectangles.

In $S_2$, we assume that the unique solution $u$ can be spanned by

$$
u = \sum_{i=1}^{\infty} a_i \psi_i, \quad (2.3)
$$
where $a_i$ are the expansion coefficients, and $\psi_i (i=1,2,\ldots,\infty)$ are complete and linearly independent base functions in the Sobolev space $H^1(S_2)$. \{\psi_i\} may be chosen as analytical and singular functions. Then the admissible functions of combinations of the RGM–FEM are written as

$$ v = \begin{cases} v^- = v_1 & \text{in } S_1, \\ v^+ = f_L(\tilde{a}_i) & \text{in } S_2, \end{cases} $$

(2.4)

where $v_1$ is the piecewise bilinear function on $S_1$, $f_L(a_i) = \sum_{i=1}^{L} a_i \psi_i$ and $\tilde{a}_i$ are unknown coefficients to be sought. If the particular solutions of (2.1) and (2.2) are chosen as $\psi_i$, the total number of $\psi_i$ used will greatly decrease for a given accuracy of solutions. Considering the discontinuity of the admissible solutions on $\Gamma_0$, i.e.,

$$ v^+ \neq v^- \quad \text{on } \Gamma_0, $$

(2.5)

we define another space

$$ H = \{ v \mid v \in L^2(S), \, v \in H^1(S_1) \text{ and } v \in H^1(S_2) \}, $$

where $H^1(S_1)$ is the Sobolev space. Let $V_h(\subseteq H)$ denote a finite-dimensional collection of the function $v$ in (2.4) satisfying (2.2). The combinations of the RGM–FEMs involving integral approximation on $\Gamma_0$ can be expressed by

$$ \hat{a}_h(u_h,v) = f(v), \quad \forall v \in V_h, $$

(2.6)

where

$$ \hat{a}_h(u,v) = \int_{S_1} \nabla u \nabla v \, ds + \int_{S_2} \nabla u \nabla v \, ds + \hat{D}(u,v), $$

(2.7)

$$ f(v) = \int_{S} f v \, ds, $$

(2.8)

$$ \hat{D}(u,v) = \frac{P_c}{\Delta^s} \int_{\Gamma_0} (u^+ - u^-)(v^+ - v^-) \, dl - \int_{\Gamma_0} \left( \alpha \frac{\partial u^+}{\partial n} + \beta \frac{\Delta u^+}{\Delta n} \right) (v^+ - v^-) \, dl $$

$$ - \int_{\Gamma_0} \left( \alpha \frac{\partial v^+}{\partial n} + \beta \frac{\Delta v^-}{\Delta n} \right) (u^+ - u^-) \, dl, $$

(2.9)

where $\partial_1 n$ on $\Gamma_0$ is the outer normal of $\partial S_2$, and $\Delta u^-/\Delta x_{(x_i,y_j)} \in f_0 \cap (x_i+h_i,y_j) \in S_1 = (u^-(x_i+h_i,y_j) - u^-(x_i,y_j))/h_i$.

In the coupling integrals (2.9), $P_c(>0)$ is the penalty constant, $\sigma$ is the penalty power, and two parameters $\alpha(\geq 0)$ and $\beta(\geq 0)$ satisfy $\alpha + \beta = 1$ or 0. The first term on the right-hand side of (2.9) is called the penalty integral, and the second and third terms are called the hybrid integrals. Four combinations of (2.6) are obtained from different parameters in (2.9). [5–7].

(I) Combination I : $\alpha = 0$ and $\beta = 1$,

(II) Combination II : $\alpha = 1$ and $\beta = 0$,

(III) Symmetric Combination : $\alpha = \beta = \frac{1}{2}$,

(IV) Penalty Combination : $\alpha = \beta = 0$.

Besides, a direct continuity constraint at the difference nodes $Z_k$ on $\Gamma_0$ is also given in [5–7, 9] as

$$ v^+(Z_k) = v^-(Z_k), \quad \forall Z_k \in \Gamma_0 $$

(2.10)
where the interface difference nodes $Z_k$ are located just on $\Gamma_0$. We then obtain the nonconforming combination:

$$ I(u_h^N, v) = f(v), \quad \forall v \in \mathcal{V}_h, \tag{2.11} $$

where

$$ I(u, v) = \int_{S_1} \nabla u \nabla v \, ds + \int_{S_2} \nabla u \nabla v \, ds \tag{2.12} $$

and $\mathcal{V}_h(\subset H)$ is a finite-dimensional collection of functions defined in (2.4) satisfying both (2.10) and (2.2).

The approximate integrals in $\tilde{D}(u, v)$ on $\Gamma_0$ are given by using the following integration rules:

$$ \int_{I_0} \xi \eta \, dl \approx \int_{I_0} \hat{\xi} \hat{\eta} \, dl = \sum_{k=1}^{N_1} \frac{Z_{k-1} - Z_k}{6} [2 \xi(Z_{k-1}) \eta(Z_k) + \xi(Z_k) \eta(Z_{k-1}) + 2 \xi(Z_k) \eta(Z_k)]. \tag{2.13} $$

where $I_0 = \bigcup_{k=1}^{N_1} I_0^k$, $I_0^k = Z_{k-1}Z_kZ_kZ_k$ denotes the length of $Z_{k-1}Z_k$, and $\hat{\xi}$ and $\hat{\eta}$ are the piecewise linear interpolatory functions on $I_0$. For the interior boundary $I_0$ we have

$$ \int_{I_0} \frac{\partial u^-}{\partial n} (v^+ - v^-) \, dl = \int_{I_0} \frac{\partial u^-}{\partial x} (v^+ - v^-) \, dx \int_{I_0} \frac{\partial u^-}{\partial y} (v^+ - v^-) \, dy \int_{I_0} \frac{\partial u^-}{\partial n} (v^+ - v^-) \, dl $$

$$ \approx \int_{I_0} \frac{\Delta u^-}{\Delta x} (v^+ - v^-) \, dy + \int_{I_0} \frac{\Delta u^-}{\Delta y} (v^+ - v^-) \, dx $$

$$ = \int_{I_0} \frac{\Delta u^-}{\Delta n} (v^+ - v^-) \, dl. \tag{2.14} $$

Note that the above integration is also suited for the slant-up boundary $I_0$, while using triangular elements (see [6,7]).

Define

$$ \|v\|_I = (\|v\|^2_{1, S_1} + \|v\|^2_{1, S_2})^{1/2}, \quad |v|_I = (\|v\|^2_{1, S_1} + \|v\|^2_{1, S_2})^{1/2}, \tag{2.15} $$

where $\|v\|_{1, S_1}$ and $|v|_{1, S_1}$ are the Sobolev norm and semi-norm (see [1,16]). Optimal convergence rates of numerical solutions $\|e\|_I = O(h)$ have been obtained in [8,9], where $e = u - u_h$ is error of the solution. In this paper, we pursue global superconvergence based on the new norms

$$ \|v\|_h = \left( \|v\|^2_{1, S_1} + \|v\|^2_{1, S_2} + \frac{P}{h^g} \|v^+ - v^-\|^2_{0, I_0} \right), \tag{2.16} $$

where the norms with discrete summation in $S_1$ are assigned on $I_0$ only

$$ \|v^+ - v^-\|^2_{0, I_0} = \int_{I_0} (v^+ - v^-)^2 \, dl. \tag{2.17} $$

Note that the norms defined in [5,6] also involve the discrete summation in $S_1$, denoted as $\|\cdot\|_{1, S_1}$. An equivalence between these two norms will be discovered in the last section.

The stability analysis is given in [9] for five combinations. In this paper, the global superconvergence rates, $\|u - \Pi^2_h u\|_h = O(h^{2-\delta})$, can be achieved by all five combinations for quasiuniform rectangles; detailed proofs are presented by Theorems 3.3 and 4.3 in the continued sections.
3. The nonconforming combination

Let $S_1$ be partitioned into quasiuniform rectangles in $2 \times 2$ fashion, see Fig. 1. $S_A$ is a boundary layer consisting of $\Box_{ij}$, to separate $S_2$ and $S^*$ in Fig. 2, where

$$S_1 = S^* \cup S_A, \quad S^* \cap S_A = \emptyset$$

such that $S^* \subset S_1$. To construct the following interpolant:

$$\hat{u}_{I,L} = \begin{cases} u_I^ = = u_I & \text{in } S^*, \\ \hat{u}_I & \text{in } S_A, \\ u_L^ = = u_L & \text{in } S_2, \end{cases}$$

(3.1)

where $u_L = \sum_{i=1}^L \hat{a}_i \psi_i$, $\hat{a}_i$ are the unknown coefficients and $u_I$ is the piecewise bilinear interpolant of the true solution on rectangulation of $S_1$. We choose $S_A$ with width of just one rectangle $\Box_{ij}$, and the function $\hat{u}_I$ in $S_A$ is also the piecewise bilinear function with the corner values at $P_i$, defined by

$$\hat{u}_I(P_i) = \begin{cases} u(P_i) & \text{if } P_i \in \Gamma_1 = S' \cap S_A, \\ u_L^ = (P_i) & \text{if } P_i \in \Gamma_0, \\ 0 & \text{if } P_i \in \Gamma. \end{cases}$$

(3.2)
Hence,

\[ \hat{u}_{L,L} \subseteq \mathcal{V}_h. \]  

(3.3)

First, let us prove a basic theorem.

**Theorem 3.1.** There exist the error bounds between the numerical and interpolant solutions, \( u_h^N \) given in (2.11) and \( \hat{u}_{L,L} \) given in (3.1)

\[ \| u_h^N - \hat{u}_{L,L} \|_1 \leq C \sup_{w \in \mathcal{V}_h} \| w \|_1 \left\{ \left\| \int_{S_1} \nabla (u - u_I) \nabla w \, ds \right\|_1 + \left\| \int_{S_1} \nabla (u - u_I) \nabla w \, ds \right\|_1 + \left\| \int_{S_3} \frac{\partial u}{\partial n} (w^+ - w^{-}) \, dl \right\|_1 \right\}, \]  

(3.4)

where \( C \) is a bounded constant independent of \( h, L, u \) and \( w \), and the norm \( \| \cdot \|_1 \) is defined in (2.15).

**Proof.** For the true solution \( u \), we have

\[ I(u - u_h^N, v) = \int_{\Gamma_0} \frac{\partial u}{\partial n} (v^+ - v^-) \, dl, \quad \forall v \in \mathcal{V}_h. \]  

(3.5)

Let \( w = u_h^N - \hat{u}_{L,L} \in \mathcal{V}_h \). Then from definition (3.1)

\[ |w|_1^2 \leq I(u_h^N - \hat{u}_{L,L}, w) = I(u - \hat{u}_{L,L}, w) - \int_{\Gamma_0} \frac{\partial u}{\partial n} (w^+ - w^-) \, dl \]

\[ = \int_{S_1} \nabla (u - u_I) \nabla w \, ds + \int_{S_1} \nabla (u - u_I) \nabla w \, ds + \int_{S_1} \nabla (u - u_I) \nabla w \, ds + \int_{S_3} \frac{\partial u}{\partial n} (w^+ - w^-) \, dl. \]  

(3.6)

Since \( u - \hat{u}_I = u - u_I + u_I - \hat{u}_I \) in \( S_3 \), we have

\[ \int_{S_3} \nabla (u - u_I) \nabla w \, ds + \int_{S_3} \nabla (u - \hat{u}_I) \nabla w \, ds \]

\[ = \int_{S_1} \nabla (u - u_I) \nabla w \, ds + \int_{S_3} \nabla (u_I - \hat{u}_I) \nabla w \, ds. \]  

(3.7)

Desired results (3.4) are obtained from (3.5)–(3.7) and the Poincare–Friedricks inequality

\[ \| w \|_1 \leq C |w|_1 \]

with a bounded constant independent of \( w \). This completes the proof of Theorem 3.1. \( \square \)

Below, we will derive the bounds of all terms on the right-hand side of (3.4). We then have the following lemma.
Lemma 3.1. Let the rectangles \( \Box_{ij} \) be quasiuniform, then for \( w \in V_h \)
\[
\left| \int \int_{\Box_{ij}} \nabla (u_i - \hat{u}_i) \nabla w \, ds \right| \leq C h^{-(1/2)} \| R_{ij} \|_{0, \Gamma_0} \| w \|_1.
\] (3.8)

Proof. From the Schwarz inequality,
\[
\left| \int \int_{\Box_{ij}} \nabla (u_i - \hat{u}_i) \nabla w \, ds \right| \leq |u_i - \hat{u}_i|_{1, \Box} |w|_1.
\] (3.9)

Denoting \( \phi = u_i - \hat{u}_i \), we obtain from the inverse estimates
\[
|u_i - \hat{u}_i|_{1, \Box} = |\phi|_{1, \Box} \leq C h^{-1} \| \phi \|_{0, \Gamma_0}.
\] (3.10)

Note that \( \phi \) is the piecewise bilinear function on \( S_{ij} \) and the fact that \( \phi(Z_i) = 0 \) for all element nodes \( Z_i \notin \Gamma_0 \) in \( S_{ij} \), the maximal values of \( |\phi| \) along any vertical and horizontal lines in \( S_{ij} \) are just located on \( \Gamma_0 \), i.e.,
\[
\| \phi \|_{0, S_{ij}}^2 = \int_{S_{ij}} \phi^2 \, ds \leq C h \int_{\Gamma_0} \phi^2 \, dl = C h \| u_i - \hat{u}_i \|_{0, \Gamma_0}^2.
\] (3.11)

Moreover, since \( u_i - \hat{u}_i = u_i - \hat{u}_L = \hat{R}_L \) on \( \Gamma_0 \).

Since \( \hat{R}_L \) is the piecewise linear interpolant of the remainder \( R_L(u - u_L) \) on \( \Gamma_0 \), we obtain from the triangle inequality
\[
|u_i - \hat{u}_i|_{0, \Gamma_0} = \| R_L \|_{0, \Gamma_0}
\]
\[
\leq \| R_L \|_{0, \Gamma_0} + h \| R_L - \hat{R}_L \|_{0, \Gamma_0}
\]
\[
\leq \| R_L \|_{0, \Gamma_0} + C h \| R_L \|_{0, \Gamma_0} \leq C \| R_L \|_{0, \Gamma_0},
\] (3.12)

where \( \varepsilon > 0 \) and \( \varepsilon \to 0 \). Hence, we have from (3.10) and (3.11)
\[
|u_i - \hat{u}_i|_{1, \Box} = |\phi|_{1, \Box} \leq C h^{-1/2} \| R_L \|_{0, \Gamma_0} = C h^{-1/2} \| R_L \|_{0, \Gamma_0}.
\] (3.13)

Combining (3.9) and (3.13) yields (3.8). This completes the proof of Lemma 3.1. \( \square \)

Lemma 3.2. Let \( u \in H^3(S_1) \), then
\[
\left| \int \int_{S_1} \nabla (u - u_I) \nabla w \, ds \right| \leq C h^2 |u|_{3, S_1} \| w \|_1, \quad \forall w \in V_h,
\] (3.14)

where \( |u|_{3, S_1} \) is the Sobolev seminorm over the space \( H^3(S_1) \).

Proof. We may follow Lin [10, p. 5], to prove (3.14), but rather provide a straightforward argument below. Denote the second-order interpolant of \( u \) by
\[
u_i^{(2)} = \sum_{i+j=2}^2 z_{ij} x^i y^j = u_I + z_{20} x^2 + z_{02} y^2,
\] (3.15)

where \( z_{ij} \) are coefficients. By virtue of the following equality (shown later):
\[
\int \int_{D_s} \nabla (u - u_I) \nabla w \, ds = \int \int_{D_s} \nabla (u - u_i^{(2)}) \nabla w \, ds,
\] (3.16)
we obtain
\[
\left|\int_S (u - u_i) \nabla w \, ds\right| = \left|\sum_{ij} \int_D (u - u_i) \nabla w \, ds\right| = \left|\sum_{ij} \int_{D_{ij}} (u - u_i^{(2)}) \nabla w \, ds\right| = \left|\int_{S_i} (u - u_i^{(2)}) \nabla w \, ds\right| \leq |u - u_i^{(2)}|_{1,S_i} |w|_{1,S_i} \leq C h^2 |u|_{3,S_i} |w|_{1}.
\]
This is desired result (3.14).

Now we show (3.16). Denote
\[
\hat{\Omega} = \{(x, y), -\frac{1}{2} \leq x \leq \frac{1}{2}, -\frac{1}{2} \leq y \leq \frac{1}{2}\}.
\]
Then by the linear transformations
\[
\hat{x} = -\frac{1}{2} + (x - x_i)/h_i, \quad \hat{y} = -\frac{1}{2} + (y - y_j)/k_j,
\]
the integrals on \(\hat{\Omega}\)
\[
\int_{\hat{\Omega}} (u - u_{ij}^{(2)}) \xi w \, d\hat{s} = \frac{h_i k_j}{h^2} \int_{\hat{\Omega}} (\hat{u} - \hat{u}_{ij}^{(2)}) \xi \hat{w} \, d\hat{s} = \frac{k_j}{h_i} \int_{\hat{\Omega}} (\hat{u} - \hat{u}_{ij}) \xi \hat{w} \, d\hat{s} - 2\hat{a}_{20} \int_{\hat{\Omega}} \hat{x} \hat{w} \, d\hat{s}.
\]
where \(\hat{a}_{20}\) and \(\hat{a}_{02}\) are also constant. In the last equality of (3.18), we have used the following equation for any bilinear functions \(\hat{w}\) on \(\hat{\Omega}\):
\[
\int_{\hat{\Omega}} \hat{x} \hat{w} \, d\hat{s} = \left(\int_{-1/2}^{1/2} \hat{x} \, d\hat{x}\right) \left(\int_{-1/2}^{1/2} \hat{w} \, d\hat{y}\right) = 0.
\]
By the inverse transformation of (3.17),
\[
\int_{\hat{\Omega}} (\hat{u} - \hat{u}_{ij}) \xi \hat{w} \, d\hat{s} = \frac{h_i}{k_j} \int_{\Omega} (u - u_{ij}) \xi w \, ds.
\]
Combining (3.18) and (3.20) yields
\[
\int_{\Omega} (u - u_{ij}^{(2)}) \xi w \, ds = \int_{\Omega} (u - u_{ij}) \xi w \, ds.
\]
Similarly, we have
\[
\int_{\Omega} (u - u_{ij}^{(2)}) \eta w \, ds = \int_{\Omega} (u - u_{ij}) \eta w \, ds.
\]
Adding (3.21) and (3.22) leads to (3.16). This completes the proof of Lemma 3.2. \(\square\)
Lemma 3.3. Assume \( u \in H^2(S) \) and that there exists a bounded power \( \mu(>0) \) independent of \( L, h, v \) such that
\[
|v^+|_{I,R_0} \leq CL^\mu |v^+|_{0,R_0}, \quad I = 1, 2, \quad \forall v \in \mathcal{V}_h,
\]
where \( |v^+|_{I,R_0} \) is the Sobolev seminorm over the space \( H^I(\Gamma_0) \). Then for \( \forall w \in \mathcal{V}_h \)
\[
\int_{R_0} \frac{\partial u}{\partial n} (w^+ - w^-) \, dl \leq C(hL^\mu)^2 \left\| \frac{\partial u}{\partial n} \right\|_{0,R_0} \| w \|_1. \tag{3.24}
\]

Proof. From the Schwarz inequality, we have
\[
\int_{R_0} \frac{\partial u}{\partial n} (w^+ - w^-) \, dl \leq \left\| \frac{\partial u}{\partial n} \right\|_{0,R_0} \| w^+ - w^- \|_{0,R_0}. \tag{3.25}
\]

Since the continuity of \( w \) at the nodes on \( \Gamma_0, w^- \) is, in fact, regarded as the piecewise linear interpolant of \( w^+ \). Hence we obtain from assumption (3.23),
\[
\| w^+ - w^- \|_{0,R_0} = \| w^+ - \hat{w}^+ \|_{0,R_0} \leq C h^2 \| w^+ \|_{2,R_0} \leq C h^2 L^\mu \| w^+ \|_{0,R_0} \leq C (hL^\mu)^2 \| w^+ \|_{1,S_2} \leq C (hL^\mu)^2 \| w \|_1. \tag{3.26}
\]

Desired results (3.24) follows from (3.25) and (3.26).

Now we provide an important theorem.

Theorem 3.2. Let \( u \in H^3(S_1) \) and (3.23) be given. Then there exist the error bounds between \( u^N_h \) and \( u_{I,L} \),
\[
\| u^N_h - u_{I,L} \|_1 \leq C \left\{ h^2 |u|_{3,S_1} + \| R_L \|_{1,S_2} + (hL^\mu)^2 \left\| \frac{\partial u}{\partial n} \right\|_{0,R_0} + h^{-(1/2)} \| R_L \|_{0,R_0} \right\}. \tag{3.27}
\]

Proof. We have
\[
\int \int_{S_2} \nabla (u - u_L) \nabla w \, ds \leq \| u - u_L \|_{1,S_2} \| w \|_{1,S_2} \leq \| R_L \|_{1,S_2} \| w \|_1. \tag{3.28}
\]

The bounds of other terms in the right-hand side of (3.24) are given in Lemmas 3.1–3.3, to lead to (3.27) directly. This completes the proof of Theorem 3.2.

Theorem 3.2 provides the global errors between \( u^N_h \) and \( u_{I,L} \), but not the true global errors between \( u^N_h \) and \( u \). The key idea in raising convergence rates is to constitute an a posteriori interpolant, \( \Pi_\mu u^N_h \), based on \( u^N_h \). The global errors \( \| u - \Pi_\mu u^N_h \|_1 \) can attain the same global superconvergence.

The solution of (2.11) is denoted as
\[
u^N_h = \begin{cases} (u^N_h)^- \quad \text{in } S_1, \\ (u^N_h)^+ \quad \text{in } S_2. \end{cases}
\]

Based on Lin’s techniques [10] for the FEM solutions \((u^N_h)^-\) in \( S_1 \), the bi-quadratic interpolant, \( \Pi_{2h}(u^N_h)^- \), can be formed in \( 2 \times 2 \) neighbouring rectangles shown in Fig. 1. The a posteriori process
is made only in $S_1$. Denote
\[
\Pi_p v = \begin{cases} 
\Pi_{2h}^v & \text{in } S_1, \quad \forall v \in \mathcal{P}_h, \\
v^+ & \text{in } S_2, \quad \forall v \in \mathcal{P}_h.
\end{cases}
\] (3.30)

We obtain the following theorem.

**Theorem 3.3.** Let all conditions in Theorem 3.2 hold. Then there exist the error bounds
\[
\|u - \Pi_p u_h^N\|_1 \leq C \varepsilon_1,
\] (3.31)
where $\varepsilon_1$ is given in Theorem 3.2.

**Proof.** By noting the interpolation operation $\Pi_p$ in (3.30) we have
\[
\|u - \Pi_p u_h^N\|_1^2 \leq \|u - \Pi_p \hat{u}_{h,L}\|_1^2 + \|\Pi_p (\hat{u}_{h,L} - u_h^N)\|_1^2
\]
\[
= \|u - \Pi_{2h}^v u_i\|_{1,S_i}^2 + \|\Pi_{2h}^v (\hat{u}_i - u_i)\|_{1,S_i}^2
\]
\[
+ \|\Pi_p (\hat{u}_{h,L} - u_h^N)\|_{1,S_i}^2 + \|R_L\|_{1,S_i}^2,
\] (3.32)

where $S_i^* \subseteq S^{*}$ is a continued extension of $S_d$ due to the $2 \times 2$ rectangles.

Based on the equivalence of finite-dimensional norms, we have
\[
\|\Pi_{2h}^v\|_{l,\Box} \leq C \|v\|_{l,\Box},
\]

to lead to
\[
\|\Pi_{2h}^v\|_{l,S_i} \leq C \|v\|_{l,S_i}, \quad \forall v \in \mathcal{P}_h.
\] (3.33)

Also from Theorem 3.2,
\[
\|\Pi_p (\hat{u}_{h,L} - u_h^N)\|_{1,S_i} + \|R_L\|_{1,S_i} \leq C (\|\hat{u}_{h,L} - u_h^N\|_{1,S_i} + \|R_L\|_{1,S_i})
\]
\[
\leq C \|\hat{u}_{h,L} - u_h^N\|_1 \leq C \varepsilon_1.
\] (3.34)

Let $v = \hat{u}_i - u_i$. We then have from (3.13)
\[
\|\Pi_{2h}^v (\hat{u}_i - u_i)\|_{1,S_i} = C \|\hat{u}_i - u_i\|_{l,S_i} \leq C \|\hat{u}_i - u_i\|_{1,S_i} \leq C h^{-1/2} \|R_L\|_{0,R}.
\] (3.35)

Since $\Pi_{2h}^v u_i$ is the bi-quadratic interpolant of $u$, then we have
\[
\|u - \Pi_{2h}^v u_i\|_{1,S_i} \leq C h^2 \|u\|_{3,S_i}.
\] (3.36)

Combining (3.32), (3.34)–(3.36) yields the desired results (3.31). This completes the proof of Theorem 3.3. $\Box$

Based on Theorems 3.2 and 3.3 we obtain the following corollary.

**Corollary 3.1.** Let the conditions in Theorem 3.1 hold. Assume the expansion term $L$ is chosen such that
\[
|R_L|_{1,S_i} = O(h^2), \quad \|R_L\|_{0,R} = O(h^{5/2}).
\] (3.37)
Then
\[ \| u_h^N - \hat{u}_{I,L} \|_1 = O(h^2) + O(h^2 L^{2\nu}) \] (3.38)
and
\[ \| u - \Pi h u_h^N \|_1 = O(h^2) + O(h^2 L^{2\nu}). \] (3.39)
Also if such \( L \) is so small as to satisfy (see [4])
\[ L = O(|\ln h|), \] (3.40)
then
\[ \| u_h^N - \hat{u}_{I,L} \|_1 = O(h^{2-\delta}) \] (3.41)
and
\[ \| u - \Pi h u_h^N \|_1 = O(h^{2-\delta}), \] (3.42)
where \( \delta \to 0 \) as \( h \to 0 \).

Corollary 3.1 displays the global superconvergence in the entire solution domain \( S \). In particular in \( S_1 \),
\[ \| u - \Pi h^2 u_h^N \|_{1,S_1} = O(h^{2-\delta}). \]

In fact, the a posteriori quadratic interpolation \( \Pi h^2 \) on the solution \( u_h^N \) costs a little more computation. The analytic results can be applied directly to Motz’s problem, see Section 6.

4. The penalty combination

In this section, we will derive the error bounds for the solution \( u_P^h \) by (2.6) as \( \alpha = \beta = 0 \). We first give a basic theorem in the norm \( \| \cdot \|_h \) defined in (2.16).

Theorem 4.1. There exist the error bounds between the solution \( u_P^h \) and \( \hat{u}_{I,L} \) defined in (3.1),
\[ \| u_h^P - u_{I,L} \|_h \leq C \sup_{w \in \mathcal{V}_h} \left\{ \frac{1}{\| w \|_h} \left( \int_{S_1} \nabla (u - u_I) \nabla w \, ds \right) + \int_{S_1} \nabla (u_I - \hat{u}_I) \nabla w \, ds \right\}, \] (4.1)
where \( C \) is a bounded constant independent of \( h, L, u \) and \( w \).

Proof. By the following arguments similarly in Theorem 3.1, we also have:
\[ \hat{a}_h(u - u_h^P, v) = \int_{E_0} \frac{\partial u}{\partial n} (v^+ - v^-) \, dl, \quad \forall v \in V_h, \] (4.2)
Then for \( w = u_p^p - \hat{u}_{I,L} \in V_h \),

\[
C_0 \| w \|_{H^1}^2 \leq \hat{a}_h(u_p^p - \hat{u}_{I,L}, w) = \hat{a}_h(u - \hat{u}_{I,L}, w) - \int_{\Gamma_0} \frac{\partial u}{\partial n} (w^+ - w^-) \, dl,
\]

where \( C_0(> 0) \) is constant independent of \( h, L, u \) and \( w \). We have

\[
\hat{a}_h(u - \hat{u}_{I,L}, w) = \int_{S_1} \nabla (u - \hat{u}_{I,L}) \nabla w \, ds + \int_{S_2} \nabla (u - \hat{u}_{I,L}) \nabla w \, ds + \hat{D}(u - \hat{u}_{I,L}, w)
\]

\[
= \int_{S_1} \nabla (u - \hat{u}_{I,L}) \nabla w \, ds + \int_{S_2} \nabla (\hat{u}_I - u_L) \nabla w \, ds + \hat{D}(u - \hat{u}_{I,L}, w).
\]

The penalty term

\[
\hat{D}(u - \hat{u}_{I,L}, w) = \frac{P_c}{h^p} \int_{\Gamma_0} (u_I^+ - \hat{u}_L^-)(w^+ - w^-) \, dl = 0,
\]

because of (3.2) and integration rule (2.13). Combining (4.4) and (4.5) leads to the desired results (4.1).

The bounds in Theorem 4.1 are analogous to those in Theorem 3.1, but the key difference is that the space \( V_h \) does not offer the explicit relation between \( w^+ \) and \( w^- \) along \( \Gamma_0 \). Hence the bounds of the last term in (4.1) should be estimated in different manner from the above. We can prove the following lemma.

**Lemma 4.1.** Let (3.23) be given, then for \( v \in V_h \),

\[
\| v^+ - v^- \|_{0, I_0} \leq C \{ \| v^+ - v^- \|_{0, I_0} + (hL^\mu)^2 \| v \|_{1, S_1} \}
\]

and

\[
\| v^+ - v^- \|_{0, I_0} \leq C \{ \| v^+ - v^- \|_{0, I_0} + (hL^\mu)^2 \| v \|_{1, S_1} \}.
\]

**Lemma 4.2.** Let (3.23) be given then for \( v \in V_h \),

\[
\left| \int_{\Gamma_0} \frac{\hat{u}}{\partial n} (w^+ - w^-) \, dl \right| \leq C \left\| \frac{\hat{u}}{\partial n} \right\|_{0, I_0} (h^{p/2} + (hL^\mu)^2) \| w \|_h.
\]

**Proof.** We have from Schwarz’s inequality and (4.6)

\[
\left| \int_{\Gamma_0} \frac{\hat{u}}{\partial n} (w^+ - w^-) \, dl \right| \leq \left\| \frac{\hat{u}}{\partial n} \right\|_{0, I_0} \| w^+ - w^- \|_{0, I_0}
\]

\[
\leq C \left\| \frac{\hat{u}}{\partial n} \right\|_{0, I_0} \{ \| w^+ - w^- \|_{0, I_0} + (hL^\mu)^2 \| w \|_{1, S_1} \}
\]
\[
\leq C \left\| \frac{\partial u}{\partial n} \right\|_{0, T_0} \left\{ \| h^{\gamma/2} \| \nu_{\delta} \| w \|_h + (hL)\nu \| w \|_{1, S_1} \right\}
\]
\[
\leq C \left\| \frac{\partial u}{\partial n} \right\|_{0, T_0} \left\{ \| h^{\gamma/2} \| (hL)\nu \| w \|_h \right\},
\]

(4.9)

This completes the proof of Lemma 4.2. □

Similarly, we obtain the following theorem from Lemmas 4.2, 3.1, 3.2 and (3.28).

**Theorem 4.2.** Let all conditions in Theorem 3.1 hold. Then there exist the error bounds of \( u_h^p \)
\[
\| u_h^p - \hat{u}_{l,L} \|_h \leq e_2
\]
\[
\leq C \left\{ h^2 \| u \|_{3, S_1} + \| R_L \|_{1, S_1} + \left\| \frac{\partial u}{\partial n} \right\|_{0, T_0} (h^{\gamma/2} + (hL)\nu) + h^{-1/2} \| R_L \|_{0, T_0} \right\}.
\]

(4.10)

Moreover for the a posteriori interpolant \( \Pi_p u_h^p \), we also have the following theorem.

**Theorem 4.3.** Let all conditions in Theorem 3.1 hold. Then
\[
\| u - \Pi_p u_h^p \|_h \leq C e_2.
\]

(4.11)

**Proof.** We have
\[
\| u - \Pi_p u_h^p \|_h \leq \| u - \Pi_p \hat{u}_{l,L} \|_h + \| \Pi_p (\hat{u}_{l,L} - u_h^p) \|_h.
\]

(4.12)

Since interpolation rule (3.30) gives
\[
(\Pi_{2h} v)^- = (v^+) \quad \text{on } P_i \subset \Gamma_0,
\]
then
\[
(\Pi_p v)^+ = (\Pi_p v^-) \quad \text{on } P_i \subset \Gamma_0.
\]

(4.13)

This leads to
\[
\| u - \Pi_p \hat{u}_{l,L} \|_{0, T_0} = 0.
\]

(4.14)

Also,
\[
\| \Pi_p (\hat{u}_{l,L} - u_h^p) \|_{0, T_0} = \| \hat{u}_{l,L} - u_h^p \|_{0, T_0}.
\]

(4.15)

We then have
\[
\| u - \Pi_p \hat{u}_{l,L} \|_h \leq C \left\{ \| u - \Pi_{2h} \hat{u}_{l,L} \|_{1, S_1} + \| u - u_L \|_{1, S_1} + \left( \frac{P_c}{h^\alpha} \right)^{1/2} \| u - \Pi_p \hat{u}_{l,L} \|_{0, T_0} \right\}
\]
\[
\leq C \left\{ \| u - \Pi_{2h} \hat{u}_{l,L} \|_{1, S_1} + \| R_L \|_{1, S_1} \right\}
\]
\[
\leq C \left\{ \| u - \Pi_{2h} u_i \|_{1, S_1} + \| \Pi_{2h} (\hat{u}_{l,L} - u_i) \|_{1, S_1} + \| R_L \|_{1, S_1} \right\}.
\]

(4.16)

From (3.35) and (3.36)
\[
\| u - \Pi_p \hat{u}_{l,L} \|_h \leq C \left\{ h^2 \| u \|_{3, S_1} + h^{-1/2} \| R_L \|_{0, T_0} + \| R_L \|_{1, S_1} \right\}.
\]

(4.17)
On the other hand, we obtain from (4.15) and Theorem 4.2

\[
\| \Pi_p(\hat{u}_{I,L} - u_h^p) \|_h \leq C \left\{ \| \Pi^2_{hL}(\hat{u}_{I,L} - u_h^p) \|_{1,S_I} + \| u_{I,L} - u_h^p \|_{1,S_I} + \left( \frac{P_c}{H^p} \right)^{1/2} \| \Pi_p(\hat{u}_{I,L} - u_h^p) \|_{0,R_0} \right\}
\]

\[
\leq C \left\{ \| \hat{u}_{I,L} - u_h^p \|_{1,S_I} + \| u_h^p \|_{1,S_1} + \left( \frac{P_c}{H^p} \right)^{1/2} \| \hat{u}_{I,L} - u_h^p \|_{0,R_0} \right\}
\]

\[
\leq C \| \hat{u}_{I,L} - u_h^p \|_h \leq C_{c_2}.
\]

(4.18)

Combining (4.17) and (4.18) leads to (4.11).

**Corollary 4.1.** Let all conditions in Theorem 3.1 hold. Also assume Eq. (3.37) be given. Then

\[
\| u - \Pi_p u_h^p \|_h = O(h^2) + O(h^2 L^2 u) + O(h^{\sigma/2}).
\]

(4.19)

Moreover if \( \sigma \geq 4 \) and (3.40) are given,

\[
\| u - \Pi_p u_h^p \|_h = O(h^{2-\delta}), \quad 0 < \delta \ll 1.
\]

The penalty combination leads to the same global convergence rates as the nonconforming combination does because

\[
\| u - \Pi_p u_h^p \|_1 \leq \| u - \Pi_p u_h^p \|_h = O(h^{2-\delta}).
\]

(4.21)

Moreover, we have for \( \sigma \geq 4 \)

\[
\| u - \Pi_p u_h^p \|_{0,R_0} \leq C h^{\sigma/2} \| u - \Pi_p u_h^p \|_h = O(h^{2+(\sigma/2)-\delta}).
\]

(4.22)

Finally let us derive the relation of the penalty coupling to the nonconforming constraints (2.10).

**Corollary 4.2.** Let all conditions in Corollary 4.1 hold. Then when \( \sigma \geq 4 \) the average jumps of the solution \( v = \Pi_p u_h^p \) at the element nodes \( Z_i \in \Gamma_0 \) have the convergence rates

\[
E(v^+ - v^-) = \frac{1}{N+1} \sum_{i=0}^{N} |v^+(Z_i) - v^-(Z_i)| = O(h^{2+\sigma/2-\delta}),
\]

(4.23)

where \( N \) is the number of element nodes on \( \Gamma_0 \).

**Proof.** Denote first the trapezoidal rule

\[
\int_{\Gamma_0} v^2 \, dl = \sum_{k=1}^{N} \frac{Z_{k-1}Z_k}{2} [v^2(Z_{k-1}) + v^2(Z_k)].
\]

Since \( \frac{1}{2}(x^2 + y^2) \leq (x^2 + xy + y^2) \), we have from (2.13),

\[
\int_{\Gamma_0} v^2 \, dl \leq 3 \sum_{k=1}^{N} \frac{Z_{k-1}Z_k}{3} [v^2(Z_{k-1}) + v(Z_{k-1})v(Z_k) + v^2(Z_k)] = 3 \int_{\Gamma_0} v^2 \, dl.
\]

(4.24)
Since the rectangles are quasiuniform, we obtain from the Schwarz inequality
\[
\left( \sum_{k=0}^{N} \frac{v(Z_k)}{N+1} \right)^2 \leqslant \sum_{k=0}^{N} \frac{1}{N+1} v^2(Z_k) \leqslant C \left( \sum_{k=1}^{N} \frac{Z_k-Z_{k-1}}{2} [v^2(Z_{k-1}) + v^2(Z_k)] \right).
\]
Let \( v = \Pi_p u_h^+ \) and \( w = u - \Pi_p u_h^+ \)
\[
E(v^+ - v^-) = \frac{1}{N+1} \sum_{i=0}^{N} |(\Pi_p u_h^+)_Z - (\Pi_p u_h^+)_Z| \leqslant C \| (\Pi_p u_h^+)^+ - (\Pi_p u_h^+)^- \|_{0,t_0}
\]
when \( \sigma \geqslant 4 \),
\[
E(v^+ - v^-) \leqslant C h^{2+\frac{n}{2} - \delta}
\]
from Corollary 4.1. This completes the proof of Corollary 4.2. □

Corollary 4.2 displays an interesting fact that the average jumps between \((\Pi_p u_h^+)^+\) and \((\Pi_p u_h^+)^-\) are \(O(h^{2+\frac{n}{2} - \delta})\). Hence when \( \sigma \to \infty \), the penalty combination leads to the nonconforming combination. Note that the large values of \( \sigma(\geqslant 4) \) do not incur the reduced convergence rates.

**Remark.** We may derive similarly the same global superconvergence \(O(h^{2-\delta})\) for Combinations I, II and symmetric combination under the condition \( \sigma \geqslant 2 \). Although their algorithms are more complicated, using smaller values of \( \sigma \) will lead to better stability.

5. Comparisons and computations

Now, let us clarify the relation between [5,6] and this paper, provide some numerical experiments and make some remarks.

In [5,6], the semi-norm in discrete summation is defined as
\[
\|v\|_{1,S_i} = \left\{ \sum_{ij} \int_{\partial ij} (v_x^2 + v_y^2) \, ds \right\}^{1/2},
\]
where
\[
\int_{\partial ij} v_x^2 \, ds = \frac{h_{ij}}{2} (v_x(A) + v_x(B)),
\]
\[
\int_{\partial ij} v_y^2 \, ds = \frac{h_{ij}}{2} (v_y(C) + v_y(D))
\]
and \( A, B, C \) and \( D \) are the mid-points of edges of \( \square_{ij} \), shown in Fig. 3. Then we have the following lemma.
Lemma 5.1. Let
\[ u \in C^3(S_1) \] (5.3)
and assume the numerical solutions \( u_h \in V_h \) have the errors
\[ |u - u_h|_{1,S_1} = O(h^{2-\delta}), \quad 0 \leq \delta < 1. \] (5.4)
Then the global errors between \( u_h \) and the solution interpolant \( u_I \) have also the error bounds
\[ |u_I - u_h|_{1,S_1} = O(h^{2-\delta}). \] (5.5)

Proof. For the piecewise bilinear functions, \( v \in V_h \), we have
\[ |v|_{1,S_1} = \sum \int_{\mathcal{D}_i} (v_x^2 + v_y^2) \, ds, \]
where
\[ \int_{\mathcal{D}_i} v_x^2 \, ds = \frac{h_i k_j}{3} [v_x^2(A) + v_x^2(B) + v_x(A)v_x(B)], \]
\[ \int_{\mathcal{D}_i} v_y^2 \, ds = \frac{h_i k_j}{3} [v_y^2(C) + v_y^2(D) + v_y(C)v_y(D)]. \]
Since \( x^2 + y^2 + xy \leq \frac{3}{2}(x^2 + y^2) \), we obtain for \( v \in V_h \),
\[ \int_{\mathcal{D}_i} v_x^2 \, ds \leq \frac{h_i k_j}{2} [v_x^2(A) + v_x^2(B)] = \int_{\mathcal{D}_i} v_x^2 \, ds. \]
Hence
\[ |v|_{1,S_1} \leq \bar{|v|}_{1,S_1}, \quad \forall v \in V_h. \] (5.6)
We then have
\[ |u_I - u_h|_{1,S_1} \leq |u_I - u|_{1,S_1} + |u - u_h|_{1,S_1}, \] (5.7)
Also,
\[ |u - u_I|_{1,S_1} = Ch^2 |u|_{3,\infty,S_1} = O(h^2), \] (5.8)
where \( |u|_{3,\infty,S_1} \) is also the Sobolev \( p \)-norm with \( p = \infty \). Desired results (5.5) follow (5.4), (5.7) and (5.8). This completes the proof of Lemma 5.1. \( \square \)
Lemma 5.2. Let (5.3) and
\[ |u_l - u_h|_{1,S_1} = O(h^{2-\delta}), \quad 0 \leq \delta < 1 \]
be given. Then
\[ |u - u_h|_{1,S_1} = O(h^{2-\delta}). \]

Proof. Similarly, from the arguments in Lemma 5.1 we have
\[ |u_l - u_h|_{1,S_1} \leq \sqrt{3} |u_l - u_h|_{1,S_1}. \]
Then
\[
|u - u_h|_{1,S_1} \leq |u - u_h|_{1,S_1} + |u_l - u_h|_{1,S_1} \leq O(h^2) + \sqrt{3} |u_l - u_h|_{1,S_1} \\
= O(h^2) + O(h^{2-\delta}) = O(h^{2-\delta}). \quad \Box
\]

Lemmas 5.1 and 5.2 imply that under assumption (5.3), the superconvergence in this paper and in [5,6] is equivalent to each other. We then conclude that the order \(O(h^{2-\delta})\) also holds for superconvergence of the solutions of RGM–FEM in this paper, to the average nodal derivatives at the edge mid-points of \(\square_{ij}\), so does for the maximal nodal derivatives in majority.

On the other hand, based on Lemma 5.1, the solutions in [6] by the combinations of Ritz–Galerkin and finite-difference methods also have the global superconvergence as (3.41) and (3.42). However, the proofs in this paper using Lin’s techniques are simpler so as to be extend to other kinds of singularity problems, i.e., in biharmonic equations and elastic plates with cracks. Details appear elsewhere. Also note that the assumption \(u \in H^3(S_1)\) in Theorem 3.2 is weaker than (5.3).

In this section, numerical experiments are also carried out to confirm the global superconvergence \(O(h^{2-\delta})\) made in Section 3 by the nonconforming combination. Other kinds of combinations can also be verified by numerical experiments. Let us consider the typical Motz problem (see Fig. 4):

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } S, \tag{5.10}
\]

\[
u_{|x<0 \land y=0} = 0, \quad u_{|x=1} = 500, \tag{5.11}
\]

\[
\frac{\partial u}{\partial y} \bigg|_{y=1} = \frac{\partial u}{\partial y} \bigg|_{x>0 \land y=0} = \frac{\partial u}{\partial x} \bigg|_{x=-1} = 0, \tag{5.12}
\]
Table 1
Errors norms and condition number of nonconforming combination of RGM–FEM

<table>
<thead>
<tr>
<th>Divisions</th>
<th>Max $|e|_{a,S}$</th>
<th>$|e|_a$</th>
<th>$|\bar{e}|_1$</th>
<th>$|\bar{e}|_1$</th>
<th>$|A|_1$</th>
<th>$|A|_1$</th>
<th>$|u - \Pi_{L} u_{h}|_1$</th>
<th>Con.</th>
</tr>
</thead>
<tbody>
<tr>
<td>MS = 1</td>
<td>8.10</td>
<td>3.77</td>
<td>42.8</td>
<td>15.6</td>
<td>15.9</td>
<td>12.6</td>
<td>13.0</td>
<td>8</td>
</tr>
<tr>
<td>$L + 1 = 3$</td>
<td>3.23</td>
<td>1.29</td>
<td>22.4</td>
<td>9.66</td>
<td>9.75</td>
<td>8.72</td>
<td>8.82</td>
<td>33</td>
</tr>
<tr>
<td>MS = 3</td>
<td>1.33</td>
<td>0.559</td>
<td>14.3</td>
<td>4.19</td>
<td>4.22</td>
<td>3.93</td>
<td>3.97</td>
<td>82</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td>0.779</td>
<td>0.315</td>
<td>10.6</td>
<td>2.33</td>
<td>2.35</td>
<td>2.22</td>
<td>2.23</td>
<td>163</td>
</tr>
<tr>
<td>MS = 6</td>
<td>0.358</td>
<td>0.141</td>
<td>6.97</td>
<td>1.04</td>
<td>1.05</td>
<td>1.01</td>
<td>1.02</td>
<td>462</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td>0.214</td>
<td>0.0794</td>
<td>5.20</td>
<td>0.583</td>
<td>0.587</td>
<td>0.567</td>
<td>0.572</td>
<td>470</td>
</tr>
</tbody>
</table>

where $S$ is a rectangle ($-1 \leq x \leq 1$, $0 \leq y \leq 1$). The origin $(0,0)$ is a singular point with the solution behaviour $u = O(r^{1/2})$ as $r \to 0$ due to the intersection of the Neumann and Dirichlet conditions.

Divide $S$ by $\Gamma_0$ into $S_2$ and $S_1$. The subdomain $S_2$ is chosen as a smaller rectangle ($-1/2 \leq x \leq 1/2$, $0 \leq y \leq 1$). Also the subdomain $S_1$ is again split into uniform rectangular elements shown in Fig. 4. The admissible functions are chosen as

$$v = \begin{cases} v^- = v_1, \\ v^+ = \sum_{l=0}^{L} \tilde{D}_l r^{l+1/2} \cos(l + \frac{1}{2}) \theta, \end{cases} \quad (5.13)$$

where $\tilde{D}_l$ are unknown coefficients, and $(r, \theta)$ are the polar coordinates with origin $(0,0)$.

Let $MS$ denote the difference division number along $DB$. Based on the good matching between $L + 1$ (the total number of basis functions used) and $MS$ given in [4], we will choose

$$MS = 2 \text{ and } L + 1 = 4; \quad MS = 3, 4 \text{ and } L + 1 = 5; \quad MS = 6, 8 \text{ and } L + 1 = 6. \quad (5.14)$$

Numerical solutions are conducted by the nonconforming combinations of RGM–FEM; and their error norms and the coefficients are provided in Tables 1 and 2. “Con.” in Table 1 denotes the condition number of the associated matrix resulting from the nonconforming combination, and other error norms are defined by

$$\|e\|_{a,S} = \left( \int_S e^2 \, ds \right)^{1/2}, \quad \text{max} = \max_S |e|,$$

$$\|\bar{e}\|_1 = (\|\bar{e}\|_{1,S_1}^2 + |\bar{e}|_{1,S_1}^2)^{1/2}, \quad \|\bar{e}\|_1 = (\|\bar{e}\|_{1,S_1}^2 + |\bar{e}|_{1,S_1}^2)^{1/2},$$

$$|A|_1 = (|u_h - u_I|_{1,S_1}^2 + |u_L - u_I|_{1,S_1}^2)^{1/2},$$

$$\|A\|_1 = (\|u_h - u_I\|_{1,S_1}^2 + \|u_L - u_I\|_{1,S_1}^2)^{1/2}.$$
Table 2
Approximate coefficients by nonconforming combination of RGM–FEM

<table>
<thead>
<tr>
<th>Coe.</th>
<th>$\tilde{D}_0$</th>
<th>$\tilde{D}_1$</th>
<th>$\tilde{D}_2$</th>
<th>$\tilde{D}_3$</th>
<th>$\tilde{D}_4$</th>
<th>$\tilde{D}_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MS = 1</td>
<td>403.724</td>
<td>81.257</td>
<td>2.670</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$L + 1 = 3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MS = 2</td>
<td>399.364</td>
<td>86.172</td>
<td>13.759</td>
<td>$-23.291$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$L + 1 = 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MS = 3</td>
<td>400.593</td>
<td>86.998</td>
<td>15.677</td>
<td>$-15.336$</td>
<td>2.936</td>
<td>–</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MS = 4</td>
<td>400.847</td>
<td>87.278</td>
<td>16.350</td>
<td>$-12.169$</td>
<td>2.199</td>
<td>–</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MS = 6</td>
<td>401.023</td>
<td>87.585</td>
<td>16.773</td>
<td>$-9.898$</td>
<td>1.743</td>
<td>1.080</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MS = 8</td>
<td>401.085</td>
<td>87.616</td>
<td>16.962</td>
<td>$-9.097$</td>
<td>1.601</td>
<td>0.740</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>True</td>
<td>401.162</td>
<td>87.656</td>
<td>17.238</td>
<td>$-8.071$</td>
<td>1.440</td>
<td>0.331</td>
</tr>
</tbody>
</table>

where $\varepsilon = u - \tilde{u}_h$ and $\tilde{u}_h = \tilde{u}_L$ in $S_2$. It is easy to see from the data in Tables 1 and 2 that

$$\|\tilde{u}\|_h = O(h), \quad \|\tilde{u}\|_{0,5} = O(h^2), \quad \max = O(h^{2-\delta}),$$

(5.15)

$$|\Delta|_1 = O(h^{2-\delta}), \quad \|\Delta\|_1 = O(h^{2-\delta}),$$

(5.16)

$$\|u - \Pi_{2h}^2 u_h\|_1 = O(h^{2-\delta}),$$

(5.17)

$$\|\tilde{u}\|_1 = O(h^{2-\delta}), \quad \|\tilde{u}\|_1 = O(h^{2-\delta}),$$

(5.18)

$$|D_0 - \tilde{D}_0| = O(h^2),$$

(5.19)

where $D_0$ and $\tilde{D}_0$ are the true and approximate coefficients, respectively. In Table 1, the data for $\|u - \Pi_{2h}^2 u_h\|_1$ are not available when using odd MS = 1, 3. Note that Eqs. (5.15) and (5.19) coincide with the optimal results in [4]; Eqs. (5.16), (5.17) and (5.18) verify perfectly the analysis in Section 3 and this section, respectively.

Finally let us make a few remarks.

1. The rectangular elements are discussed on this paper. In fact, when the uniform right triangle elements with the interior angle $\pi/4$ are chosen, only the global superconvergence rates $O(h^{3/2})$ can be gained, based on the analysis on this paper and Lin and Yan’s [12]. Once the solution domain is not very complicated, the rectangles and such simple triangles can be employed. Both global and nodal superconvergence can be achieved simultaneously. Note that for the global superconvergence, the a posteriori interpolant on the numerical solution costs a little more computation effort.

2. The nonconforming combination will deal with the unknown constraints (2.10). Instead, we may solicit other combinations. Corollary 4.2 implies that when $\sigma$ is large, the penalty combination approaches, indeed, the nonconforming combination. In fact, such a large weight technique has already been applied in engineering computations. Among five combinations, the nonconforming combination is still basic.
3. The different coupling strategies are studied in [7,9] for Ritz–Galerkin–FEM and in [5,6] for Ritz–Galerkin–FDM. The optimal convergence $O(h)$ and the nodal superconvergence $O(h^{2-\delta})$ have been proven. In this paper the global superconvergence $O(h^{2-\delta})$ has been exploited together. Note that the high superconvergence of combinations yields the higher accuracy of both $u_h$ in $S_1$ and the coefficients $a_i$ of the solution in $S_2$. The leading coefficient $a_1$ is, in fact, the stress intensity factor at the singularity, which has important application in fracture mechanics.

Acknowledgements

The authors are grateful to the referee for his/her valuable comments and careful reading.

References