On the double points of a Mathieu equation

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Abstract

For a Mathieu equation with parameter \( q \), the eigenvalues can be regarded as functions of the variable \( q \). Our aim is to find \( q \) when adjacent eigenvalues of the same type become equal yielding double points of the given Mathieu equation. The problem reduces to an equivalent eigenvalue problem of the form \( BX = \lambda X \), where \( B \) is an infinite tridiagonal matrix. A method is developed to locate the first double eigenvalue to any required degree of accuracy when \( q \) is an imaginary number. Computational results are given to illustrate the theory for the first double eigenvalue. Numerical results are given for some subsequent double points. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The motivation for this work comes from the discussion of eigenvalues for the Mathieu equation

\[
\frac{d^2y}{dx^2} + (\lambda - 2q \cos 2x)y = 0. \tag{1.1}
\]

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In general, Mathieu equations do not have periodic solutions (see [3]). If \( q \) is a real number, there exist infinitely many distinct eigenvalues \( \lambda \) corresponding to periodic solutions. For a detailed account of eigenvalues of (1.1) when \( q \) is real see [6].

Our interest is in the case when two consecutive eigenvalues merge and become equal for some values of parameter \( q \). This pair of merging points is called a double point of (1.1) for that value of \( q \). In this paper, we will discuss only real double points. It has been studied in [1] that for real double points to occur, the parameter \( q \) must attain some pure imaginary value. A comprehensive account of the eigenvalues of (1.1) and its branch points are given in [4]. They give an account of asymptotic expansions for eigenvalues. Blanch and Clemm [2] use truncated power series expansions for the eigenvalues and evaluate approximate values for \( q \) when it results in a double eigenvalue. Their result, while giving excellent numerical values for a number of particular cases, does not have a computable error analysis. There does not seem to be any earlier results concerning double points which include an error analysis. For finite matrices, Lippert and Edelman [5] develop heuristical and geometrical techniques to locate double eigenvalues. In this paper, we shall develop an algorithm to compute some special double points. Theoretically, this method can achieve any required accuracy. For these special double points, the numerical results compare very well with that of Blanch and Clemm.

Throughout the paper we concentrate on the solution of (1.1) with the boundary conditions

\[
y(0) = y(\pi/2) = 0.
\]

Using an even solution of the form

\[
y(x) = \sum_{r=1}^{\infty} x_r q^{-r} \sin 2rx,
\]

equation (1.1) yields the infinite linear algebraic system

\[
BX = \lambda X, \quad (1.2)
\]

where \( B \) is an infinite tridiagonal matrix \( B = (b_{ij}) \) given by

\[
b_{ij} = \begin{cases}
-Q, & j = i - 1, \ i \geq 2, \\
4i^2, & j = i, \\
1, & j = i + 1, \ i \geq 1.
\end{cases}
\]

Here \( Q = -q^2 > 0 \) and \( X = (x_1, x_2, \ldots)^T \). This means \( \lambda \) is an eigenvalue of the infinite matrix \( B \). In Section 2, we give details of the notation used and some preliminary results concerning finite truncation of the system (1.2). Existence and uniqueness of a unique double point in a stated interval \([4, 16]\) is derived in Section 3 with bounds for \( Q \). Section 4 establishes the nonexistence of a double point in \([16, 36]\), while Section 5 develops an algorithm to compute the values of \( Q \) and \( \lambda \) for that first double point by using the method of bisection. The techniques developed are also used to numerically find the values of \( Q \) and for the double point in \( \lambda \in (36, 64) \).
2. Notations and preliminaries

We use the notation $A_{k,n}$ to denote the principle submatrix of $A = B - I$ from the $k$th row to the $n$th row, i.e.,

$$A_{k,n} = \begin{pmatrix} 4k^2 - \lambda & 1 \\ -Q & 4(k+1)^2 - \lambda & 1 \\ \vdots & \vdots & \ddots & \ddots \\ -Q & 4n^2 - \lambda \end{pmatrix}.$$  \hfill (2.1)

We define

$$a_{k,n} = a_{k,n}(\lambda, Q) = \det A_{k,n}, \quad d_{k,n} = \prod_{j=k}^{n} 4j^2$$

and

$$f_{k,n} = f_{k,n}(\lambda, Q) = a_{k,n}(\lambda, Q)/d_{k,n},$$

where $a_{k,k-1} = a_{n+1,n} = 1$.

First we give an expansion formula for $a_{k,n}$.

**Lemma 2.1.** If $k \leq l \leq n - 1$, then

$$a_{k,n} = a_{k,l}a_{l+1,n} + Qa_{k,l-1}a_{l+2,n}.$$  \hfill (2.2)

**Proof.** Expanding the determinant $a_{k,n}$ along the first row, we get

$$a_{k,n} = a_{k,k}a_{k+1,n} + Qa_{k+2,n}.$$  

Clearly (2.2) holds when $l = k$. Assuming that (2.2) holds for $l = j \leq n - 2$ and expanding the determinant $a_{j+1,n}$ along the first column, we obtain

$$a_{k,n} = a_{k,k}a_{j+1,n} + Qa_{k,j-1}a_{j+2,n}$$

$$= a_{k,k}a_{j+1,j+1}a_{j+2,n} + Qa_{j+3,n} + Qa_{k,j-1}a_{j+2,n}$$

$$= a_{j+2,n}(a_{k,k}a_{j+1,j+1} + Qa_{k,j-1}) + Qa_{k,j}a_{j+3,n}$$

$$= a_{k,k}a_{j+1,j+2,n} + Qa_{k,j}a_{j+3,n}.$$  

Hence, Eq. (2.2) holds for $l = j + 1$, which completes the proof. \hfill \Box

It follows from (2.2) that

$$f_{k,n} = f_{k,l}f_{l+1,n} + \frac{Q}{16(l+1)^2f_{l-1}f_{l+2,n}},$$  \hfill (2.3)

Also it is easy to show by induction that if $4k^2 - \lambda > 0$ and $Q > 0$, then

$$0 < a_{k,l} \leq a_{k+1,l+1} \quad \text{and} \quad \frac{a_{k,l}a_{l+2,j}}{a_{k,l}} \leq \frac{1}{4(l+1)^2 - \lambda}, \quad j \geq k.$$  

The next lemma bounds the first partial derivatives of $a_{k,n}$.
Lemma 2.2. If $4k^2 - \lambda > 0$ and $Q > 0$, then

$$0 < \frac{\partial a_{k,n}}{\partial Q} < g(\lambda, k+1)a_{k+1,n},$$  \hspace{1cm} (2.4)

$$0 < -\frac{\partial a_{k,n}}{\partial \lambda} < g(\lambda, k)a_{k,n},$$  \hspace{1cm} (2.5)

where $g(\lambda, k) = \sum_{j=k}^{\infty} \frac{1}{4j^2 - \lambda}$.

Proof. With $a_{k,k-1} = 1$ and $a_{k+1,n} = 1$, we have

$$\frac{\partial a_{k,n}}{\partial Q} = \sum_{j=k}^{n-2} a_{k,j}a_{j+3,n}$$

$$\leq \sum_{j=k}^{n-2} a_{k+1,j+1}a_{j+3,n}$$

$$\leq \sum_{j=k}^{n-2} \frac{a_{k+1,n}}{4(j+2)^2 - 1}$$

$$< g(\lambda, k+1)a_{k+1,n}.$$

Similarly,

$$-\frac{\partial a_{k,n}}{\partial \lambda} = \sum_{j=k}^{n-1} a_{k,j}a_{j+2,n} \leq a_{k,n} \sum_{j=k}^{n-1} \frac{1}{4(j+1)^2 - \lambda} < g(\lambda, k)a_{k,n}.$$  \hspace{1cm} (2.6)

On using (2.4) and (2.5), we can bound second partial derivatives of $a_{k,n}$. We get

$$\frac{\partial^2 a_{k,n}}{\partial Q^2} = \sum_{j=k}^{n-2} \left( \frac{\partial a_{k,j}}{\partial Q} a_{j+3,n} + a_{k,j} \frac{\partial a_{j+3,n}}{\partial Q} \right)$$

$$\leq \sum_{j=k}^{n-2} (g(\lambda, k+1)a_{k+1,j}a_{j+3,n} + g(\lambda, j+4)a_{k,j}a_{j+4,n})$$

$$\leq 2g(\lambda, k+1) \sum_{j=k}^{n-2} a_{k+1,j+1}a_{j+3,n}$$

$$< 2g(\lambda, k+1)^2 a_{k+1,n}.$$  \hspace{1cm} (2.6)

Similarly we can show that

$$0 < \frac{\partial^2 a_{k,n}}{\partial \lambda^2} < 2g(\lambda, k)^2 a_{k,n}$$  \hspace{1cm} (2.7)

and

$$0 < \frac{\partial^2 a_{k,n}}{\partial \lambda \partial Q} < 2g(\lambda, k)g(\lambda, k+1)a_{k,n}.$$  \hspace{1cm} (2.8)

We will now prove that as $n$ tends to $\infty$, $f_{k,n}$ converges uniformly in each bounded and closed region in $(\lambda, Q) \in (-\infty, +\infty) \times (0, +\infty)$. 

Lemma 2.3. Let \( \Omega \) be a bounded and closed region in \( (\lambda, Q) \in (-\infty, +\infty) \times (0, +\infty) \). For each \( k \) and \( n \), \( f_{k,n} \) uniformly converges in \( \Omega \) as \( n \) tends to \( \infty \).

Proof. Let
\[
p = \max_{(\lambda, Q) \in \Omega} \frac{Q + \lambda + 1}{4}.
\]
If \( k > p \), then \( 4(k + 1)^2 - \lambda - (4k^2 - \lambda) > 2(Q + 1) \). It follows from Gershgorin Theorem [6] that the \( j \)th eigenvalue of \( A^{k,n} \) lies in the interval
\[
[4(k + j - 1)^2 - \lambda - Q - 1, 4(k + j - 1)^2 - \lambda + Q + 1].
\]
Thus
\[
\prod_{j=k}^{\infty} \left( 1 - \frac{p}{j^2} \right) \leq \prod_{j=k}^{n} \left( 1 - \frac{\lambda + Q + 1}{4j^2} \right) \leq f_{k,n} \leq \prod_{j=k}^{\infty} \left( 1 - \frac{\lambda - Q}{4j^2} \right) \leq \prod_{j=k}^{\infty} \left( 1 + \frac{p}{j^2} \right).
\]
If \( n_1 > n_0 > k \), \( \varpi = \prod_{j=k}^{\infty} \left( 1 + p/j^2 \right) \) and letting \( l = n_0 \), \( n = n_1 \) in (2.3), we have
\[
|f_{k,n_1} - f_{k,n_0}| = \left| f_{k,n_0} (f_{n+1,n_1} - 1) + \frac{Q}{16n_0^2(n_0 + 1)^2} f_{k,n_0} - f_{k+2,n_1} \right| \
\leq \varpi \left( \prod_{j=n_0+1}^{\infty} \left( 1 + \frac{p}{j^2} \right) - 1 \right) + \frac{px^2}{4n_0^2(n_0 + 1)^2}.
\]
The right-hand side tends to 0 independently of \( \lambda \), \( Q \) and \( n_1 \) as \( n_0 \) tends to infinity. Hence \( \{f_{k,n}\}_{n=k}^{\infty} \) is a Cauchy sequence and uniformly converges in \( \Omega \). If \( k < p \), let \( l = \lceil p \rceil + 1 \). From (2.3) it is easy to show that \( f_{k,n} \) also converges uniformly. \( \square \)

Defining
\[
f_k = \lim_{n \to \infty} f_{k,n},
\]
it is clear that \( f_k \) is continuous in \( (\lambda, Q) \in (-\infty, +\infty) \times (0, +\infty) \). Using the method of proof Lemma 3, we can prove that the partial derivatives
\[
\frac{\partial^j f_{k,n}}{\partial \lambda^i \partial Q^{j-i}}, \quad 1 \leq j \leq 2, \quad 0 \leq i \leq j
\]
also converge uniformly in each closed and bounded region in \( (\lambda, Q) \in (-\infty, +\infty) \times (0, +\infty) \). Therefore,
\[
\frac{\partial^j f_k}{\partial \lambda^i \partial Q^{j-i}} = \lim_{n \to \infty} \frac{\partial^j f_{k,n}}{\partial \lambda^i \partial Q^{j-i}}.
\]
It is stated in [6] that \( \lambda \) is an eigenvalue of \( B \) if and only if it satisfies \( f_1(\lambda, Q) = 0 \). It is clear that \( \lambda \) is a double point of (1.1) if and only if \( \lambda \) satisfies
\[
f_1(\lambda, Q) = 0,
\]
\[
\frac{\partial f_1(\lambda, Q)}{\partial \lambda} = 0.
\] (2.9)
3. The first double point

Since \( f(\lambda, Q) > 0 \) if \( \lambda < 4 \) and \( Q > 0 \), the real double points lies in the interval \([4, \infty)\), which we partition into separate sub-intervals

\[
[0, \infty) = \bigcup_{k=1}^{\infty} I_k,
\]

where \( I_k = [4k^2, 4(k+1)^2)\).

In this section, we will prove that there exists a unique double point in the interval \([4, 16]\). In other words, we prove that there is only one solution to Eq. (2.9).

**Lemma 3.1.** If \( \lambda \in [4, 16] \) and \( a_{1,2} = Q + (\lambda - 4)(\lambda - 16) \geq 0 \), then

\[
\frac{\partial f_1}{\partial Q} > 0.
\]

**Proof.** Noting that

\[
g(\lambda, 5) = \sum_{j=5}^{\infty} \frac{1}{4j^2 - \lambda} \leq \sum_{j=5}^{\infty} \frac{1}{4j(j - 1)} \leq \frac{1}{16}
\]

and

\[
a_{1,n} = a_{1,2}a_{3,n} + Qa_{1,1}a_{4,n},
\]

we have

\[
\frac{\partial f_1}{\partial Q} = \lim_{n \to \infty} \frac{1}{d_{1,n}} \frac{\partial a_{1,n}}{\partial Q} = \lim_{n \to \infty} \frac{1}{d_{1,n}} \left\{ \frac{\partial a_{1,2}}{\partial Q} a_{3,n} + a_{1,2} \frac{\partial a_{4,n}}{\partial Q} + a_{1,1} a_{4,n} + Qa_{1,1} \frac{\partial a_{4,n}}{\partial Q} \right\}
\]

\[
\geq \lim_{n \to \infty} \frac{1}{d_{1,n}} \left\{ a_{3,n} + (4 - \lambda)a_{4,n} + \frac{4 - \lambda}{16} Qa_{5,n} \right\}
\]

\[
= \lim_{n \to \infty} \frac{1}{d_{1,n}} \left\{ 2(20 - \lambda)a_{4,n} + \frac{Q(20 - \lambda)}{16} a_{5,n} \right\}
\]

\[
\geq \lim_{n \to \infty} 8a_{4,n} > 0. \quad \square
\]

Choosing \( k = 1 \) and letting \( n \to \infty \) in (2.3) yields

\[
f_1 = f_1, f_{i+1} + \frac{Q}{16f_1^2(l+1)^2} f_1, f_{i-1} f_{i+2},
\]

(3.1)

from which we can establish

**Lemma 3.2.** If \( Q > 0 \) and \( f_{1,i-1} f_{1,i} \geq 0 \), then \( f_1 < 0 \) if \( f_{1,i-1} < 0 \) or \( f_{1,i} < 0 \); and \( f_1 > 0 \) if \( f_{1,i-1} > 0 \) or \( f_{1,i} > 0 \).
Proof. Since
\[ f_{1,j} = f_{1,j-1} f_{j,j} + \frac{Q}{16j^2(j-1)^2} f_{1,j-2} \quad \text{for } j \leq l, \]
by induction we can show that sign\( (f_{1,j}) = (-1)^j \) for \( \lambda > 4l^2 \). From \( f_{1,l-1} f_{1,l} \geq 0 \), we have \( \lambda \leq 4l^2 \), which means \( f_2 \geq 0 \) for \( k \geq l + 1 \). Using (3.2) completes the proof. \( \square \)

If \( \lambda \in (4,16) \), \( f_{1,1} < 0 \). Consequently, if \( f_{1,2} \leq 0 \), i.e., \( Q \leq (\lambda - 4)(16 - \lambda) \), then \( f_1 < 0 \). If \( f_{1,2} \geq 0 \), then \( f_{1,2} > 0 \), \( f_1 > 0 \) and
\[ Q < \frac{(36 - \lambda)(16 - \lambda)(\lambda - 4)}{2(20 - \lambda)}. \]
Since \( f_1 \) is a continuous function, from Lemma 3.2 there exists a unique \( Q \) for each \( \lambda \) such that \( f(\lambda, Q) = 0 \). We regard this \( Q \) as a function of \( \lambda \) and write it as \( Q = Q(\lambda) \). Obviously,
\[
(\lambda - 4)(16 - \lambda) \leq Q(\lambda) \leq \frac{(36 - \lambda)(16 - \lambda)(\lambda - 4)}{2(20 - \lambda)}. \tag{3.2}
\]
Using Lemma 3.1 and Differentiating \( f(\lambda, Q) = 0 \) with respect to \( \lambda \), we have
\[
Q'(\lambda) = -\frac{\partial f_1/\partial \lambda}{\partial f_1/\partial Q}. \tag{3.3}
\]
Because \( Q(4) = Q(16) = 0 \), there exists at least one point \( \lambda_0 \) in interval \( [4,16] \) such that \( Q'(\lambda_0) = 0 \), i.e., \( \partial f_1/\partial \lambda(\lambda_0, Q(\lambda_0)) = 0 \), which means that \( \lambda_0 \) is a double point and \( Q(\lambda_0) \) is the corresponding singular point.

Now we show that there is a unique double point in \( [4,16] \). To do this, we first need the following lemma.

**Lemma 3.3.** If \( \lambda \) is a double point in \( [4,16] \), then
\[ \lambda < \frac{55}{4}. \]

**Proof.** From
\[
f_1 = \frac{4 - \lambda}{4} f_2 + \frac{Q}{64} f_3 = 0,
\]
\[
f_2 f_3 = \frac{1}{16} \frac{Q}{(\lambda - 4)}. \tag{3.4}
\]
We can easily check that \( g(\lambda, 3) \leq 2/15 \) and \( g(\lambda, 4) \leq 1/12 \). Thus
\[
-\frac{\partial f_3}{\partial \lambda} = \lim_{n \to \infty} \frac{1}{d_{3,n}} \left( -\frac{\partial a_{3,n}}{\partial \lambda} \right) < g(\lambda, 3) \cdot \lim_{n \to \infty} \frac{a_{3,n}}{d_{3,n}} \leq \frac{2}{15} f_3
\]
and
\[
-\frac{\partial f_4}{\partial \lambda} \leq \frac{1}{20} f_4.
\]
We have

\[
0 = -\frac{\partial f_1}{\partial \lambda} = \frac{f_2}{4} - \frac{4}{4} \frac{\partial f_2}{\partial \lambda} - \frac{Q}{64} \frac{\partial f_3}{\partial \lambda} = \frac{f_2}{4} + \frac{\lambda - 4}{4} \left\{ -\frac{f_3}{16} + \frac{16 - \lambda}{16} \cdot \frac{\partial f_3}{\partial \lambda} + \frac{\partial f_4}{64 \partial \lambda} \right\} - \frac{Q}{64} \frac{\partial f_3}{\partial \lambda}
\]

\[
\leq \frac{f_2}{4} - \frac{\lambda - 4}{4} \cdot \frac{f_3}{16} - \frac{Q}{64} \frac{\partial f_3}{\partial \lambda}
\]

\[
< \frac{f_2}{4} + \frac{1}{64} \left( \frac{2Q}{15} - (\lambda - 4) \right) f_3,
\]

i.e.,

\[
\frac{f_2}{f_3} \geq \frac{1}{16} \left( \lambda - 4 - \frac{2}{15} Q \right).
\]

Combining (3.4) and (3.5) produces

\[
Q(\lambda) \geq \frac{15(\lambda - 4)^2}{2\lambda + 7}.
\]

Using the right inequality of (3.2), we have the inequality

\[
\frac{(2\lambda + 7)(36 - \lambda)(16 - \lambda)}{30(20 - \lambda)(\lambda - 4)} > 1,
\]

which can easily be shown that it does not hold if \( \lambda \geq \frac{55}{7} \).

Using Lemma 3.3, we prove the following

**Lemma 3.4.** If \( \lambda \) is a double point in \([4, 16]\), then

\[Q'(\lambda) < 0.\]

**Proof.** Differentiating (3.3) gives

\[
Q''(\lambda) = -\frac{(\partial^2 f_1/\partial \lambda^2 + (\partial^2 f_1/\partial Q \partial \lambda) Q'(\lambda)) \partial f_1/\partial Q - \partial f_1/\partial \lambda (\partial^2 f_1/\partial Q^2) Q'(\lambda) + \partial^2 f_1/\partial Q \partial \lambda)}{\partial f_1/\partial Q}.
\]

Since

\[
Q'(\lambda) = \frac{\partial f_1/\partial \lambda}{\partial f_1/\partial Q} = 0 \quad \text{and} \quad f_1 = 0,
\]

\[
Q''(\lambda) = -\frac{\partial^2 f_1/\partial \lambda^2}{\partial f_1/\partial Q},
\]

the problem reduces to establishing that \( \partial^2 f_1/\partial \lambda^2 > 0 \). From (2.7),

\[
0 < -\frac{\partial f_3}{\partial \lambda} \leq \frac{2}{15} f_3, \quad \frac{\partial^2 f_4}{\partial \lambda} \leq -\frac{1}{6} \frac{f_4}{\partial \lambda},
\]
we can write
\[
\frac{\partial^2 f_1}{\partial \lambda^2} = \frac{\partial^2 f_{1,2}}{\partial \lambda^2} f_3 + 2 \frac{\partial f_{1,2}}{\partial \lambda} \frac{\partial f_3}{\partial \lambda} + f_{1,2} \frac{\partial^2 f_3}{\partial \lambda^2} + \frac{Q}{576} \left( -2 \frac{\partial f_4}{\partial \lambda} + (4 - \lambda) \frac{\partial^2 f_4}{\partial \lambda^2} \right)
\]
\[
\geq \frac{f_3}{32} + \frac{\lambda - 10}{16} \cdot \frac{\partial f_3}{\partial \lambda} + \frac{Q}{576} \left( 2 - \frac{4 - \lambda}{6} \right) \left( - \frac{\partial f_4}{\partial \lambda} \right)
\]
\[
\geq \frac{f_3}{32} \left( 1 - \frac{4|\lambda - 10|}{15} \right) f_3 > 0,
\]
which completes the proof. □

Now we present the main result of this section.

**Theorem 3.1.** There exists a unique double point in \([4, 16]\).

**Proof.** We only need to prove uniqueness. If there are two double points \(\lambda_0 < \lambda_1\) in \([4, 16]\), from Lemma 7, \(Q''(\lambda_0) < 0, Q''(\lambda_1) < 0\). Without loss of generality, we assume that there are no other double points in \((\lambda_0, \lambda_1)\), which means that \(Q'(\lambda)\) remains positive or negative. We assume it is positive. Then, \(Q'(\lambda_0) \geq 0\), which contradicts with \(Q'(\lambda_0) < 0\). □

In Section 5, we will present an algorithm to compute this double point in \([4, 16]\).

**4. The interval [16, 36]**

In this section, we will prove that there are no double points in the interval \([16, 36]\). To do this, it suffices to show that if \(\lambda \in [16, 36]\) and \(Q > 0\) satisfying \(f_1(\lambda, Q) = 0\), then
\[
\frac{\partial f_1}{\partial \lambda} \bigg|_{(\lambda, Q)} < 0.
\]
First we need to bound \(Q\) in terms of \(\lambda\).

Since
\[
0 = f_1 = f_{1,3} f_4 + \frac{Q}{16 \cdot 3^2 \cdot 4^2} f_{1,2} f_5
\]
and \(f_{1,2}, f_4, f_5\) are all positive,
\[
f_{1,3} < 0,
\]
which implies that
\[
(36 - \lambda)[(\lambda - 4)(\lambda - 16) + Q] + (4 - \lambda)Q < 0.
\]
It follows that \(\lambda > 20\),
\[
Q \geq \frac{(36 - \lambda)(\lambda - 4)(\lambda - 16)}{2(\lambda - 20)}
\]
and
\[
0 < - f_{1,3} < (\lambda - 4)Q.
\]
On the other hand,
\[
\frac{f_4}{f_5} = \frac{1}{4 \cdot 4^2} \left[ (64 - \lambda) + \frac{Q}{4 \cdot 5^2} \cdot \frac{f_6}{f_5} \right]
\]

\[
\leq \frac{1}{4 \cdot 4^2} \left[ 64 - \lambda + \frac{Q}{100 - \lambda} \right]
\]
yields
\[
Q = -\frac{16 \cdot 3^2 \cdot 4^2 f_{1,3} f_4}{f_{1,2} f_5}
\]
\[
\leq \frac{(\lambda - 4)Q}{(\lambda - 4)(\lambda - 16) + Q} \left[ 64 - \lambda + \frac{Q}{100 - \lambda} \right],
\]
which on simplification gives
\[
Q \leq \frac{(\lambda - 4)(40 - \lambda)(100 - \lambda)}{52 - \lambda}.
\]

**Lemma 4.1.** Let \( \lambda \in [16, 36] \) and \( Q > 0 \) satisfying \( f_1(\lambda, Q) = 0 \), then for \( k \geq 4 \)
\[
a_{1,3} a_{4,k} + Q a_{1,2} a_{5,k} = (-Q)^{k-3} a_{1,3} \lim_{n \to \infty} \frac{a_{k+2,n}}{a_{5,n}}.
\]

**Proof.** Letting \( a_{5,3} = 0 \), we have
\[
a_{1,n} = a_{1,3} a_{4,n} + Q a_{1,2} a_{5,n}
\]
\[
= a_{1,3} [a_{4,k} a_{k+1,n} + Q a_{4,k-1} a_{k+2,n}] + Q a_{1,2} [a_{5,k} a_{k+1,n} + Q a_{5,k-1} a_{k+2,n}]
\]
\[
= (a_{1,3} a_{4,k} + Q a_{1,2} a_{5,k}) a_{k+1,n} + Q (a_{1,3} a_{4,k-1} + Q a_{1,2} a_{5,k-1}) a_{k+2,n}.
\]
Since
\[
f_1 = \lim_{n \to \infty} \frac{a_{1,n}}{d_{1,n}} = 0,
\]
\[
a_{1,3} a_{4,k} + Q a_{1,2} a_{5,k} = -Q (a_{1,3} a_{k+1,n} + Q a_{1,2} a_{k-1,n}) \lim_{n \to \infty} \frac{a_{k+2,n}}{a_{k+1,n}}
\]
\[
= (-Q)^{k-3} (a_{1,3} a_{4,3} + Q a_{1,2} a_{5,3}) \lim_{n \to \infty} \frac{a_{k+2,n}}{a_{5,n}}
\]
\[
= (-Q)^{k-3} a_{1,3} \lim_{n \to \infty} \frac{a_{k+2,n}}{a_{5,n}}. \quad \square
\]

Because \( a_{1,3} < 0 \), we can conclude that \( a_{1,3} a_{4,k} + Q a_{1,2} a_{5,k} \) is positive if \( k \) is even and negative if \( k \) is odd. To prove the main result of this section, we still need the following lemma.

**Lemma 4.2.** Let \( \lambda \in [16, 36] \) and \( Q > 0 \) satisfying \( f_1(\lambda, Q) = 0 \). Then
\[
0 > -a_{1,3} \frac{\partial a_{4,n}}{\partial \lambda} - Q a_{1,2} \frac{\partial a_{5,n}}{\partial \lambda} \geq \frac{121}{120} a_{1,3} a_{5,n}.
\]
Proof. From Lemma 4.1,
\[-a_{1,1} \frac{\partial a_{k,n}}{\partial \lambda} + Q a_{1,2} \frac{\partial a_{5,n}}{\partial \lambda} = a_{1,3} \sum_{k=4}^{n} a_{k,k-1} a_{k+1,n} + Q a_{1,2} \sum_{k=5}^{n} a_{5,k-1} a_{k+1,n}\]
\[= a_{1,3} a_{5,n} + \sum_{k=5}^{n} (a_{1,3} a_{4,k-1} + Q a_{1,2} a_{5,k-1}) a_{k+1,n}\]
\[\geq a_{1,3} a_{5,n} \left[ 1 + \sum_{k=2}^{\lfloor n/2 \rfloor - 1} Q^{2k-2} a_{2k+3,n} \lim_{n \to \infty} \frac{a_{2k+3,n}}{a_{5,n}} \right].\]

Using
\[\frac{a_{2k+3,n}}{a_{5,n}} \leq \prod_{j=6}^{2k+3} \frac{1}{4j^2 - \lambda},\]
we get
\[\lim_{n \to \infty} \frac{a_{2k+3,n}}{a_{5,n}} \leq \prod_{j=6}^{2k+3} \frac{1}{4j^2 - \lambda}.\]

Using (4.2) and noting \( \lambda > 20 \), we have
\[Q^{2k-2} a_{2k+3,n} \lim_{n \to \infty} \frac{a_{2k+3,n}}{a_{5,n}} \leq Q^{2k-2} \prod_{j=6}^{2k+3} \frac{1}{4j^2 - \lambda} \leq \left( \frac{Q}{(44 - \lambda)(196 - \lambda)} \right)^{2k-2}\]
\[\leq \left( \frac{40 - \lambda}{52 - \lambda} \right)^{2k-2},\]
\[\leq \left( \frac{40 - 20}{52 - 20} \right)^{2k-2} a_{2k+3,n}.\]

Thus,
\[-a_{1,1} \frac{\partial a_{k,n}}{\partial \lambda} + Q a_{1,2} \frac{\partial a_{5,n}}{\partial \lambda} \geq a_{1,3} a_{5,n} \left( 1 + \sum_{k=2}^{\infty} \frac{1}{11^{2k-2}} \right) = \frac{121}{120} a_{1,3} a_{5,n}.\]

The following theorem is the main result of this section.

Theorem 4.1. There are no double points in \([16, 36]\).

Proof. Let \( \lambda \in [16, 36] \) and \( Q > 0 \) satisfying \( f_1(\lambda, Q) = 0 \). We have
\[\frac{\partial a_{1,3}}{\partial \lambda} = (\lambda - 4)(\lambda - 16) + 2Q - 2(36 - \lambda)(\lambda - 10).\]
Since \( \lambda > 20 \) and (4.1), it is easy to check that
\[
-\frac{\partial a_{1,3}}{\partial \lambda} > 0.
\]
From Lemma 4.2,
\[
-\frac{\partial a_{1,n}}{\partial \lambda} = -\frac{\partial a_{1,3}}{\partial \lambda} a_{4,n} - a_{1,1} \frac{\partial a_{4,n}}{\partial \lambda} - Q \frac{\partial a_{1,2}}{\partial \lambda} a_{5,n} - Q a_{1,2} \frac{\partial a_{5,n}}{\partial \lambda}
\geq \left(-\left(64 - \lambda\right) \frac{\partial a_{1,3}}{\partial \lambda} - Q \frac{\partial a_{1,2}}{\partial \lambda} + \frac{121}{120} a_{1,3}\right) a_{5,n}
= h(\lambda, Q) a_{5,n},
\]
where
\[
h(\lambda, Q) = (\lambda - 4)(\lambda - 16) \left(64 - \frac{121(36 - \lambda)}{120}\right) - 2(36 - \lambda)(\lambda - 10)(64 - \lambda)
+ 2Q \left(74 + \frac{121}{6} - \frac{361}{120}\right).
\]
We construct
\[
h(\lambda) = \begin{cases} h \left(\lambda, \frac{(36 - \lambda)(\lambda - 16)}{2(20 - \lambda)}\right), & 74 + \frac{121}{6} - \frac{361}{120}\lambda > 0, \\
\left(\lambda, \frac{(\lambda - 4)(\lambda - 10)(100 - \lambda)}{52 - \lambda}\right), & 74 + \frac{121}{6} - \frac{361}{120}\lambda < 0.
\end{cases}
\]
Clearly,
\[
h(\lambda) \leq h(\lambda, Q).
\]
Using MATLAB, we can check that the function \( h(\lambda) \) is positive in the interval \((20, 36]\). Thus
\[
-\frac{\partial a_{1,n}}{\partial \lambda} \geq h(\lambda) a_{5,n}, \quad \text{i.e.,} \quad \frac{\partial a_{1,n}}{\partial \lambda} \leq -h(\lambda) a_{5,n},
\]
and
\[
\frac{\partial f_1}{\partial \lambda} = \lim_{n \to \infty} \frac{\partial a_{1,n}}{d_{1,n}} \leq -h(\lambda) \lim_{n \to \infty} \frac{a_{5,n}}{d_{1,n}} < 0,
\]
which completes the proof. \( \square \)

5. The algorithm

In this section, we will discuss on how to compute the first double point. Our first task is to
design an algorithm to compute \( Q(\lambda) \) for \( \lambda \in (4, 16) \). From Lemma 3.1, if there exist
\[
Q_1 > (\lambda - 4)(16 - \lambda), \quad Q_2 > (\lambda - 4)(16 - \lambda),
\]
such that
\[
f_1(\lambda, Q_1) < 0 \quad \text{and} \quad f_1(\lambda, Q_2) > 0,
\]
then
\[
Q_1 < Q(\lambda) < Q_2,
\]
which suggests the bisection method to compute $Q(\lambda)$. To achieve this, we should be able to decide the sign of $f_1$ for any $(\lambda, Q)$ in the region $(4, 16) \times (0, \infty)$.

Consider the following recursive procedure:

\begin{align}
  b_1 &= 1 - \frac{\lambda}{4}, \\
  b_{k+1} &= 1 - \frac{\lambda}{4(k+1)^2} + \frac{Q}{4(k+1)^2 \cdot 4k^2} \cdot \frac{1}{b_k}. 
\end{align}

(5.1)

It is easy to show that $b_1, b_2, \ldots, b_k$ are the diagonal pivots in $k-1$ steps of Gaussian elimination on matrix $(D^{1,k})^{-1}A^{1,k}$. Obviously, $f_1(\lambda, Q) = \prod_{j=1}^{\infty} b_j$.

**Lemma 5.1.** Let $\lambda \in (4, 16)$ and $Q > 0$. If $f_1(\lambda, Q) \neq 0$, then there exists $k$ such that $b_k \geq 0$. Let $k_0$ be the smallest integer such that $b_{k_0} \geq 0$, then $f_1(\lambda, Q) > 0$ if $k_0$ is odd and $f_1(\lambda, Q) < 0$ if $k_0$ is even.

**Proof.** If there is no integer $k$ such that $b_k \geq 0$, we get from (5.1),

$$1 - \frac{\lambda}{4(k+1)^2} + \frac{Q}{4(k+1)^2 \cdot 4k^2} \cdot \frac{1}{b_k} < 0,$$

which yields

$$0 > b_k > \frac{1}{4k^2\lambda - 16(k+1)^2k^2},$$

and hence

$$\lim_{k \to \infty} b_k = 0.$$

Since $f_{1,n}(\lambda, Q) = \prod_{i=1}^{n} b_i$,

$$f_1(\lambda, Q) = \lim_{n \to \infty} f_{1,n}(\lambda, Q) = \prod_{i=1}^{\infty} b_i = 0.$$

This contradicts with $f_1(\lambda, Q) \neq 0$.

If $k_0$ is even, then $f_{1,k_0-1}(\lambda, Q) < 0$ and $f_{1,k_0}(\lambda, Q) < 0$, which yields

$$f_1(\lambda, Q) < 0.$$

Similarly, if $k_0$ is odd, $f_1(\lambda, Q) > 0$. $\square$

According to this lemma, we design an algorithm to compute $Q(\lambda)$, which we will later include in the algorithm to find the double point.

We now turn to find the double point. From the discussion in Section 3, the double point $\lambda_0$ satisfies

$$Q(\lambda_0) = \max_{\lambda \in [4,16]} Q(\lambda).$$

Obviously, if $\lambda < \lambda_0$, then $(\lambda + z, Q(\lambda))$ lies in the region $\Omega$:

$$\Omega: 4 \leq \lambda \leq 16, \quad 0 < Q < Q(\lambda),$$
which implies

\[ f_1(\lambda + \varepsilon, Q(\lambda)) < 0. \]

Here, \( 0 < \varepsilon < \lambda_0 - \lambda \). On the other hand, if \( \lambda > \lambda_0 \), then \( f_1(\lambda + \varepsilon, Q(\lambda)) > 0 \) for any \( \varepsilon > 0 \). Using this fact, we can find the double point by bisection method.

Let tol be the required accuracy. The following algorithm in Matlab is to compute the double point in \([4, 16]\) and the corresponding singular point.

**Algorithm**

```matlab
function [a, q] = db(a1, a2, tol)
    while (a2 - a1 > tol)
        a = (a2 + a1)/2;
        q1 = (16 - a) * (a - 4);
        q2 = (36 - a) * (16 - a) * (a - 4) / (20 - a);
        while (q2 - q1) > 1.0e-12
            q = (q1 + q2) / 2;
            d = (4 - a) / 4;
            n = 1;
            while d < 0
                n = n + 1;
                d = 1 - a / (4 * n^2) + q / (16 * d * n^2 * (n - 1)^2);
            end
            n = n - 1;
            if rem(n, 2) == 0
                q2 = q;
            else
                q1 = q;
            end
        end
        a = a + 0.5 * tol;
        d = (4 - a) / 4;
        n = 1;
        while d < 0
            n = n + 1;
            d = 1 - a / (4 * n^2) + q / (16 * d * n^2 * (n - 1)^2);
        end
        n = n - 1;
        if rem(n, 2) == 0
            a2 = a;
        else
            a1 = a;
        end
    end
```
We perform our algorithm on an IBM Workstation TTX. The results are listed in Table 1.

To test our results, we truncate the infinite matrix $A$ of order 100 with $Q = 48.01041405$. Using Matlab we have that its first two eigenvalues are

$$\lambda_1 = \lambda_2 = 11.1904735991294,$$

These two eigenvalues merge and are very near to the double point we compute, which shows our results are credible.

We can also use the idea of our algorithm to compute the subsequent double points, though it is lacking theoretical analysis. Modifying the choice of $a_1$ and $a_2$ in Algorithm 1 as $a_1 = 0$ and $a_2 = 2000$, we have the numerical results in Table 2.

Also if we truncate $A$ of order 100 with $Q = 905.81573524$, we find the third and fourth eigenvalue merge:

$$\lambda_3 = \lambda_4 = 50.47198768.$$