Less simple contiguous function relations for hypergeometric functions

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Abstract

Several results for contiguous function relations for hypergeometric functions of several variables are obtained. In these recurrences the coefficients are not all constant, but instead contain linear functions of the variables. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Gauss [5] obtained the contiguous function relations for the hypergeometric function. This was extended by Rainville [6] to the generalized hypergeometric function; he introduced the terminology simple if the coefficients of the relation are constants, otherwise less simple. For hypergeometric functions of several variables, we have shown in [1] how a basis for the set of simple contiguous function relations can be obtained from the null space of the associated matrix for the function. In [2] we have listed simple and less simple contiguous function relations for Horn’s functions, the properties of which can be found in Erdélyi et al. [3]. A list of triple hypergeometric functions can be found in Srivastava and Karlsson [7], the numbering of which we use as subscripts. Several quadruple hypergeometric functions are defined in Exton [4]. Our references are to these lists.

We consider hypergeometric functions of several variables and we approach the problem by searching for recurrences which are satisfied by classes of functions, instead of investigating a particular function and obtaining its set of recurrences. The results include all of the three term linear coefficient contiguous function relations for the Horn functions and most of the four term relations. By a contiguous function we mean a function in which exactly one parameter is shifted by ±1 from those of the given function. In this work a linear recurrence which involves some subset of the set
containing the function along with all of its contiguous functions is called a contiguous function relation.

Here we shall consider several results which produce less simple relations for hypergeometric functions of any number of variables. They do not apply to all functions and one of them does not really show up well for Horn’s functions, however they do provide a number of results for individual multiple hypergeometric functions. In one case the subset does not contain the function itself; that is, the relation contains only functions contiguous to the given function.

In order to simplify the statements, we prove our results for a special set of indices. Then, through a set of remarks, we indicate how various changes can be made to extend the results by alteration of the indices.

2. Result for repeated index, numerator–numerator

If, in the series coefficients for a Gaussian function \( F \), the index \( m \) is repeated on two numerator parameters; for example, if the index appears only in connection with these parameters and in the following format:

\[(a_j)_m(a_k)_m(a_r)_m \]

then there is a linear coefficient contiguous function relation of the form

\[ F[a_j - 1] - F[a_k - 1] + x(a_k - a_j) \frac{F[a_r - 1]}{(a_r - 1)} = 0 \]

This can be seen by writing out the products from the series coefficients and noting that only three factors mismatch. We have collected all of the common factors into the symbol \( \Omega \). Thus we have

\[
\frac{F[a_j - 1]}{(a_j - 1)} \leftrightarrow (a_k + m - 1)\Omega, \\
\frac{F[a_k - 1]}{(a_k - 1)} \leftrightarrow (a_j + m - 1)\Omega, \\
x \frac{F[a_r - 1]}{(a_r - 1)} \leftrightarrow (m)\Omega.
\]

From these expressions the result readily follows.

**Remark 1.** In all of our results clearly we can replace the \( m \) by \( n \) provided we also replace \( x \) by \( y \), etc.

As an example from Horn’s list in [3] we have the function \( H_7 \) for which we display the parameters in the series coefficients

\[ H_7[a_1, a_2, a_3; \gamma_1] \leftrightarrow \frac{(a)_{2m-n}(a_2)_n(a_3)_n}{(\gamma_1)_m}. \]

This function satisfies

\[ H_7[a_2 - 1] - H_7[a_3 - 1] + \gamma(a_3 - a_2) \frac{H_7[a_1 - 1]}{(a_1 - 1)} = 0. \]
Remark 2. If the repeated index occurs in the following form:

\[
\frac{(a_j)_m(a_k)_m}{(\gamma_r)_{a+m}},
\]

then the variable coefficient term must be replaced by

\[
x(a_k - a_j) \frac{F[\gamma_r + 1]}{\gamma_r}.
\]

The Appell function \(F_3\) serves as an example; it has two recurrences of this form.

Remark 3. If the repeated index \(m + n\) appears in the form

\[
(a_j)_{m+n}(a_k)_{m+n}(a_r)_{a-m}(a_s)_{b-n},
\]

where \(a\) and \(b\) do not involve \(m\) and \(n\), then two variable coefficient terms of the appropriate form appear.

Taking into consideration the earlier remarks, there is also a suitable alteration where one or more unpaired occurrences of involved indices involves a denominator parameter. An example is provided by \(F_4\) of [3] for which we display the coefficients

\[
F_4[a_1, a_2, \gamma_1, \gamma_2] \leftrightarrow \frac{(a_j)_{m+n}(a_k)_{m+n}}{(\gamma_2)_m(\gamma_2)_n}.
\]

The resulting recurrence is

\[
F_4[a - 1] - F_4[a - 1] + x(a_j - a_k) \frac{F[\gamma_1 + 1]}{\gamma_1} + y(a_j - a_k) \frac{F[\gamma_2 + 1]}{\gamma_2} = 0.
\]

As an evident further extension, in the list of functions of three variables, \(F_{22a}\) has a sum of three indices and consequently three of the less simple terms appear in the relation.

In Horn’s list, if we include the various modifications, there are 7 functions with repeated numerator indices to which these results apply. Similarly in the triple function list of Srivastava and Karlsson [7] we find 47 such functions; in the list of quadruple functions in Exton [4], 4 such functions.

3. Result for repeated index, numerator–denominator

If we have parameters and indices in the form

\[
\frac{(a_r)_{m}(a_s)_{m+a}}{(\gamma_r)_m}
\]

then there is a four term relation of the form

\[
(2a_r - \gamma_r)F + x(a_jF[a_j + 1] - a_rF[a_r + 1] + (\gamma_r - a_r)F[a_r - 1] = 0.
\]

The proof is slightly more complicated than in the previous section. If the second and fourth contiguous terms are combined first, then a factor common to the other terms appears, and the coefficient
of $F$ can readily be determined. Two such recurrences appear for $F_{4d}$ of [7]. If $m+n$ is the repeated index, as in $F_{1}$, then an additional less simple contiguous term must be adjoined.

From the list of triple functions we find 47 with such indices, some of which satisfy more than one recurrence. There are 4 Horn functions, plus several confluent cases, as well as 5 quadruple $K$ functions which possess this property.

4. Result for paired indices

If the sum $m+n$ appears in two places in either of the forms

$$(x_{r})_{m+n+a}(x_{s})_{b-m-n}(x_{j})_{m}(x_{k})_{n}, \quad (y_{r})_{m+n+a}(x_{j})_{n}(x_{k})_{n},$$

(where $a, b, c$, and $d$ do not involve $m$ and $n$) then there is a less simple contiguous function relation of the form

$$(x - y)F - xF[x_{k} - 1] + yF[x_{j} - 1] = 0.$$

A method of derivation similar to that of Section 2 can be applied. We first note that the parameters associated with the sum indices are altered in the same manner with either $x$ or $y$ multipliers. With similar notations as before, we write out

$$xF \leftrightarrow (x_{k} + n - 1)(m)\Omega,$$

$$yF \leftrightarrow (x_{j} + m - 1)(n)\Omega,$$

$$xF[x_{k} - 1] \leftrightarrow (x_{k} - 1)(m)\Omega,$$

$$yF[x_{j} - 1] \leftrightarrow (x_{j} - 1)(n)\Omega,$$

from which we can now see how the combination collapses to produce the desired result.

The only double functions with this property is $F_{1}$, but it is shared by 10 triple functions in the list of Srivastava and Karlsson [7] and 11 of the $K$ and $D$ quadruple functions listed in Exton [4].

5. Result for indices of opposite signs

Another result is worth mentioning because of the simplicity of the relations. If we have parameters of the form

$$(x_{j})_{m}(x_{k})_{m-a}(x_{r})_{-(a-1)},$$

where $a$ does not involve $m$, then there is a less simple contiguous function relation of the form

$$(1 + x)F - F[x_{r} - 1] - x(x_{k} + x_{r} - 1) \left[F[x_{r} - 1] \over (x_{r} - 1)\right] = 0.$$

The triple function $F_{29j}$ of [7] supplies an example.

The proof goes in much the same manner as for previous recurrences. If we first compute $F - F[x_{j} - 1]$, then adjoin $xF$, it is not difficult to see what coefficient is needed for the other linear coefficient term.
In connection with Remark 2, a companion situation occurs for the parameter and index arrangement

\[(x_j)_{m+a} (x_k)_{m+a} \]

which leads to the linear coefficient recurrence

\[(1-x)F - F[x_j - 1] + x(\gamma_r - x_k) \frac{F[\gamma_r + 1]}{\gamma_r} = 0.\]

One or more of these relations are found for 6 double functions, 7 triple functions, and 4 quadruple function of the lists.

6. Result for Kummer functions

It is further to be noted that most of the triple Kummer functions listed in Srivastava and Karlsson [7] satisfy similar relations which can be obtained from the same type of computations as in the previous sections. If we have parameters and indices arranged in the form

\[(x_k)_{m+a} (x_r)_{a-m},\]

where \(a\) does not involve \(m\), then the relations take the form

\[K - K[x_k - 1] - x \frac{K[x_r - 1]}{(x_r - 1)} = 0,\]

with suitable modifications in accordance with the Remark 2 of the previous sections. Further, 11 of the confluent double functions also possess this property.

References