Advanced combinations of splitting–shooting–integrating methods for digital image transformations

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Abstract

An image consists of many discrete pixels with greyness of different levels, which can be quantified by greyness values. The greyness values at a pixel can also be represented by an integral as the mean of continuous greyness functions over a small pixel region. Based on such an idea, the discrete images can be produced by numerical integration; several efficient algorithms are developed to convert images under transformations. Among these algorithms, the combination of splitting–shooting–integrating methods (CSIM) is most promising because no solutions of nonlinear equations are required for the inverse transformation. The CSIM is proposed in [6] to facilitate images and patterns under a cycle transformations $T^{-1}T$, where $T$ is a nonlinear transformation. When a pixel region in two dimensions is split into $N^2$ subpixels, convergence rates of pixel greyness by CSIM are proven in [8] to be only $O(1/N)$. In [10], the convergence rates $O_p(1/N^{1.5})$ in probability and $O_p(1/N^2)$ in probability using a local partition are discovered. The CSIM is well suited to binary images and the images with a few greyness levels due to its simplicity. However, for images with large (e.g., 256) multi-greyness levels, the CSIM still needs more CPU time since a rather large division number is needed.

In this paper, a partition technique for numerical integration is proposed to evaluate carefully any overlaps between the transformed subpixel regions and the standard square pixel regions. This technique is employed to evolve the CSIM such that the convergence rate $O(1/N^2)$ of greyness solutions can be achieved. The new combinations are simple to carry out for image transformations because no solutions of nonlinear equations are involved in, either. The computational figures for real images of $256 \times 256$ with 256 greyness levels display that $N=4$ is good enough for real applications. This clearly shows validity and effectiveness of the new algorithms in this paper. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Usually, the approaches studying the discrete and continuous topics are quite different due to different natures, such as those in discrete and analytic mathematics. This paper demonstrates an example how to study discrete images by integrals and their numerical approximation. The key idea is as follows. An image consists of many discrete pixels with greyness at different levels, which can be quantified by greyness values. The greyness values at a pixel can also be represented by an integral as the mean of continuous greyness functions over a small pixel region. Based on such an idea, the discrete images can be produced by numerical techniques. However, it is due to special nature of the integration from image transformations that renovation of the existing integration rules is necessary. Consequently, new discrete algorithms have been developed. In our past research on image transformation in [6–11], the study on discrete algorithms is, indeed, the study of numerical integration for the integrand without uniform smoothness. This paper also reveals how to employ numerical methods and error analysis to discrete topics effectively. Note that our research process looks pass a long, cycle road: from the discrete to the continuous, and then from the continuous back to the discrete, our destination. But a number of amazing results have been found, see [6–11]; one of them is reported in this paper.

Several combined methods are proposed in [6] to facilitate restoration of digital images and patterns under $T^{-1}T$, where $T$ is a nonlinear transformation defined by

$$T : (\xi, \eta) \rightarrow (x, y), \quad x = x(\xi, \eta), \quad y = y(\xi, \eta)$$

and $\xi\oy$ and $\xi\on$ are two Cartesian coordinate systems. Let us assume that the functions $x(\xi, \eta)$ and $y(\xi, \eta)$ in (1.1) are explicit and known. To bypass solving nonlinear equations, the combination CSIM is proposed in [6], in which we employ the splitting–shooting method for $T$ given and the splitting–integrating method for $T^{-1}$ given. An error analysis is made in [8] for estimating consecutive errors of pixel greyness solutions, to show that only a low convergence rate $O(1/N)$ can be obtained, where a pixel is split into $N^2$ subpixels. A low convergence rate implies that $N$ must be chosen large for a small tolerable error of greyness to result, thus consuming a large amount of CPU time; this drawback is more severe in images with multiple (i.e., 256) greyness-levels and in three-dimensional images.

The question asked here is: can we raise the convergence rates of pixel greyness solutions by CSIM? Paper [10] responds to this question. In [10], we employ probabilistic analysis, to discover that the convergence rates, $O_p(1/N^{1.5})$ in probability, can be obtained. Moreover, the high convergence rate, $O_p(1/N^2)$ in probability, can be achieved, if using a local partition. When $N \geq 32$ good figures of images are produced. This, however, implies that a 2D pixel is split into at least $32 \times 32 = 1024$ subpixels for real images. Obviously, such a computational effort seems to be exhausted if we recall the fact that the pixel number of usually images is huge, e.g., from $256 \times 256$ to $1024 \times 1024$. Hence, the performance of the CSIM in [10] is still not very satisfactory to 256 greyness level images. Our motivation of this paper is to decrease the computation work per pixel as much as possible, i.e., to reduce the applicable division number $N$ as much as possible for images with 256 greyness levels.

In this paper, we will analyze again the algorithmic nature of CSIM, and explore a new technique that can identify and evaluate carefully the overlaps between the distorted subpixel regions and the standard square pixel regions. This new technique is adopted in CSIM to lead to two new
combinations C\text{SIM} and CS\text{IM}, where the notations ‘\text{S}’ and ‘\text{I}’ denote the renovated splitting–shooting method and the renovated splitting–integrating method, respectively. Both C\text{SIM} and CS\text{IM} can grant the pixel image greyness under $T^{-1}T$ to have the convergence rate $O(1/N^2)$, based on strict error analysis without probability. The new combinations are simple and easy to carry out because no solutions of nonlinear equations are needed, either. Surprisingly, when the division number is chosen to be $N = 4$, good image pictures of images with 256 greyness levels can be produced by the techniques given in this paper.

This paper can be read as one of numerical analysis if the terminologies, “pixel” and “greyness” (or “greyness value”) are replaced by “point” and “value”. Consequently, this paper also demonstrates a good example for numerical methods applied to image processing. Many interesting applications of our discrete algorithms can be found in [11].

Below, we describe and analyze the combination CS\text{IM} in Section 2, propose the new partition technique in Section 3, to lead to the new combinations C\text{SIM} and CS\text{IM}, then derive error bounds of transformed images by C\text{SIM} and CS\text{IM} in Section 4, and finally in Section 5 provide numerical and graphical results to verify the convergence rate $O(1/N^2)$. Some real images of $256 \times 256$ pixels with 256 greyness levels display significance of the new algorithms in this paper.

2. Numerical algorithms

Let a given standard image undergo a cycle conversion (see [6]).

\[
\hat{W} \overset{T}{\rightarrow} \hat{Z} \overset{T^{-1}}{\rightarrow} \hat{W}, \quad \hat{W} = \{\hat{W}_{ij}\}, \quad \hat{Z} = \{\hat{Z}_{IJ}\},
\]

where the pixels $\hat{W}_{ij}$ and $\hat{Z}_{IJ}$ are located at the points $(i,j)$ and $(I,J)$, respectively,

\[
(i,j) = \{(\xi,\eta), \quad \xi = iH, \quad \eta = jH\}, \quad (I,J) = \{(x,y), \quad x = IH, \quad y = JH\}
\]

and $H$ is the mesh resolution in an optical scanner.

We will apply numerical approaches to perform (2.1), illustrated in Fig. 1 with eight steps.

In Steps 1 and 5, we convert image pixels and their greyness to each other. For the sake of simplicity, we assume the binary images, and choose

\[
\Phi_{ij} = \begin{cases} 1 & \text{if } W_{ij} = \ast, \\ 0 & \text{if } W_{ij} = \ast', \end{cases} \quad B_{ij} = \begin{cases} 1 & \text{if } Z_{IJ} = \ast, \\ 0 & \text{if } Z_{IJ} = \ast'. \end{cases}
\]

Furthermore, if the values of greyness $\hat{\Phi}_{ij}$ and $\hat{B}_{IJ}$ have been obtained, in Steps 4 and 8 we may obtain image pixels by

\[
\hat{W}_{ij} = \begin{cases} \ast & \text{when } \hat{\Phi}_{ij} \geq \frac{1}{2}, \\ + & \text{when } \frac{1}{4} \leq \hat{\Phi}_{ij} < \frac{1}{2}, \\ \ast' & \text{when } 0.1 \leq \hat{\Phi}_{ij} < \frac{1}{4}, \\ \ast'' & \text{when } \hat{\Phi}_{ij} < 0.1, \end{cases} \quad \hat{Z}_{IJ} = \begin{cases} \ast & \text{when } \hat{B}_{IJ} > \frac{1}{2}, \\ + & \text{when } \frac{1}{4} \leq \hat{B}_{IJ} < \frac{1}{2}, \\ \ast' & \text{when } 0.1 \leq \hat{B}_{IJ} < \frac{1}{4}, \\ \ast'' & \text{when } \hat{B}_{IJ} < 0.1. \end{cases}
\]

In Step 2, the following simplest piecewise constant and bilinear interpolatory functions are adopted.
Fig. 1. Schematic steps in digital images under transformations by numerical approaches.

I. The piecewise constant interpolation \((\mu = 0)\):

\[
\hat{\Phi}_0(\xi, \eta) = \hat{\Phi}_{ij} \quad \text{in } \Box_{ij}, \quad \text{where}
\]

\[
\Box_{ij} = \begin{cases} 
(\xi, \eta), & (i - \frac{1}{2})H \leq \xi < (i + \frac{1}{2})H, \\
(j - \frac{1}{2})H \leq \eta < (j + \frac{1}{2})H 
\end{cases}
\]  

(2.6)

and the total domain \(\Omega\) of the standard image \(\hat{W}\) in \(\xi\eta\) is \(\Omega = \bigcup_{ij} \Box_{ij}\).

II. The piecewise bilinear interpolation \((\mu = 1)\):

\[
\hat{\Phi}_1(\xi, \eta) = \frac{1}{H^2} [ \Phi_{ij}((i + 1)H - \xi)((j + 1)H - \eta) + \Phi_{i+1,j}(\xi - iH)((j + 1)H - \eta) \\
+ \Phi_{ij+1}((i + 1)H - \xi)(\eta - jH) + \Phi_{i+1,j+1}(\xi - iH)(\eta - jH)] \quad \text{in } \Box_{ij},
\]  

(2.7)

where

\[
\Box_{ij} = \begin{cases} 
(\xi, \eta), & iH \leq \xi < (i + 1)H, \\
jH \leq \eta < (j + 1)H 
\end{cases}
\]  

and \(\Omega = \bigcup_{ij} \Box_{ij}\).

A pixel can be viewed as the representation of the mean greyness over \(\Box_{ij}\), given by

\[
\Phi_{ij}^M = \frac{1}{H^2} \int_{\Box_{ij}} \Phi(\xi, \eta) \, d\xi \, d\eta.
\]  

(2.9)
Similarly, we have
\begin{equation}
B_{M}^{IJ} = \frac{1}{H^2} \int_{\Omega_{ij}} b(x, y) \, dx \, dy,
\end{equation}
where
\begin{equation}
\phi(\xi, \eta) = b(x(\xi, \eta), y(\xi, \eta))
\end{equation}
and the standard square pixel region
\begin{equation}
\square_{ij} = \left\{ (x, y), \ (I - \frac{1}{2})H \leq x < (I + \frac{1}{2})H, \right. \\
\left. (J - \frac{1}{2})H \leq y < (J + \frac{1}{2})H. \right\}
\end{equation}

Note that the representation of image as the integrals in (2.9) and (2.10) is a key idea that enable us to develop new discrete algorithms by numerical approximation and to evaluate greyness errors by numerical analysis. The diagram of Fig. 1 also illustrates our research process how to deal with discrete topics by continuous treatments and how to solicit numerical methods.

We assume that the Jacobian determinants
\begin{equation}
J(\xi, \eta) = \left| \begin{array}{cc}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}
\end{array} \right|
\end{equation}
always satisfy
\begin{equation}
0 < J_0 \leq J(\xi, \eta) \leq J_M,
\end{equation}
where \(J_0\) and \(J_M\) are two bounded constants independent of \(\xi, \eta, x\) and \(y\). For the inverse transformation \(T^{-1}\), the integral (2.10) is reduced to
\begin{equation}
B_{M}^{ij} = \frac{1}{H^2} \int_{\Omega_{ij}} \phi(\xi, \eta) J(\xi, \eta) \, d\xi \, d\eta,
\end{equation}
under
\begin{equation}
\square_{ij} \rightarrow \Omega_{ij}, \text{ i.e., } \Omega_{ij} \rightarrow \square_{ij}.
\end{equation}

Let the pixel region \(\square_{ij}\) in \(\xi\eta\) of \(\hat{W}_{ij}\) be split into \(N^2\) small subregions \(\square_{ij,kl}\), i.e., \(\square_{ij} = \bigcup_{k,l=1}^{N} \square_{ij,kl}\), where
\begin{equation}
\square_{ij,kl} = \left\{ (\xi, \eta), \ (i - \frac{1}{2})H + (k - 1)h \leq \xi < (i - \frac{1}{2})H + kh, \right. \\
\left. (j - \frac{1}{2})H + (l - 1)h \leq \eta < (j - \frac{1}{2})H + lh \right\}
\end{equation}
and \(h\) is the boundary length of \(\square_{ij,kl}\) given by \(h = H/N\).

The splitting–shooting method given in [6] collects the contribution of such subpixels \(\square_{ij,kl}\) that whose transformed centroid by \(T\) falls into the identifying pixel region \(\square_{ij}\). As a result, we have
\begin{equation}
B_{M}^{ij} = \frac{1}{H^2} \sum_{ij,kl} \int_{\Omega_{ij,kl} \cap \Omega_{ij}} \phi(\xi, \eta) J(\xi, \eta) \, d\xi \, d\eta.
\end{equation}

Denote
\begin{equation}
\square_{ij,kl} \rightarrow \square_{ij,kl}, \quad \hat{G}_{ij,kl} \rightarrow \hat{G}_{ij,kl}^*.
\end{equation}
where $\hat{G}_{ij,kl}$ is the centroid of $\Box_{ij,kl}$, we can see

$$\Box_{ij,kl} \cap \Omega_U \xrightarrow{T} \Box_{ij,kl} \cap \Box_{IJ}.$$  

(2.20)

The following approximate integration can be obtained

$$B_{IJ}^M \approx \hat{B}_{IJ}^{(N)} = \left(\frac{h}{H}\right)^2 \sum_{\gamma=2,3} \hat{\phi}_\mu(\hat{G}_{ij,kl})J(\hat{G}_{ij,kl}), \quad \mu = 0, 1,$$  

(2.21)

where $\hat{\phi}_0$ and $\hat{\phi}_1$ are given in (2.5) and (2.7); and $\phi(\hat{G}) = \phi(\xi(\hat{G}), \eta(\hat{G}))$. The coordinates of $\hat{G} = \hat{G}_{ij,kl}$ are given by

$$\xi(\hat{G}) = (i - \frac{1}{2})H \pm (k - \frac{1}{2})h, \quad \eta(\hat{G}) = (j - \frac{1}{2})H \pm (l - \frac{1}{2})h.$$  

(2.22)

When the transformed centroid $\hat{G}^* = G^*_{ij,kl}$ falls into the standard square pixel region $\Box_{IJ}$ defined in (2.12), the values of $(I, J)$ can be computed by

$$I = \lfloor x(\hat{G}^* + \frac{1}{2}) \rfloor, \quad J = \lfloor y(\hat{G}^* + \frac{1}{2}) \rfloor,$$  

(2.23)

where $\lfloor x \rfloor$ is the floor function, and

$$x(\hat{G}^*) = x(\xi(\hat{G}), \eta(\hat{G})), \quad y(\hat{G}^*) = y(\xi(\hat{G}), \eta(\hat{G})).$$  

(2.24)

Based on the greyness $B_{IJ}$ obtained, we also construct the constant and bilinear functions $b_\mu(x, y)$ on the analogy of (2.5) and (2.7), where $\hat{b}_0(x, y)$ in $\Box_{IJ}$ and $\hat{b}_1(x, y)$ in $\Box_{IJ}$ and

$$\tilde{\Box}_{IJ} = \begin{cases} (x, y), & \text{IH} \leq x < (I + 1)H, \\ \text{JH} \leq y < (J + 1)H. \end{cases}$$  

(2.25)

The restored greyness (2.9) can be evaluated by the simplest centroid rule [2,14].

$$\phi_{ij} \approx \Phi_{ij}^M = \frac{1}{H^2} \sum_{k,l=1}^N \int_{\Box_{ij,kl}} \hat{\phi}(\xi, \eta) \, d\xi \, d\eta$$

$$\approx \hat{\Phi}_{ij}^{(N)} = \left(\frac{h}{H}\right)^2 \sum_{k,l=1}^N \hat{\phi}(\xi(\hat{G}), \eta(\hat{G})).$$  

(2.26)

where $\hat{\phi}(\xi, \eta) = \hat{b}_\mu(x, y), \quad \mu = 0, 1.$

The evaluations (2.21) and (2.26) for pixel greyness are called the splitting–shooting method (SSM) and the splitting–integrating method (SIM), respectively. The combination of SSM and SIM is referred to CSIM, which will be discussed in the following two cases (see Fig. 1). Case II consists of steps 1–8; Case I consists of Steps 1–4 and 6–8. In Case I, the greyness $B_{IJ}$ after Step 3 will be used directly for $T^{-1}$ without any changes. The distorted image $\{\hat{Z}\}$ may be obtained from $\{\hat{B}_{IJ}\}$ in Step 4, but no feedback (i.e., from $Z_{IJ}$ to $B_{IJ}$ as in Step 5) is carried out.

Combination CSIM given above is remarkably advantageous over other methods in image transformation of [3–5,12,13], since no nonlinear solutions are involved in for the cycle conversion $T^{-1}T$ of images.
3. Numerical integration using partition technique

We now intend to improve the integration approximations in (2.21) and (2.26) by using new partition techniques, to provide more accurate images. Denote

\[ \square_{ij,kl}^* \rightarrow \square_{ij,kl} \]  \hspace{1cm} (3.1)

A drawback of (2.21) is that all the contribution of the entire subpixel \( \square_{ij,kl} \) with \( \hat{G}^* \in \square_{ij,kl} \) is counted no matter how large a portion (e.g., even near a half) of \( \square_{ij,kl}^* \) is located outside \( \square_{ij,kl} \). Also the effect of \( \square_{ij,kl} \) with \( \hat{G}^* \notin \square_{ij,kl} \) is ignored even though \( \square_{ij,kl} \) falls partially into \( \square_{ij,kl} \). This drawback results in a low convergence rate \( O(1/N) \) of image greyness under transformations.

In order to obtain a better approximation of the integration, we have to distinguish carefully the parts of \( \square_{ij,kl} \) that are located inside and outside the standard square pixel region \( \square_{ij,kl} \), that is, to evaluate the overlaps of

\[ \square_{ij,kl} \cap \square_{ij,kl} \]  \hspace{1cm} (3.2)

There exist two different cases.

- **Case A**: The entire \( \square_{ij,kl} \) falls into \( \square_{ij,kl} \),
  \[ \square_{ij,kl} \subseteq \square_{ij,kl} \]  \hspace{1cm} (3.3)

- **Case B**: A part of \( \square_{ij,kl} \) falls into \( \square_{ij,kl} \),
  \[ |\square_{ij,kl} \cap \square_{ij,kl}| < |\square_{ij,kl}| \]  \hspace{1cm} (3.4)

where \( |\square| \) denotes the area of \( \square \). For Case A, the centroid rule is still employed for (2.18), to get

\[ \iint_{\square_{ij,kl} \subseteq \square_{ij,kl}} \hat{\phi}(\xi, \eta)J(x, y) \, d\xi \, d\eta = \iint_{\square_{ij,kl}} \hat{\phi}(\xi, \eta)J(\xi, \eta) \, d\xi \, d\eta \approx h^2 \hat{\phi}_{ij,kl}(\hat{G})J(\hat{G}). \]  \hspace{1cm} (3.5)

For Case B, however, the following new refined technique is proposed.

We have from (2.18) and (3.1)

\[ \iint_{\square_{ij,kl} \cap \square_{ij,kl}} \hat{\phi}(\xi, \eta)J(x, y) \, d\xi \, d\eta = \iint_{\square_{ij,kl} \cap \square_{ij,kl}} \hat{b}(x, y) \, dx \, dy, \]  \hspace{1cm} (3.6)

where the portion (3.2) can be carefully evaluated through three steps described below.

**Step I. Choice of \( N \)** to simplify the partition situation. We choose \( N \) so that any \( \square_{ij,kl}^* \) is located, at most within the following four pixel regions:

\[ \square_{ij,kl} \subseteq (\square_{ij,kl} + 1) \cup \square_{ij,kl} + 1 \cup \square_{ij,kl} \cup \square_{ij,kl} + 1. \]  \hspace{1cm} (3.7)

**Step II (Partitions of Squares)**. Divide a subregion, \( \square_{ij,kl} \) in \( \xi \) by a diagonal into two triangular elements (see Fig. 2)

\[ \square_{ij,kl} = \bigcup_{t=1,2} \Delta_{ij,kl,t}. \]  \hspace{1cm} (3.8)

Denote

\[ \Delta_{ij,kl,t} \rightarrow \tilde{\Delta}_{ij,kl,t} \approx \hat{\Delta}_{ij,kl,t} \]  \hspace{1cm} (3.9)
which is also represented in Fig. 2 by

\[ \triangle abc \rightarrow \tilde{\triangle} ABC \approx \hat{\triangle} ABC \]  
(3.10)

with \( a \rightarrow A, \overline{ab} \rightarrow \overline{AB} \) etc.

Consequently, the overlaps (3.2) lead to

\[ \mathcal{D}_{ij,kl} \cap \mathcal{D}_{H} = \bigcup_{t=1,2} (\hat{\mathcal{D}}_{ij,kl,t} \cap \mathcal{D}_{H}) \approx \bigcup_{t=1,2} (\hat{\mathcal{D}}_{ij,kl,t} \cap \mathcal{D}_{H}). \]  
(3.11)

**Step III (Partitions of triangles).** Based on the chosen \( N \) in Step I, for any \( \hat{\mathcal{D}}_{ij,kl,t} \) there exists, at most, one boundary line of

\[ x = (I \pm \frac{1}{2})H, \text{ or } y = (J \pm \frac{1}{2})H, \]  
(3.12)

along \( x \) or \( y \) that can pass through its middle. Moreover, let \( A, B, \) and \( C \) denote the top, middle, and bottom vertices of \( \triangle ABC \). For instance, we partition \( \triangle ABC \) by a horizontal boundary line,

\[ y = \overline{y} = (J + \frac{1}{2})H \]  
(3.13)

such that

\[ \triangle ABC \cap (y \geq \overline{y}) \text{ or } \triangle ABC \cap (y \leq \overline{y}). \]  
(3.14)

For simplicity, we may partition \( \triangle ABC \) into sub-triangles in such a way that each sub-triangle is located either above or under the boundary line (3.13).

The following three situations will occur that lead to different partitions of triangles, due to different locations of the boundary line (3.13) as illustrated in Fig. 3.
I. When the middle vertex $B$ is just on line (3.13), we may split $\triangle ABC$ into two triangles,

$$\triangle ABC = \triangle^+ ABD \cup \triangle^- BCD,$$

where $\triangle^+$ and $\triangle^-$ denote the upper triangle and the lower triangle respectively, with respect to (3.13).

II. When line (3.13) is located between the vertices $A$ and $B$, $\triangle ABC$ is split into three triangles:

$$\triangle ABC = \triangle^- AED \cup \triangle^+ DEC \cup \triangle^+ EBC,$$

where $E$ is the intersection point of $AB$ and line (3.13), with the coordinates,

$$y_E = \bar{y}, \quad x_E = x_A + \frac{\bar{y} - y_A}{y_B - y_A} (x_B - x_A).$$

III. When line (3.13) is located between the vertices $B$ and $C$, then

$$\triangle ABC = \triangle^- ABD \cup \triangle^- BDE \cup \triangle^+ DEC.$$

By (3.15), (3.16) and (3.18), we split $\triangle ABC$ into the sub-triangles which are no longer traversed by the horizontal boundary line (3.13).

Furthermore, some of these sub-triangles may still be traversed by a vertical boundary coordinate line

$$x = \bar{x} - (I + \frac{1}{2})H.$$

By means of the same technique as in Steps I–III, we can split such a sub-triangle into smaller sub-triangles again so that none of the sub-triangles is crossed by all the boundary lines, (3.12), of $\square_{ij}$.

Let us summarize the partition of triangle $\hat{\triangle}^*_{ij,kl,t}$ by Steps I–III. If regarding $\hat{\triangle}^*_{ij,kl,t}$ as $\triangle ABC$ in $xoy$ in Fig. 3, we obtain

$$\hat{\triangle}^*_{ij,kl,t} = \bigcup_m \hat{\triangle}^*_{ij,kl,t,m}, \quad m \leq 9,$$

where all the sub-triangles will fall into just one of the following pixel regions:

$$\hat{\triangle}^*_{ij,kl,t,m} \subseteq \square_{I_0+J_0,J_0}, \quad I_0, J_0 = 0 \text{ or } 1.$$
Applying the above technique, we can improve evaluation of integration. First we have from (2.18), (3.5) and (3.6)

\[
\begin{align*}
B_{ij}^M &= \frac{1}{H^2} \sum_{ij,kl} \int_{\Omega_{ij} \cap \Omega_{kl}} \phi(\xi, \eta) J(\xi, \eta) \, d\xi \, d\eta \\
&= \frac{1}{H^2} \sum_{ij,kl} \int_{\Omega_{ij} \cap \Omega_{kl}} \phi(\xi, \eta) J(\xi, \eta) \, d\xi \, d\eta \\
&+ \frac{1}{H^2} \sum_{ij,kl} \int_{\Omega_{ij} \cap \Omega_{kl}} \phi(\xi, \eta) J(\xi, \eta) \, d\xi \, d\eta \\
&\approx \left( \frac{h}{H} \right)^2 \sum_{ijkl} \phi(\hat{G}) J(\hat{G}) + \frac{1}{H^2} \sum_{ijkl} \int_{\Omega_{ij} \cap \Omega_{kl}} \phi(\xi, \eta) J(\xi, \eta) \, d\xi \, d\eta. \quad (3.22)
\end{align*}
\]

Next, we obtain from (3.6) and (3.20) for Case B,

\[
\begin{align*}
\int_{\Omega_{ij} \cap \Omega_{kl}} \phi(\xi, \eta) J(\xi, \eta) \, d\xi \, d\eta &= \int_{\Omega_{ij} \cap \Omega_{kl}} b(x, y) \, dx \, dy \\
&= \sum_{t=1}^{2} \int_{\Delta_{ij,kl}} \hat{b}(x, y) \, dx \, dy \\
&= \sum_{t=1}^{2} \int_{\Delta_{ij,kl}} \hat{b}(x, y) \, dx \, dy \\
&\approx \sum_{t=1}^{2} \int_{\Delta_{ij,kl}} \hat{b}(x, y) \, dx \, dy \\
&= \sum_{l,m} \int_{\Delta_{ij,kl,m}} \hat{b}(x, y) \, dx \, dy \\
&\approx b(\hat{G}^*) \sum_{l,m} |\Delta_{ij,kl,m}|, \quad (3.23)
\end{align*}
\]

where \( \hat{G}^* \) denotes the center of gravity of \( \Delta_{ij,kl,m} \) in \( xy \), satisfying

\[
\hat{\Delta}_{ij,kl,m} \subseteq \Omega_{ij} \cap \Omega_{kl}. \quad (3.24)
\]

The area of a triangle in (3.23) can be computed by the formula (see [11])

\[
| \Delta ABC | = \frac{1}{2} \left| \begin{array}{ccc} 1 & 1 & 1 \\
ax & bx & cx \\
ay & by & cy \end{array} \right|. \quad (3.25)
\]

Consequently, the renovated splitting–shooting method (called \( \tilde{SSM} \)) using the partition technique from (3.22) and (3.23) yields greyness \( B_{ij} \) under \( T \) by

\[
B_{ij} \approx \tilde{B}_{ij}^{(N)} = \left( \frac{h}{H} \right)^2 \sum_{ijkl} \hat{\phi}(\hat{G}) J(\hat{G}) + \frac{1}{H^2} \sum_{ijkl} \hat{b}(\hat{G}^*) \sum_{l,m} |\Delta_{ij,kl,m}|. \quad (3.26)
\]

As to the splitting–integrating method (SIM) for \( T^{-1} \), the convergence rates of pixel greyness solutions can reach \( O(1/N^2) \) only when \( \mu = 1 \). When using the piecewise constant interpolation \( (\mu = 0) \), the low convergence rate \( O(1/N) \) still occurs. Therefore in this case, the partition technique should also be adopted to modify SIM as well. In fact, the greyness (2.9) leads to

\[
\phi_{ij}^M \approx \frac{1}{H^2} \sum_{ijkl} \int_{\Omega_{ijkl}} \hat{\phi}(\xi, \eta) \, d\xi \, d\eta + \frac{1}{H^2} \sum_{ijkl} \int_{\Omega_{ijkl}} \hat{\phi}(\xi, \eta) \, d\xi \, d\eta. \quad (3.27)
\]
For Case A, the centroid rule is also valid, yielding
\[ \int \int_{Q_{ij}^{kl} \text{ Case A}} \hat{\phi}(\xi, \eta) \, d\xi \, d\eta \approx h^2 \hat{\phi}(\hat{G}). \] (3.28)

On the other hand, for Case B as \( \mu = 0 \), we have from (3.8) and (3.20)
\[ \int \int_{Q_{ij}^{kl} \cap Q_{ij}^{kl}} \hat{\phi}(\xi, \eta) \, d\xi \, d\eta = \int \int_{Q_{ij}^{kl} \cap Q_{ij}^{kl}} \hat{b}_0(x, y)J^{-1} \, dx \, dy \]
\[ = \sum_{r=1}^{2} \int \int_{\Delta_{ij}^{r} \cap Q_{ij}^{kl}} \hat{b}_0(x, y)J^{-1} \, dx \, dy \approx \sum_{r=1}^{2} \int \int_{\Delta_{ij}^{r} \cap Q_{ij}^{kl}} \hat{b}_0(x, y)J^{-1} \, dx \, dy \]
\[ = \sum_{t,m} \int \int_{\Delta_{ij}^{t,m} \cap Q_{ij}^{kl}} \hat{b}_0(x, y)J^{-1} \, dx \, dy = \sum_{t,m} B_{ij} \int \int_{\Delta_{ij}^{t,m} \cap Q_{ij}^{kl}} J^{-1} \, dx \, dy, \] (3.29)
where \( J \) is the Jacobian determinant given in (2.13). Since
\[ \frac{1}{J} \approx \frac{|\hat{\Delta}_{ij,kl,t}|}{|\hat{\Delta}_{ij,kl,t}|} = \frac{h^2}{2} / |\hat{\Delta}_{ij,kl,t}|, \] (3.30)
we have
\[ \int \int_{\Delta_{ij,kl,t} \cap Q_{ij}^{kl}} J^{-1} \, dx \, dy \approx \frac{h^2}{2} \frac{|\hat{\Delta}_{ij,kl,t,m}|}{|\hat{\Delta}_{ij,kl,t}|}. \] (3.31)
Consequently, from (3.27)–(3.31) the splitting–integrating method (SIM) using the partition technique seeks image greyness under \( T^{-1} \) when \( \mu = 0 \), by
\[ \Phi_{ij} \approx \Phi_{ij}^{(N)} = \sum_{k,l \text{ Case A}} \left( \frac{h}{H} \right)^2 \hat{\phi}(\hat{G}) + \sum_{k,l \text{ Case B}} \sum_{m \text{ (3.24)}} \frac{1}{2} \left( \frac{h}{H} \right)^2 B_{ij} \frac{|\hat{\Delta}_{ij,kl,t,m}|}{|\hat{\Delta}_{ij,kl,t}|}. \] (3.32)
Combining (3.26) and (2.21) leads to C SIM, and combining (3.26) and (3.32) leads to C SIM as \( \mu = 0 \). Note that both C SIM and C SIM do not require solutions of nonlinear equations either.

4. Error bounds of integration approximation and image greyness

It is clear that the discrete algorithms in Section 3 are of numerical integration, basically. However, the integration approximation, (3.26) and (3.32), are not the same as the traditional methods in [2,14]. Such a distinctness results from different regularities of the integrand in different subregions due to piecewise bilinear interpolation. Therefore, error analysis on new algorithms is necessary and important.

We will define some error norms to measure the approximation degree of greyness solutions. Choose the division number
\[ N = N_p = 2^p \quad \text{where} \quad p = p_0, \quad p_0 + 1 \text{ integer} \quad p_0 > 0. \] (4.1)
Define the consecutive errors of image greyness under $T^{-1}$ or $T^{-1}T$ with the two division numbers $N_p$ and $N_{p-1}$:

$$\Delta E^{(N_p)}(\Phi) = \sum_{ij} \left| \frac{\Phi_{ij}^{(N_p)} - \Phi_{ij}^{(N_{p-1})}}{I_{\max}(\tilde{W})} \right|, \quad \Delta E_2^{(N_p)}(\Phi) = \left\{ \sum_{ij} \left( \frac{\Phi_{ij}^{(N_p)} - \Phi_{ij}^{(N_{p-1})}}{I_{\max}(\tilde{W})} \right)^2 \right\}^{1/2},$$

(4.2)

where $I_{\max}(\tilde{W})$ is the total number of nonempty pixels, defined by

$$I_{\max}(\tilde{W}) = \sum_{ij} N_t(\tilde{W}_{ij}^{(N_p)}), \quad N_t(W_{ij}) = \begin{cases} 1 & \text{if } W_{ij} \neq \cdot \cdot \cdot, \\ 0 & \text{if } W_{ij} = \cdot \cdot \cdot. \end{cases}$$

(4.3)

In fact, errors (4.2) are the mean and the standard squares deviation of greyness errors respectively. Moreover, when the original values $\Phi_{ij}$ are known in the cycle version $T^{-1}T$, we can also compute the absolute errors,

$$E^{(N_p)}(\Phi) = \sum_{ij} \left| \frac{\Phi_{ij}^{(N_p)} - \Phi_{ij}}{I_{\max}(\tilde{W})} \right|, \quad E_2^{(N_p)}(\Phi) = \left\{ \sum_{ij} \left( \frac{\Phi_{ij}^{(N_p)} - \Phi_{ij}}{I_{\max}(\tilde{W})} \right)^2 \right\}^{1/2}.$$

(4.4)

Similarly, define the sequential errors of image greyness under $T$

$$E^{(N_p)}(\tilde{B}) = \sum_{IJ} \left| \frac{\tilde{B}_{IJ}^{(N_p)} - B_{IJ}}{I_{\max}(\tilde{Z})} \right|, \quad \Delta E_2^{(N_p)}(\tilde{B}) = \sum_{IJ} \left| \frac{\tilde{B}_{IJ}^{(N_p)} - \tilde{B}_{IJ}^{(N_{p-1})}}{I_{\max}(\tilde{Z})} \right|.$$

(4.5)

$E_2^{(N_p)}(\tilde{B})$ and $\Delta E_2^{(N_p)}(\tilde{B})$, where $I_{\max}(\tilde{Z}) = \sum_{IJ} N_t(\tilde{Z}_{IJ}^{(N_p)})$.

The transformation $T$ is said to be regular if the following three conditions are satisfied:

1. The functions in (1.1)

$$x(\xi,\eta), y(\xi,\eta) \in C^2(\Omega),$$

(4.6)

where $C^k(\Omega)$ denotes the space of functions having $k$-order continuous derivatives over $\Omega$.

2. Eq. (2.14) holds true.

3. The transformed elements $\square_{ii,k}$ under (3.1) are quasiuniform; this leads to the bounded ratios

$$0 < C_0 \leq \frac{dl_i}{dl_j} \leq C_1, \quad 1 \leq i,j \leq 3,$$

(4.7)

where $C_0$ and $C_1$ are bounded constants independent of $\xi,\eta,x$ and $y$, and the boundary length under $T$

$$dl_1 = \sqrt{\left( \frac{\partial x}{\partial \xi} \right)^2 + \left( \frac{\partial y}{\partial \xi} \right)^2}, \quad dl_2 = \sqrt{\left( \frac{\partial y}{\partial \eta} \right)^2 + \left( \frac{\partial y}{\partial \eta} \right)^2},$$

(4.8)

$$dl_3 = \sqrt{\left( \frac{\partial x}{\partial \xi} \pm \frac{\partial x}{\partial \eta} \right)^2 + \left( \frac{\partial y}{\partial \xi} \pm \frac{\partial y}{\partial \eta} \right)^2}.$$

(4.9)
Lemma 4.1. Let $T$ be regular, and $M_A$ and $M_B$ denote the numbers of $\Box_{ij,kl}$ in Cases A and B given in (3.3) and (3.4), respectively. Then there exists a bounded constant $C$ independent of $N$ such that

$$M_A \leq N^2/J_0, \quad M_B \leq CN/J_0^{1/2},$$

(4.10)

where $J_0$ is the lower bound of (2.14).

Proof. From (2.14) and (2.16) we have $J_0|\omega| \leq |\Box_{ij,kl}| \leq J_M|\omega|$. It then follows that

$$M_A \leq |\omega|/h^2 \leq \frac{1}{J_0} |\Box_{ij,kl}|/h^2 = \frac{N^2}{J_0^{1/2}}.\quad (4.11)$$

The first inequality in (4.10) is obtained. Next, it is due to the quasi-uniform elements $\Box_{ij,kl}$ such that $M_B = O(M_A^{1/2})$. Then we obtain the second inequality in (4.10) from the first inequality, thus completing the proof of Lemma 4.1.

We have the following lemma.

Lemma 4.2. Let $T$ be regular, (3.10) be given, and $\Delta abc$ be one of $\Delta_{ij,kl,t}$ in (3.8) under (3.9). Then the area differences between $\Delta ABC$ and $\Delta ABC$ have the bounds

$$|\Delta ABC \oplus \Delta ABC| \leq C h^3 J_M^{1/2} M_2,\quad (4.12)$$

where

$$M_2 = |x|_{\infty,2,\Omega} + |y|_{\infty,2,\Omega}, \quad |v|_{k,\infty,\Omega} = \max_{a,b} |D^2 v|.$$

(4.13)

Here and below, $C$ always denotes a bounded constant independent of $h$, $H$, $\xi$, $\eta$, $x$ and $y$; the values of $C$ may be different in different contexts.

Proof. Since the shaded area is denoted by $|\Delta ABC \oplus \Delta ABC|$ consisting of three parts along three boundaries of $\Delta ABC$, it suffices to show that one part of the shaded area has the bounds

$$\delta_{AB} \leq C h^3 J_M^{1/2} M_2,\quad (4.14)$$

where $\delta_{AB}$ denotes the shaded area along the boundary $AB$ of $\Delta ABC$ in Fig. 2. Denote by $d(\tilde{AB},\tilde{AB})$ the distance between $\tilde{AB}$ and $\tilde{AB}$ since $ab \rightarrow \tilde{AB}$, we have

$$\delta_{AB} \leq |AB| + d(\tilde{AB},\tilde{AB}) \leq |AB| \max_{\xi \leq \xi \leq \zeta} \left( ((x(\xi, \bar{\eta}) - \hat{x}(\xi, \bar{\eta}))^2 + (y(\xi, \bar{\eta}) - \hat{y}(\xi, \bar{\eta}))^2 \right)^{1/2},\quad (4.15)$$

where $\bar{\eta}$=constant, and $\hat{x}(\xi, \bar{\eta})$ is the linear interpolation function of $x(\xi, \bar{\eta})$, which passes two points $(\xi, x(a, \bar{\eta}))$ and $(\xi, x(b, \bar{\eta}))$ (see Fig. 4). The truncation errors of the linear interpolation give the bounds

$$|x(\xi, \bar{\eta}) - \hat{x}(\xi, \bar{\eta})| \leq \frac{|ab|}{2} |x|_{\infty,2,\Omega} \leq \frac{h^2}{2} M_2.$$

(4.16)

Moreover, we have from the quasiuniform element $\Box_{ij,kl}$,

$$|AB| \leq C |ab| J_M^{1/2} = Ch J_M^{1/2}.$$

(4.17)
From (4.15)–(4.17) we obtain \[ \delta_{AB} \leq |AB| h^2 M_2 \leq Ch^2 J_{M}^{1/2} M_2, \] This is (4.14), thus completing the proof of Lemma 4.2. \( \Box \)

We have the following lemma from [2,14].

**Lemma 4.3.** Let \( \Box \) be a square with the boundary length \( h \), and \( f \in C^k(\Box) \), \( k = 0, 1, 2 \). Then the centroid rule of integration has the error bounds,
\[
\left| \int \int_{\Box} f(x, y) \, dx \, dy - f(\hat{G}) h^2 \right| \leq Ch^{2+k}|f|_{k, \infty, \Box},
\] (4.18)
where \( \hat{G} \) is the centroid of \( \Box \).

Denote
\[
\hat{B}_{IJ} = \frac{1}{H^2} \int_{\Box} \hat{b} \, dx \, dy,
\] (4.19)
where \( \hat{b} = \hat{\phi}_\mu (\xi, \eta) \), \( \mu = 0, 1 \), \( \hat{\phi}_0 \) and \( \hat{\phi}_1 \) are the constant and bilinear interpolation functions (2.5) and (2.7). Now we will prove an important theorem.

**Theorem 4.1.** Let \( T \) be regular, and
\[
x(\xi, \eta), y(\xi, \eta) \in C^3(\Omega).
\] (4.20)

Then the greyness (3.26) under \( T \) of \( \tilde{SSM} \) using the partition technique has the error bounds
\[
\left| \tilde{\Delta}_{IJ} - \hat{B}_{IJ} \right| \leq C \left\{ \frac{1}{J_0} \left( \frac{H}{N} \right)^2 |\hat{\phi}|_{2, \infty, \tilde{\Omega}_{IJ}} + \frac{1}{J_0^{1/2}} \frac{H}{N^2} J_{M}^{1/2} M_2 + \mu J_M |\hat{\phi}|_{1, \infty, \tilde{\Omega}_{IJ}} \right\},
\] (4.21)
where \( \tilde{\Omega}_{IJ} = \bigcup_{\text{Case A}} \square_{ij, kl}, \hat{\Delta}_{IJ} = \bigcup_{\text{Case B}} \square_{ij, kl}. \)

---

**Fig. 4.** The functions \( x(\xi) \) and \( y(\xi) \) along the curved boundary \( \hat{AB} \) of \( \hat{\Delta} ABC. \)
Proof. From (3.26) and (4.19) we have
\[ | \hat{B}_{ij} - \hat{B}_{ij}^{(N)} | \leq D_I + D_{II}, \]  
(4.22)
\[ D_I = \frac{1}{H^2} \sum_{\text{Case A}} \int_{x_i,kl} \hat{b} J d \xi d \eta \left| \hat{\phi} J d \xi d \eta - h^2 \hat{\phi}(\hat{G}) J(\hat{G}) \right|, \]  
(4.23)
\[ D_{II} = \frac{1}{H^2} \sum_{\text{Case B}} \int_{x_i,kl} \hat{b} d x d y - \hat{b}_\mu(\hat{G}^*) \sum_{\ell,m} | \hat{\Delta}^*_{ij,kl,t,m} |. \]  
(4.24)
From Lemmas 4.1 and 4.3, we obtain the bounds
\[ D_I \leq \frac{C}{H^2} \sum_{\text{Case A}} h^4 | \hat{\phi} J |_{2,\infty,D_{ij}} \]  
\[ \leq \frac{C}{H^2} M h^4 | \hat{\phi} J |_{2,\infty,\partial U} \leq C \frac{1}{J_0} \left( \frac{H}{N} \right)^2 | \hat{\phi} J |_{2,\infty,\partial U}. \]  
(4.25)
Also from (3.11) and (4.24) we can see that
\[ \int_{x_i,kl} \hat{b}_\mu d x d y - \hat{b}_\mu(\hat{G}^*) \sum_{\ell,m} | \hat{\Delta}^*_{ij,kl,t,m} | \]  
\[ \leq \sum_{t=1}^{2} \int_{x_i,kl} \hat{b}_\mu d x d y + \sum_{t=1}^{2} \int_{x_i,kl} \hat{b}_\mu d x d y - \hat{b}_\mu(\hat{G}^*) | \hat{\Delta}^*_{ij,kl,t} |. \]  
(4.26)
Moreover, applying Lemma 4.2 yields
\[ \int_{\Delta_{ij,kl} \cap \Delta_{ik,kl}} \hat{b}_\mu d x d y \leq | \hat{b}_\mu |_{0,\infty,\partial} | \hat{\Delta}^*_{ij,kl,t} \cap \hat{\Delta}^*_{ik,kl,t} | \]  
\[ \leq C h^3 J_{ij}^{1/2} M_2. \]  
(4.27)
When \( \mu = 0 \), by noting (3.24),
\[ \hat{b}(\hat{G}^*) = \hat{b} = \text{constant}, \]  
(4.28)
to get
\[ \int_{\Delta_{ij,kl,m}} \hat{b} d x d y - \hat{b}(\hat{G}^*) | \hat{\Delta}^*_{ij,kl,t,m} | = 0. \]  
(4.29)
When $\mu = 1$, however

\[
\left| \int \int_{\Delta_{ij,kl}} \hat{b} \, dx \, dy - \tilde{b}(\hat{G}) \right| \leq \max_{\Delta_{ij,kl}} |\hat{b} - \tilde{b}(\hat{G})| \leq C b \max_{\Delta_{ij,kl}} |\hat{G} - \tilde{G}|.
\]

(4.30)

We obtain from (4.24), (4.26)–(4.30) and Lemma 4.1

\[
D_H \leq C \frac{H^3}{H^2} \sum_{i,j} \left\{ J_M^{1/2} M_2 + \mu J_M |\hat{\phi}|_{1,\infty} \right\}
\]

\[
\leq C \frac{H^3}{H^2} M_B \left\{ J_M^{1/2} M_2 + \mu J_M |\hat{\phi}|_{1,\infty} \right\}
\]

\[
\leq C \frac{H}{J_0^{1/2} N^2} \left\{ J_M^{1/2} M_2 + \mu J_M |\hat{\phi}|_{1,\infty} \right\}.
\]

(4.31)

Combining (4.22)–(4.25) and (4.31) gives (4.21). This completes the proof of Theorem 4.1.

Below, we shall provide the error bounds for the splitting–integrating method (SIM) and S\(\tilde{M}\) as $\mu = 0$ using the partition technique. Denote

\[
\hat{\phi}_{ij} = \frac{1}{H^2} \int \int_{\Omega_{ij}} \hat{\phi} \, d\xi \, d\eta,
\]

(4.32)

where $\hat{\phi} = \tilde{b}_{i\mu}$, $\mu = 0,1$, and $\tilde{b}_0$ and $\tilde{b}_1$ are the piecewise constant and bilinear interpolatory functions. We cite the following theorem from [8].

**Theorem 4.2.** Let $T$ be regular and the piecewise constant and bilinear interpolations be used. Then the greyness (2.26) under $T^{-1}$ by SIM has the error bounds

\[
|\hat{\phi}_{ij} - \tilde{\phi}_{ij}^{(N)}| \leq C \left\{ \left( \frac{H}{N} \right)^2 |\hat{\phi}|_{1,\infty,\Omega_{ij}} + \frac{1}{H} \left( \frac{H}{N} \right)^{\mu+1} |\hat{\phi}|_{\mu,\infty,\Omega_{ij}} \right\}, \quad \mu = 0,1,
\]

(4.33)

where $\hat{\Omega}_{ij} = \bigcup_{k,l} \square_{ij,kl}$, and $\tilde{\hat{\Omega}}_{ij} = \bigcup_{k,l} \square_{ij,kl}$.

Note that when $\mu = 0$ only the low convergence rate $O(1/N)$ can be reached. We shall prove the following theorem to provide a better convergence rate.

**Theorem 4.3.** Let $T$ be regular and the piecewise constant interpolation $\tilde{b}_0$ be used; then the greyness (3.32) under $T^{-1}$ by S\(\tilde{M}\) using the partition technique has the error bounds

\[
|\hat{\phi}_{ij} - \tilde{\phi}_{ij}^{(N)}| \leq C \left\{ \left( \frac{H}{N} \right)^2 |\hat{\phi}|_{1,\infty,\Omega_{ij}} + \frac{H}{N^{\mu}} \left( \frac{J_M^{1/2} M_2 + J_M}{J_0 M_2 + J_M} |\hat{\phi}|_{1,\infty,\Omega_{ij}} \right) \right\},
\]

(4.34)

where $M_2$ is defined by (4.13).
Proof. We have from (3.32) and (4.28)
\[ |\hat{\Phi}_{ij} - \hat{\Phi}^{(N)}_{ij}| \leq D^*_I + D^*_II, \]  
where
\[ D^*_I = \sum_{k,l} \frac{1}{H^2} \int \int_{D_{ijkl}} \hat{\phi} d\xi d\eta - \hat{h}^2 \hat{\phi}(\hat{G}) \]  
(4.35)
\[ D^*_II = \sum_{k,l} \frac{1}{H^2} \int \int_{D_{ijkl} \cap \Omega_{ij}} \hat{\phi} d\xi d\eta - \sum_{t,m} \frac{1}{2} \left( \frac{h}{H} \right)^2 B_{ij} \frac{|\hat{\Delta}_{ij,l}^*|}{|\hat{\Delta}_{ijkl}^*|}. \]  
(4.36)
From Lemma 4.3 we have similarly
\[ D^*_I \leq C \sum_{k,l} \frac{h^2}{H^2} \sum_{t,m} J(u, \xi, \eta) \leq C \left( \frac{H}{N} \right)^2 |\hat{\phi}|_{2, \infty, \Omega_{ij}}. \]  
(4.37)
Next, since there exists a point \( u \in \Delta_{ijkl} \) in \( \xi \eta \) such that
\[ \frac{|\hat{\Delta}_{ijkl}^*|}{|\hat{\Delta}_{ijkl}^*|} = \frac{h^2}{2} \frac{1}{|\hat{\Delta}_{ijkl}^*|} = \frac{1}{J(u)}, \]  
(4.38)
we see from Lemma 4.2
\[ |\hat{\Delta}_{ijkl}^*| \left( \frac{1}{|\hat{\Delta}_{ijkl}^*|} - \frac{1}{|\hat{\Delta}_{ijkl}^*|} \right) \leq C \left( \frac{\hat{\Delta}_{ijkl}^* + \hat{\Delta}_{ijkl}^*}{|\hat{\Delta}_{ijkl}^*|} \right) \leq C \frac{hJ_{ijkl}^{1/2} M_2}{J_0}. \]  
(4.39)
Consequently, we obtain from (4.37), (4.40) and Lemma 4.2
\[ D^*_II \leq \sum_{k,l} \frac{1}{H^2} \left\{ \sum_{t=1}^{2} \int \int_{\Delta_{ijkl}^* + \Delta_{ijkl}^*} |\hat{\phi}J^{-1}| dx dy \right. 
+ \sum_{t=1}^{2} \int \int_{\Delta_{ijkl}^* + \Delta_{ijkl}^*} J^{-1} dx dy - \frac{h^2}{2} \frac{|\hat{\Delta}_{ijkl}^*|}{|\hat{\Delta}_{ijkl}^*|} \right\} 
+ \sum_{t=1}^{2} \int \int_{\Delta_{ijkl}^* + \Delta_{ijkl}^*} \frac{h^2 B_{ij}}{2} \left\{ \frac{|\hat{\Delta}_{ijkl}^*|}{|\hat{\Delta}_{ijkl}^*|} - \frac{|\hat{\Delta}_{ijkl}^*|}{|\hat{\Delta}_{ijkl}^*|} \right\} \right\} \right\} 
\leq C \frac{1}{H^2} \left\{ \sum_{k,l} \frac{|\hat{\phi}|_{2, \infty, \Omega_{ij}}}{J_0} \Delta_{ijkl}^* + \Delta_{ijkl}^* \right\} 
+ \sum_{t=1}^{2} \int \int_{\Delta_{ijkl}^* + \Delta_{ijkl}^*} \frac{h^2 B_{ij}}{2} \left\{ \frac{|\hat{\Delta}_{ijkl}^*|}{|\hat{\Delta}_{ijkl}^*|} + \frac{h^2 B_{ij} J_{ijkl}^{1/2} M_2}{J_0} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\}
\begin{align*}
\leq C \frac{1}{H^2} \left\{ \sum_{\substack{i,j \in h \cap \Omega}} \frac{h^3 J_{M}^{1/2} M_2}{J_0} + h^3 J_M |J^{-1}|_{1,\infty, \Omega} \right\} \\
\leq CN \frac{h^3}{H^2} \left\{ \frac{J_{M}^{1/2} M_2}{J_0} + J_M |J^{-1}|_{1,\infty, \Omega} \right\} \leq C \frac{H}{N^2} \left\{ \frac{J_{M}^{1/2} M_2}{J_0} + J_M \frac{|J|_{1,\infty, \Omega}}{J_0^2} \right\}. \quad (4.41)
\end{align*}

Combining (4.35)–(4.38) and (4.41) yields (4.34). This completes the proof of Theorem 4.3. ∎

From Theorems 4.1–4.3, we obtain the following asymptotic relations:

\begin{align*}
|\hat{B}_{IJ} - \bar{B}_{IJ}^{(N)}| &= O(1/N^2) \quad \text{as } N \to \infty \quad \text{by } \overline{SSM} \quad \text{as } \mu = 0, 1, \quad (4.42) \\
|\hat{\Phi}_{ij} - \bar{\Phi}_{ij}^{(N)}| &= O(1/N^2) \quad \text{as } N \to \infty \quad \text{by } \overline{SIM} \quad \text{as } \mu = 0. \quad (4.43)
\end{align*}

By using the partition technique, the high convergence rate as O(1/N^2) can always be achieved for images under transformation by \overline{SSM}, and \overline{SIM} as \mu = 0. This is a great improvement from SSM and SIM as \mu = 0, which produce the transformation images with only the low convergence rate O(1/N) (see [8]), and O_p(1/N^{1.5}) in probability (see [10]).

Based on the above error estimates, we can provide other important error bounds by following the arguments in [8], accompanied with outlines of proofs only.

**Lemma 4.4.** Let \(T\) be regular, and

\[ b(x,y) \in C^2(S). \quad (4.44) \]

There exist the bounds

\[ \sum_{IJ} |B_{IJ}^T - \hat{B}_{IJ}| \leq CI_{tot}(W)H^2J_M|\phi|_{2,\infty, \Omega}, \quad (4.45) \]

where \(I_{tot}(W)\) is the total number of pixels \(\{W_{ij}\}\).

**Proof.** The norm \(|\phi|_{\infty, \Omega}\) exists due to (2.11), (4.6) and (4.44). We then have from (2.10), (4.19), and the mean theorem of integration,

\[ \begin{align*}
\sum_{IJ} |B_{IJ}^T - \hat{B}_{IJ}| &= \frac{1}{H^2} \sum_{IJ} \int_{\Omega} |b(x,y) - \hat{b}(x,y)| \, dx \, dy \\
&= \frac{1}{H^2} \int_{S} |b(x,y) - \hat{b}(x,y)| \, dx \, dy = \frac{1}{H^2} \int_{\Omega} |(\varphi - \hat{\varphi})| \, d\xi \, d\eta \\
&\leq \frac{J_M}{H^2} \int_{\Omega} |(\varphi - \hat{\varphi})| \, d\xi \, d\eta = \frac{J_M}{H^2} \sum_{ij} \int_{\Omega} |(\varphi - \hat{\varphi})| \, d\xi \, d\eta \\
&\leq CJ_M \sum_{ij} H^2 |\phi|_{2,\infty, \Omega} \leq CI_{tot}(W)J_M H^2 |\phi|_{2,\infty, \Omega}. \quad (4.46)
\end{align*} \]

This completes the proof of Lemma 4.4. ∎
Below let us provide bounds for the errors $E$ defined in (4.5). We have the triangle inequality,
\[ |\bar{B}_{ij}^{(N)} - B_{ij}| \leq |\bar{B}_{ij}^{(N)} - \hat{B}_{ij}| + |\hat{B}_{ij} - B_{ij}| + |B_{ij}^{(M)} - B_{ij}|. \] (4.47)

The bounds corresponding to the first and second terms in the right-hand side of (4.47) are provided from Theorem 4.1 and Lemma 4.4. Also we obtain from Lemma 4.3
\[ |B_{ij}^{(M)} - B_{ij}| \leq \frac{1}{H^2} \left| \int_{\Omega} b(x, y) \, dx \, dy - B_{ij}H^2 \right| \leq CH^2|b|_{\infty, \Omega}. \] (4.48)

The above analyses and estimates of errors allow us to obtain the detailed bounds of $\tilde{E}(B)$ defined in (4.5). As to $\Delta E(\tilde{B})$ we may use for proofs the triangle inequality
\[ |\bar{B}_{ij}^{(Np)} - B_{ij}^{(Np-1)}| \leq |\bar{B}_{ij}^{(Np)} - \hat{B}_{ij}| + |B_{ij}^{(Np-1)} - \hat{B}_{ij}|. \] (4.49)

We only provide their asymptotic results as $N \to \infty$ in the following corollary.

**Corollary 4.1.** Let (4.1), (4.44) and all the conditions in Theorems 4.1–4.3 hold. Then the greyness (3.26) under $T$ from $\tilde{SSM}$ has the following asymptotes:
\[ E(\tilde{B}) = O(H^2) + O(1/N^2), \quad \Delta E(\tilde{B}) = O(1/N^2), \quad \mu = 0, 1. \] (4.50)

Also the greyness (3.32) under $T^{-1}$ by $\tilde{SIM}$ as $\mu = 0$ has
\[ E(\tilde{\Phi}) = O(H^2) + O(1/N^2), \quad \Delta E(\tilde{\Phi}) = O(1/N^2), \] (4.51)

where $E(\Phi)$ and $\Delta E(\Phi)$ are defined in (4.4) and (4.2), respectively.

Below, we will provide the asymptotic relations of errors for image greyness under $T^{-1}T$ from $C\tilde{SIM}$ and $C\tilde{SIM}$. Let $\Phi_{ij}$ be given; the greyness under $T^{-1}T$ is evaluated by $C\tilde{SIM}$ as $\mu = 1$ and $C\tilde{SIM}$ as $\mu = 0$:
\[ \{\Phi_{ij}\} \xrightarrow{T^{-1}} \{\tilde{B}_{ij}\} \xrightarrow{T} \{\tilde{\Phi}_{ij}\}, \quad \mu = 1, \]
\[ \{\Phi_{ij}\}, \quad \mu = 0. \] (4.52)

By following the proofs in [8], we can derive the following corollary.

**Corollary 4.2.** Let (4.1) and all conditions in Theorems 4.1–4.3 hold true; also assume $\phi(\xi, \eta) \in C^2(\Omega)$. Then when $N \to \infty$, the image greyness under $T^{-1}T$ by $C\tilde{SIM}$ in Case 1 has the asymptotic relations
\[ E(\tilde{\Phi}^*) = O\left(\frac{1}{H^2}\right) + O(1/N^{\mu+1}), \quad \Delta E(\tilde{\Phi}^*) = O(1/N^{\mu+1}), \quad \mu = 0, 1. \] (4.53)

Also when $\mu = 0$ and $N \to \infty$, the image greyness under $T^{-1}T$ by $C\tilde{SIM}$ as $\mu = 1$ in Case 0 has the asymptotic relations
\[ E(\tilde{\Phi}^*) = O\left(\frac{1}{H^2}\right) + O(1/N^2), \quad \Delta E(\tilde{\Phi}^*) = O(1/N^2). \] (4.54)
The new combinations CSI as $\mu = 1$ and CSI as $\mu = 0$ can produce the images under $T^{-1}T$ with the better convergence rate $O(1/N^2)$.

5. Numerical and graphical experiments

We may also define the pixel error under $T^{-1}T$

$$\Delta E^{(N)}(\tilde{W}) = \sum_{ij} N_{d.i}(\tilde{W}_{ij}^{(N_r)} - W_{ij}^{(N_r-1)}), \quad I_i^{(N_r)}(\tilde{W}) = \sum_{ij} N_{d.i}(\tilde{W}_{ij}^{(N_r)} - W_{ij}),$$

(5.1)

where

$$N_{d.i}(W_1 - W_2) = \begin{cases} 1 & \text{if } (W_1 \neq W_2) \land ((W_1 = G_1) \lor (W_2 = G_1)), \\ 0 & \text{otherwise}, \end{cases}$$

(5.2)

and $G_1 = '*'$, $G_2 = '+'$, $G_3 = '-'$.

Let the standard image $W$ be given in Fig. 5, and $T$ be a bi-quadratic transformation in [7]. The pixel greyness under $T^{-1}T$ as (4.52) is evaluated by CSI and CSI. Their errors are listed in Tables 1 and 2, and error curves of $\Delta E$ are depicted in Figs. 6 and 7. It can be seen that

$$\Delta E^{(N)}(\tilde{B}) = O(1/N^2) \quad \text{by CSI as } \mu = 0, 1,$$

(5.3)

$$\Delta E^{(N)}(\tilde{\Phi}) = O(1/N^{\mu+1}), \quad \text{by CSI as } \mu = 0, 1 \quad \text{in Case I},$$

(5.4)

$$\Delta E^{(N)}(\tilde{\Phi}^*) = O(1/N^2), \quad \text{by CSI as } \mu = 0 \quad \text{in Case I}.$$ 

(5.5)

All the experimental results (5.3)–(5.5) confirm the analysis in Section 4. We provide some images under transformation by CSI and CSI in Figs. 8–10. Furthermore, Fig. 10 indicates that

![Fig. 5. A standard image.](image-url)
### Table 1
Errors of pixels and greyness under transformations by CSIM as $\mu = 1$

<table>
<thead>
<tr>
<th>Case</th>
<th>N</th>
<th>Sequential error</th>
<th>$T^{-1}T$</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\Delta I_1$</td>
<td>$\Delta I_2$</td>
<td>$\Delta I_3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case I</td>
<td>4</td>
<td>898 0 0 0</td>
<td>0.00279</td>
<td>0.00339</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>979 0 0 0</td>
<td>0.6927 $\times 10^{-3}$</td>
<td>0.8066 $\times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>979 0 0 0</td>
<td>0.1707 $\times 10^{-1}$</td>
<td>0.1949 $\times 10^{-3}$</td>
</tr>
<tr>
<td>Case II</td>
<td>4</td>
<td>898 0 0 0</td>
<td>0.00279</td>
<td>0.00339</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>976 0 0 0</td>
<td>0.6927 $\times 10^{-3}$</td>
<td>0.8066 $\times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>976 0 0 0</td>
<td>0.1707 $\times 10^{-1}$</td>
<td>0.1949 $\times 10^{-3}$</td>
</tr>
</tbody>
</table>

### Table 2
Errors of pixels and greyness under transformations by CSIM as $\mu = 0$

<table>
<thead>
<tr>
<th>Method</th>
<th>N</th>
<th>Sequential error</th>
<th>$T^{-1}T$</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\Delta I_1$</td>
<td>$\Delta I_2$</td>
<td>$\Delta I_3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CSIM</td>
<td>4</td>
<td>892 0 0 0</td>
<td>0.9160 $\times 10^{-4}$</td>
<td>0.1478 $\times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>929 0 0 0</td>
<td>0.2425 $\times 10^{-4}$</td>
<td>0.3682 $\times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>929 0 0 0</td>
<td>0.6048 $\times 10^{-5}$</td>
<td>0.9030 $\times 10^{-5}$</td>
</tr>
<tr>
<td>CSIM</td>
<td>4</td>
<td>892 0 0 0</td>
<td>0.9160 $\times 10^{-4}$</td>
<td>0.1478 $\times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>929 0 0 0</td>
<td>0.2425 $\times 10^{-4}$</td>
<td>0.3682 $\times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>929 0 0 0</td>
<td>0.6048 $\times 10^{-5}$</td>
<td>0.9030 $\times 10^{-5}$</td>
</tr>
</tbody>
</table>
Fig. 6. Error curves of $\Delta E$ versus $N$ by CSIM in Case I as $\mu = 1$.

Fig. 7. Error curves of $\Delta E$ versus $N$ by CSIM and C$\tilde{S}$IM in Case I as $\mu = 0$. 
Combination CSIM is also well suited to the transformation of curve images, which often cause troubles by many other approaches.

Next, consider the different division numbers $N_{\text{ford}}$ and $N_{\text{back}}$ used for $T$ and $T^{-1}$ in CSIM, respectively. We choose $N_{\text{ford}} = 8$ in SSM; and $N_{\text{back}} = 1, 2, 4, \ldots, 32$ in SIM. Denote $S(N_{\text{ford}}, N_{\text{back}})$ as the greyness solution using $N_{\text{ford}}$ for $T$ and $N_{\text{back}}$ for $T^{-1}$. We list in Table 3 the errors between $S(32, 32)$ and $S(8, N_{\text{back}})$ and the absolute errors between $S(8, N_{\text{back}})$ and the true solutions. Table 3 indicates the optimal division number is about $N_{\text{back}} = 8$. We then conclude that an equal number, $N_{\text{back}} = N_{\text{ford}} = N$, is a good choice. As a consequence, we always choose the same division number for both $T$ and $T^{-1}$ in CSIM and their renovation.
Fig. 9. Images under $T^{-1}T$ by CSIM in Case 1 as $\mu = 0$: (a) as $N = 4$; (b) as $N = 8$.

We collect in Table 4 all the errors by different combinations CSIM, CSIM and CSIM when $N = 8$. The absolute errors $E$ obtained from both CSM (as $\mu = 1$) and CSM (as $\mu = 0$) are significantly smaller than those obtained from CSIM (as $\mu = 0, 1$) cited from [8]. For example, the ratios of restoring greyness errors under $T^{-1}T$ are

$$\frac{\Delta E|_{CSIM}}{\Delta E|_{CSIM}} = \frac{0.04054}{0.001973} = 20.55 \quad \text{as } \mu = 1,$$  \hspace{1cm} (5.6)
This clearly displays a significant advantage of CSIM and CSM over CSIM in [6,10].

The above examples are all binary images; we now apply CSIM and CSM to real images of 256×256 pixels with 256 greyness levels. Choosing $N=4$, the computer images are produced under the transformation, and illustrated in Fig. 11–13. The original and restored girl images are shown on
Table 3
Errors of pixels and greyness under $T$ by CSIM as $\mu = 1$ in Case I when $N_{\text{front}} = 8$ and $N_{\text{back}}$ varies

<table>
<thead>
<tr>
<th>$N_{\text{back}}$</th>
<th>Total</th>
<th>$\Delta I_1$</th>
<th>$\Delta I_2$</th>
<th>$\Delta I_3$</th>
<th>$\Delta E$</th>
<th>$\Delta E_2$</th>
<th>$\Delta I$</th>
<th>$E$</th>
<th>$E_2$</th>
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<tbody>
<tr>
<td>1</td>
<td>596</td>
<td>2</td>
<td>17</td>
<td>77</td>
<td>$6.369 \times 10^{-2}$</td>
<td>$6.338 \times 10^{-2}$</td>
<td>2</td>
<td>0.2493</td>
<td>0.2237</td>
</tr>
<tr>
<td>2</td>
<td>642</td>
<td>1</td>
<td>2</td>
<td>17</td>
<td>$1.264 \times 10^{-2}$</td>
<td>$1.182 \times 10^{-2}$</td>
<td>1</td>
<td>0.2987</td>
<td>0.2562</td>
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<tr>
<td>3</td>
<td>652</td>
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<td>0</td>
<td>6</td>
<td>$5.004 \times 10^{-3}$</td>
<td>$4.704 \times 10^{-3}$</td>
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<td>0.3062</td>
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<tr>
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<td>0</td>
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<td>$2.417 \times 10^{-3}$</td>
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<td>0</td>
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<td>0</td>
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<td>$8.476 \times 10^{-4}$</td>
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<td>0.3106</td>
<td>0.2649</td>
</tr>
<tr>
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<td>658</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>$6.505 \times 10^{-4}$</td>
<td>0</td>
<td>0.3112</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>$7.729 \times 10^{-4}$</td>
<td>$7.881 \times 10^{-4}$</td>
<td>0</td>
<td>0.3117</td>
<td>0.2657</td>
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<td>16</td>
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<td>0</td>
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<td>0</td>
<td>0.3112</td>
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<tr>
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<td>659</td>
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<td>0</td>
<td>1</td>
<td>$1.017 \times 10^{-3}$</td>
<td>$9.595 \times 10^{-4}$</td>
<td>0</td>
<td>0.3120</td>
<td>0.2659</td>
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</table>

Table 4
Errors of pixels and greyness under transformations by different combinations when $N = 8$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Method</th>
<th>$T$</th>
<th>$T^{-1}$</th>
<th>$T^{-1}T$</th>
<th>$T^{-1}$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta I_1$</td>
<td>$\Delta I_2$</td>
<td>$\Delta I_3$</td>
<td>$\Delta E$</td>
<td>$\Delta E_2$</td>
<td>$\Delta I_1$</td>
<td>$\Delta I_2$</td>
</tr>
<tr>
<td>-------</td>
<td>--------</td>
<td>-----</td>
<td>-----------</td>
<td>-----------</td>
<td>-----------</td>
<td>-----</td>
</tr>
<tr>
<td>$\mu = 1$</td>
<td>CSIM</td>
<td>22</td>
<td>46</td>
<td>26</td>
<td>0.07764</td>
<td>0.1044</td>
</tr>
<tr>
<td></td>
<td>CSIM</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0.00289</td>
<td>0.00339</td>
</tr>
<tr>
<td>$\mu = 0$</td>
<td>CSIM</td>
<td>11</td>
<td>35</td>
<td>46</td>
<td>0.09090</td>
<td>0.1189</td>
</tr>
<tr>
<td></td>
<td>CSIM</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.9160   $\times 10^{-4}$</td>
<td>0.1478 $\times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>CSIM</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.9160 $\times 10^{-4}$</td>
<td>0.1478 $\times 10^{-3}$</td>
</tr>
</tbody>
</table>
Fig. 11. The girl images of $256 \times 256$ pixels with 256 greyness levels under $T^{-1}T$ by CSIM in Case I as $\mu = 1$ and $N = 4$. 
Fig. 12. The girl images of 256 × 256 pixels with 256 greyness levels under $T^{-1}T$ by ČiM in Case 1 as $\mu = 0$ and $N = 4$. 
Fig. 13. The girl images of 256 × 256 pixels with 256 greyness levels under $T^{-1}T$ by CŠIM in Case I as $\mu = 0$ and $N = 4$ for the perspective transformation.

the left and right sides of the top in the figures, respectively. For Fig. 11 by using CŠIM the distorted image has about 124,000 nonempty pixels. Compared with the images by CŠIM as $N = 8$, the average levels of sequential errors are only 0.02, which are very small, indeed, in 256 levels counted. As to the restored image of Fig. 11, the sequential and absolute pixel errors are only 0.03 and 4.3, respectively. For Fig. 12 by using CŠIM as $\mu = 0$, the distorted image has 0.08 greyness
levels of sequential errors; and the restored image has 0.03 and 9.1 greyness levels of sequential and absolute errors, respectively.

An important application of nonlinear transformation is for the perspective transformation of images. Let the original image be wrapped along a cylinder in 3D, and then be photoed from a certain angle, thus to lead to a 2D picture. This is a perspective transformation from 2D to 2D. The restored image under such a nonlinear transformation is provided in Fig. 13; details appear elsewhere. The discrepancy between the original and restored images is insignificant. Hence, the discrete techniques in this paper are imperative to achieve high accuracy of distortion and normalization for image transformations.

It is worth comparing the above results with those in [10]. In [10] when $N = 128$ in Case I as $\mu = 1$, the sequential pixel errors are 0.27 and 0.12 for the same girl images under $T$ and $T^{-1}T$, respectively. Even if using the local partition, only when $N = 64$, their sequential pixel errors may reduce to 0.15 and 0.06 under $T$ and $T^{-1}T$, respectively (referring to Tables 14 and 17 in [10]). Therefore, the compatible division number to reach the same picture quality of the transformed images of Figs. 11–13 in this paper are $N = 64$ for the CSIM, and $N = 32$ for the CSIM using the local refinement of [10]. The ratios of total subpixels of the techniques of this paper to the subpixels of [10] are

$$\frac{64^2}{4^2} = 256, \quad \frac{32^2}{4^2} = 64$$

for the original CSIM and the CSIM using the local refinement, respectively. Since the total number of pixels of usual images are huge, e.g., from $256 \times 256$ to $1024 \times 1024$, the significant reduction of CPU time for each pixel is critical to real application. Consequently, the techniques in this paper are strongly recommended for real 256 level images.

Let us explain why the techniques in [10] will consume a great deal of CPU time. For 256 greyness level images, the initial sequential pixel errors of the original CSIM for $\mu = 1$ as $N = 1$ are as large as 160.5 and 31.59 for $T$ and $T^{-1}T$, respectively, cited from [10]. Although the convergence rates are $O_p(1/N^{1.5})$ and $O_p(1/N^2)$ as $N$ grows, proven in [10], a large division number is still needed to achieve the sequential errors to be less than 0.5 pixel errors.

**Final Remarks.** Since combinations $\text{C\text{SIM}}$ and $\text{C\text{SIM}}$ do not require solutions of nonlinear equations, they are highly recommended for images and patterns, in particular those with 256-greyness-levels [3,4,6,11–13]. Note that the piecewise constant interpolation is the simplest wavelets to image pictures (see [1]). Combination $C\text{SIM}$ as $\mu = 0$ is also important to image transformations including curve images. It is also worth pointing out that the new combinations in this paper have also been developed to 3D images under transformations. In summary, for binary images and the images with a few greyness levels, the original CSIM in [6,10] is suggested; however, for the images with greyness levels larger than 16, the advanced CSIM in this paper is recommended. Furthermore, variant numerical algorithms and their applications in images under geometrical transformations are exposed systematically in the coming book [9], where this paper and [10] form a cornerstone of mathematics.
Acknowledgements

I am grateful to the referee for his/her valuable comments and suggestions.

References