Asymptotics of Sobolev orthogonal polynomials for coherent pairs of Jacobi type

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Received 3 December 1998; received in revised form 20 February 1999

Abstract

Let \( \{S_n\} \) denote a sequence of polynomials orthogonal with respect to the Sobolev inner product
\[
(f, g)_S = \int_a^b f(x)g(x) d\psi_0(x) + \lambda \int_a^b f'(x)g'(x) d\psi_1(x)
\]
where \( \lambda > 0 \) and \( \{d\psi_0, d\psi_1\} \) is a so-called coherent pair with at least one of the measures \( d\psi_0 \) or \( d\psi_1 \) a Jacobi measure. We investigate the asymptotic behaviour of \( S_n(x) \), for \( n \to +\infty \) and \( x \) fixed, \( x \in \mathbb{C} \setminus [-1, 1] \) as well as \( x \in (-1, 1) \).

MSC: 33C45

Keywords: Jacobi polynomial; Sobolev orthogonal polynomial; Coherent pair; Asymptotic property

1. Introduction

Consider the Sobolev inner product
\[
(f, g)_S = \int_a^b f(x)g(x) d\psi_0(x) + \lambda \int_a^b f'(x)g'(x) d\psi_1(x)
\]  
where \( d\psi_0 \) and \( d\psi_1 \) are positive Borel measures with support \( (a, b) \) and \( \lambda > 0 \). Let \( \{S_n\}_n \) denote a sequence of polynomials orthogonal with respect to (1) normalized in some way. It is the aim of the

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1 Partially supported by Junta de Andalucía, Grupo de Investigación FQM 0229, DGES under grant PB 95 - 1205 and INTAS-93-0219-ext.
present paper to study the asymptotic behaviour of $S_n(x)$ with $x$ fixed and $n \to +\infty$. Therefore, we have to make some assumptions on $d\psi_0(x)$ and $d\psi_1(x)$. In [1] Iserles et al. introduced the concept of coherent pair for inner products of the form (1). This concept proved to be very fruitful, since in this situation algebraic and differential properties for the polynomials $\{S_n\}$ can be obtained (see [2,6]).

We define the notion here as follows. Let $\{P_n\}$ and $\{T_n\}$ denote orthogonal polynomial sequences with respect to the inner products defined by $d\psi_0$ and $d\psi_1$ respectively. The pair $\{d\psi_0, d\psi_1\}$ is called a coherent pair of measures if there exist nonzero constants $A_n$ and $B_n$ such that

$$T_n = A_n P_{n+1} + B_n P_n, \quad n \geq 1.$$ 

In [7] all coherent pairs of measures have been determined. More precisely, it has been proved that at least one of the two measures $d\psi_0$ or $d\psi_1$ has to be a Jacobi or a Laguerre measure (up to a linear change in the variable).

In the present paper, $\{d\psi_0, d\psi_1\}$ is a coherent pair where one of the measures is a Jacobi measure $(1-x)^\alpha(1+x)^\beta \, dx$, $\alpha > -1$, $\beta > -1$, on $(-1,1)$. For this situation Martínez-Finkelshtein et al. [5] proved the following result:

Let $\{\hat{S}_n\}$ and $\{\hat{T}_n\}$ denote the monic versions of $\{S_n\}$ and $\{T_n\}$, then

$$\lim_{n \to \infty} \frac{\hat{S}_n(x)}{\hat{T}_n(x)} = \frac{1}{\Phi(x)},$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$, where

$$\Phi(x) = \frac{x + \sqrt{x^2 - 1}}{2}$$

with $\sqrt{x^2 - 1} > 0$ when $x > 1$.

The proof is based on general asymptotic results for polynomials orthogonal with respect to a positive Borel measure supported on $[-1,1]$ and $x$ outside $[-1,1]$. This method does not give asymptotic results for $x \in (-1,1)$.

In a very recent paper [9] the outer relative asymptotic is obtained following a different approach as well as an estimate of $\{S_n\}$ with respect to the sup-norm on $[-1,1]$. Also, in [4], an extension of such results is obtained when $d\psi_1(x)$ is the Jacobi measure with parameters $(\alpha, \beta)$, $d\psi_0(x)$ is a measure in the Szegő’s class and the $\psi_0(x)$-norm of the orthonormal Jacobi polynomials with parameters $(\alpha - 1, \beta - 1)$ behaves like $o(n)$.

The present paper is based on a different approach. We use the fact that $\{P_n\}$ and $\{T_n\}$ can be expressed as simple linear combinations of Jacobi polynomials. Then we can apply the well-known asymptotic behaviour of Jacobi polynomials in order to obtain asymptotic results for $S_n(x)$. Our method gives the asymptotics for $x \in \mathbb{C} \setminus [-1,1]$ as well as for $x \in (-1,1)$. Moreover the order of the remainder terms is obtained.

In Section 2 we recall some well-known results for Jacobi polynomials, which will be used in the paper. In particular, we give the asymptotic behaviour of $P_n^{(\alpha,\beta)}(x)$ with $x$ fixed and $n \to \infty$, where we have to distinguish $x \in \mathbb{C} \setminus [-1,1]$ and $x \in (-1,1)$ (Lemmas 1 and 2).

In Section 3 we study coherent pairs $\{d\psi_0, d\psi_1\}$ where $d\psi_1$ is a Jacobi measure $d\psi_1 = (1-x)^\alpha(1+x)^\beta \, dx$ with $\alpha > -1$, $\beta > 0$ on $(-1,1)$. For $n \geq 2$, we choose the leading coefficient
of $S_n(x)$ equal to the leading coefficient of $P_{n}^{(x-1, \beta-1)}(x)$. We show that $S_n$ satisfies a simple relation (Lemma 3)
\[ P_{n-1}^{(x-1, \beta-1)}(x) = S_n(x) - a_n S_{n-1}(x), \quad n \geq 3, \]  
with $a_n = O(1/n^2)$. Then the asymptotics of $S_n(x)$ follows in a direct way (Theorem 4) for $x \in \mathbb{C} \setminus [-1, 1]$ as well as for $x \in (-1, 1)$.

In Section 4 the first measure $d\psi_0$ is a Jacobi measure. For a simple connection between the results of Sections 3 and 4 we take $d\psi_0 = (1-x)^\gamma (1+x)^{\beta-1} dx$ with $\gamma > -1$, $\beta > 0$ and we choose the same normalization for $S_n(x)$ as in Section 3. Relation (2) has to be replaced by a formula where the left-hand side is a linear combination of $P_n^{(x, \beta-1)}$ and $P_{n-1}^{(x, \beta-1)}$ (Lemma 9). Then, again the asymptotics of $S_n(x)$ follows in a direct way.

Finally we remark that the asymptotic behaviour for coherent pairs of Laguerre type and $x$ fixed outside the interval of orthogonality has been given in [3,8].

2. Results on Jacobi polynomials

The Jacobi polynomials, for arbitrary $\alpha$ and $\beta$, can be defined by the Rodrigues formula
\[ P_{n}^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{\alpha} (1+x)^{\beta} \left( \frac{d}{dx} \right)^n ((1-x)^{\alpha+n}(1+x)^{\beta+n}). \]  
In particular
\[ P_{0}^{(\alpha, \beta)}(x) \equiv 1, \]
\[ P_{1}^{(\alpha, \beta)}(x) = \frac{1}{2} (\alpha + \beta + 2)x + \frac{1}{2} (\alpha - \beta). \]
Observe that if $\alpha + \beta = -2$, then $P_{1}^{(\alpha, \beta)}(x)$ is a constant.

In this paper we use Jacobi polynomials $P_{n}^{(\alpha, \beta)}(x)$ where the parameters $\alpha$ and $\beta$ are restricted to $\alpha > -2$, $\beta > -1$. Then, for $n \geq 2$, $P_{n}^{(\alpha, \beta)}(x)$ is a polynomial of degree $n$ with positive leading coefficient
\[ k_{n}^{(\alpha, \beta)} = 2^{-n} \left( \frac{2n + \alpha + \beta}{n} \right) \approx \frac{2^{n+\alpha+\beta}}{\sqrt{n}}. \]  
For $\alpha > -1$, $\beta > -1$, we have
\[ \| P_{n}^{(\alpha, \beta)}(x) \|^2 = \int_{-1}^{1} (P_{n}^{(\alpha, \beta)}(x))^2 (1-x)^{\alpha} (1+x)^{\beta} dx \]
\[ = \frac{2^{2n+\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \approx \frac{2^{2n+\alpha+\beta}}{n}. \]  
For $\alpha > -1$ it holds
\[ P_{n}^{(\alpha, \beta)}(1) = \left( \frac{n+\alpha}{n} \right) \frac{n^2}{\Gamma(\alpha+1)} \left( 1 + O\left( \frac{1}{n} \right) \right). \]
For arbitrary $\alpha$ and $\beta$ the following relations are satisfied
\[
(2n + \alpha + \beta - 1)P_n^{(\alpha,\beta-1)}(x) = (n + \alpha + \beta - 1)P_n^{(\alpha,\beta-1)}(x) - (n + \beta - 1)P_{n-1}^{(\alpha,\beta-1)}(x),
\]
(7)
\[
\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n + \alpha + \beta + 1)P_{n-1}^{(\alpha+1,\beta+1)}(x).
\]
(8)

Our investigations on the asymptotics of Sobolev polynomials are based on the following asymptotic results for Jacobi polynomials (see Theorem 8.21.7 and Theorem 8.21.8 in [10, p. 196]).

**Lemma 1.** Let $\alpha$, $\beta$ be arbitrary real numbers. Let $x \in \mathbb{C} \setminus [-1,1]$. Put $\phi(x) = x + \sqrt{x^2 - 1}$ with $\sqrt{x^2 - 1} > 0$ when $x > 1$. Then
\[
P_n^{(\alpha,\beta)}(x) = \frac{\phi(x)^n}{\sqrt{n}} \left\{ c(\alpha,\beta,x) + O\left(\frac{1}{n}\right) \right\},
\]
where $c(\alpha,\beta,x)$ is a function of $\alpha,\beta$ and $x$ independent of $n$. The relation holds uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$.

**Lemma 2.** Let $\alpha$ and $\beta$ be arbitrary real numbers. Then for $0 < \theta < \pi$,
\[
P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2}k(\theta)\cos(N\theta + \gamma) + O(n^{-3/2}),
\]
where
\[
k(\theta) = \frac{1}{\sqrt{\pi}} \left( \sin \frac{\theta}{2} \right)^{-\alpha-1/2} \left( \cos \frac{\theta}{2} \right)^{-\beta-1/2},
\]
\[
N = n + \frac{1}{2}(\alpha + \beta + 1),
\]
\[
\gamma = -\left(\alpha + \frac{1}{2}\right)\pi/2.
\]
The relation holds uniformly on compact subsets of $(0,\pi)$.

### 3. Coherent pairs of Jacobi type I

In this section $\{d\psi_0, d\psi_1\}$ denotes a coherent pair, where $d\psi_1$ is a Jacobi measure on $(-1,1)$
\[
d\psi_1(x) = (1 - x)^\alpha(1 + x)^\beta\,dx
\]
with $\alpha > -1$, $\beta > 0$. For $d\psi_0$ there are three different situations, depending on $\alpha$:

1a) If $\alpha > 0$, then $d\psi_0(x) = (\xi - x)(1 - x)^{\alpha-1}(1 + x)^{\beta-1}\,dx$, with $\xi \geqslant 1$.

1b) If $\alpha = 0$, then $d\psi_0(x) = (1 + x)^{\beta-1}\,dx + M\delta(1)$, with $M \geqslant 0$.

1c) If $-1 < \alpha < 0$, then $d\psi_0(x) = (1 - x)^\alpha(1 + x)^{\beta-1}\,dx$.

In all cases, the support of $d\psi_0$ is $[-1,1]$.

It has been proved in [7] that all coherent pairs $\{d\psi_0, d\psi_1\}$ with $d\psi_1$ a Jacobi measure on $(-1,1)$ are of the above-mentioned form, or can be transformed to one of them by the transformation $x \to -x$. 
Let \( \{S_n\}_n \) denote the sequence of polynomials orthogonal with respect to the Sobolev inner product
\[
(f, g)_S = \int f(x)g(x) \, d\psi_0(x) + \lambda \int f'(x)g'(x) \, d\psi_1(x)
\] (9)

with \( \lambda > 0 \). We normalize \( S_n \) by the condition that for \( n \geq 2 \) the leading coefficient of \( S_n \) equals the leading coefficient \( k_n^{(\alpha-1,\beta-1)} \) of \( P_n^{(\alpha-1,\beta-1)} \), \( S_1 \) has leading coefficient 1 and \( S_0 \equiv 1 \).

**Lemma 3.** There exist positive constants \( a_n \) such that
\[
P_n^{(\alpha-1,\beta-1)}(x) = S_n(x) - a_{n-1}S_{n-1}(x), \quad n \geq 3.
\] (10)
Moreover
\[
a_n = O(1/n^2).
\]

**Proof.** For \( n \geq 3 \), we can write
\[
P_n^{(\alpha-1,\beta-1)}(x) = S_n(x) + \sum_{i=0}^{n-1} \gamma_i^{(n)}(x).
\]

Then, for \( 0 \leq i \leq n - 1 \), from (8)
\[
\gamma_i^{(n)}(S_n, S_i)_S = (P_n^{(\alpha-1,\beta-1)}, S_i)_S = \int_{-1}^{+1} P_n^{(\alpha-1,\beta-1)} S_n d\psi_0.(11)
\]

We will evaluate (11) in the three above-mentioned cases.

**Case Ia:** If \( 0 \leq i \leq n - 2 \) the last integral of (11) trivially equals zero. For \( i = n - 1 \) we evaluate this integral with (4) and (5)
\[
\int_{-1}^{+1} P_n^{(\alpha-1,\beta-1)} S_n d\psi_0 = \int_{-1}^{+1} P_n^{(\alpha-1,\beta-1)} S_{n-1}(\zeta - x)(1 - x)\beta^{-1}(1 + x)^{\beta-1} dx
\]
\[
= -\frac{k_n^{(\alpha-1,\beta-1)}}{k_n^{(\alpha-1,\beta-1)}} \int_{-1}^{+1} (P_n^{(\alpha-1,\beta-1)})^2(1 - x)^\beta(1 + x)^{\beta-1} dx \approx -\frac{2n^{\alpha+\beta-3}}{n}.
\]

**Case Ic:** We apply (7) to the right-hand side of (11). Then for \( 0 \leq i \leq n - 2 \) it vanishes. For \( i = n - 1 \) we obtain, using (7) as well as (4) and (5)
\[
\int P_n^{(\alpha-1,\beta-1)} S_{n-1} d\psi_0
\]
\[
= -\frac{n + \beta - 1}{2n + \alpha + \beta - 1} \int_{-1}^{+1} P_n^{(\alpha-1,\beta-1)} S_{n-1}(1 - x)^\beta(1 + x)^{\beta-1} dx
\]
\[
= -\frac{n + \beta - 1}{2n + \alpha + \beta - 1} k_n^{(\alpha-1,\beta-1)} \int_{-1}^{+1} (P_n^{(\alpha-1,\beta-1)})^2(1 - x)^\beta(1 + x)^{\beta-1} dx
\]
\[
\approx -\frac{2n^{\alpha+\beta-3}}{n}.
\]

**Case Ib:** Relation (6) implies \( P_n^{(0,-1)}(1) = 1 \) and then (7) gives \( P_n^{(-1,-1)}(1) = 0 \). Now we can proceed as in case Ic.
We have found that in all cases \( \gamma_i^{(n)} = 0, 0 \leq i \leq n - 2 \); moreover for \( i = n - 1 \) with \( \gamma_{n-1}^{(n)} = -a_{n-1} \)

\[
a_{n-1} (S_{n-1}, S_{n-1})_S = - \int P_n^{(\alpha-1, \beta-1)} S_{n-1} \, d\psi_0 \approx \frac{2^{x+\beta-3}}{n}.
\]

To complete the proof of the last assertion of Lemma 3 we use the minimal property of the norm of \( P_n^{(\alpha-1, \beta-1)} \)

\[
(S_{n-1}, S_{n-1})_S \geq \beta \int_{-1}^{1} (S'_{n-1})^2 (1-x)^{\alpha}(1+x)^{\beta} \, dx
\]

\[
\geq \beta (n - 1) \left( \frac{k_{n-1}^{(\alpha-1, \beta-1)}}{k_n^{(\alpha, \beta)}} \right)^2 \int_{-1}^{1} (P_n^{(\alpha, \beta)})^2 (1-x)^{\alpha}(1+x)^{\beta} \, dx
\]

\[
\approx \beta n 2^{x+\beta-2}.
\]

Then it follows

\[
a_n = O(1/n^2). \quad \square
\]

**Theorem 4.** Let \( \{d\psi_0, d\psi_1\} \) denote a coherent pair where \( d\psi_1 \) is the Jacobi measure \((1 - x)^{\alpha}(1 + x)^{\beta} \) dx on \((-1, 1)\). Let \( \{S_n\}_{n=1}^{\infty} \) denote the sequence of polynomials orthogonal with respect to (9), where, for \( n \geq 2 \), the leading coefficient of \( S_n \) is equal to the leading coefficient of \( P_n^{(\alpha-1, \beta-1)} \).

(a) If \( x \in \mathbb{C} \setminus [-1, 1] \), then

\[
S_n(x) = P_n^{(\alpha-1, \beta-1)}(x) + O(\phi(x)^n n^{-5/2}),
\]

uniformly on compact subsets of \( \mathbb{C} \setminus [-1, 1] \). Thus the relative asymptotic is

\[
\frac{S_n(x)}{P_n^{(\alpha-1, \beta-1)}(x)} = 1 + O\left( \frac{1}{n^2} \right).
\]

(b) If \( 0 < \theta < \pi \), then

\[
S_n(\cos \theta) = P_n^{(\alpha-1, \beta-1)}(\cos \theta) + O(n^{-5/2}),
\]

uniformly on compact subsets of \((0, \pi)\).

**Proof.** (a) We start from (10) and put

\[
S_n(x) = P_n^{(\alpha-1, \beta-1)}(x) + \phi(x)^n n^{-5/2} A_n(x), \quad n \geq 2.
\]

Then, for \( n \geq 3 \),

\[
A_n(x) = \frac{a_{n-1}}{\phi(x)} \left( \frac{n^{5/2}}{(n-1)^{5/2}} A_{n-1}(x) + a_{n-1} n^{1/2} \phi(x)^{-\alpha} P_n^{(\alpha-1, \beta-1)}(x) \right).
\]

Let \( K \) denote a compact subset of \( \mathbb{C} \setminus [-1, 1] \). From Lemma 1 we obtain that \( n^{1/2} \phi(x)^{-\alpha} P_n^{(\alpha-1, \beta-1)}(x) \) is uniformly bounded on \( K \). By Lemma 3 also the sequence \( a_{n-1} n^2 \) is bounded. Hence there exists a constant \( M \) such that

\[
|a_{n-1} n^{1/2} \phi(x)^{-\alpha} P_n^{(\alpha-1, \beta-1)}(x)| \leq M
\]
on \( K \). Let \( 0 < \varepsilon < 1 \), then there exists a positive integer \( N \), such that for \( n \geq N + 1 \)
\[ |A_n(x)| < \varepsilon |A_{n-1}(x)| + M, \]
for all \( x \in K \). By repeated application, for \( k \geq 1 \) and \( x \in K \)
\[ |A_{N+k}(x)| < \varepsilon^k |A_N(x)| + M(1 + \varepsilon + \cdots + \varepsilon^{k-1}). \]
This implies that \( A_n(x) \) is uniformly bounded on \( K \).

(b) We proceed as in the proof of (a) with \( \phi(x) \) replaced by 1 and applying Lemma 2 instead of Lemma 1. \( \square \)

4. Coherent pairs of Jacobi type II

In this section \( \{d\psi_0, d\psi_1\} \) is a coherent pair where \( d\psi_0 \) is a Jacobi measure on \((-1, 1)\)
\[ d\psi_0(x) = (1 - x)^\alpha (1 + x)^\beta \, dx \]
with \( \alpha > -1, \beta > 0 \). In [7] it has been proved that
\[ d\psi_1(x) = \frac{1}{\xi - x} (1 - x)^{\alpha + 1}(1 + x)^\beta \, dx + M\delta(\xi), \]
where \( \xi \geq 1, M \geq 0 \), the absolutely continuous part of the measure is defined on \((-1, 1)\), or that the transformation \( x \to -x \) reduces \( d\psi_1 \) to the form (12). In this section we assume \( d\psi_1 \) to be of the form (12).

Notice that for \( \xi = 1, M = 0 \) the coherent pair \( \{d\psi_0, d\psi_1\} \) becomes a coherent pair of type I studied in Section 3. This remark determines the choice of the normalization in the present section.

Let \( \{T_n\}_n \) denote the sequence of polynomials orthogonal with respect to \( d\psi_1 \), with leading coefficients equal to the leading coefficients \( k^{(x, \beta)}_n \) of \( P_n^{(x, \beta)} \).

In the special case \( \xi = 1, M = 0 \) we have, with (7)
\[ T_n = p_n^{(x, \beta)} = \frac{n + \alpha + \beta + 1}{2n + \alpha + \beta + 1} p_n^{(x+1, \beta)} - \frac{n + \beta}{2n + \alpha + \beta + 1} p_{n-1}^{(x+1, \beta)}. \]
For the general case we have the following lemma.

**Lemma 5.** There exist positive constants \( c_n \) such that
\[ T_n = \frac{n + \alpha + \beta + 1}{2n + \alpha + \beta + 1} p_n^{(x+1, \beta)} - \frac{n + \beta}{2n + \alpha + \beta + 1} c_n p_{n-1}^{(x+1, \beta)}, \quad n \geq 1. \]

**Proof.** Write
\[ T_n = \frac{n + \alpha + \beta + 1}{2n + \alpha + \beta + 1} p_n^{(x+1, \beta)} + \sum_{i=0}^{n-1} \delta_i^{(n)} p_i^{(x+1, \beta)}. \]
For \( 0 \leq i \leq n - 1 \),
\[ \delta_i^{(n)} ||p_i^{(x+1, \beta)}||^2 = \int_{-1}^{1} T_n p_i^{(x+1, \beta)} (1 - x)^{\alpha+1}(1 + x)^\beta \, dx = \int_{-1}^{1} T_n p_i^{(x+1, \beta)} (\xi - x) \, d\psi_1. \]
For $0 \leq i \leq n - 2$, the last integral is zero. For $i = n - 1$ we have

$$d_{n-1}^{(x)} \parallel P_{n-1}^{(x+1, \beta)} \parallel^2 \leq \frac{k_{n-1}^{(x+1, \beta)}}{k_n^{(x, \beta)}} \int T_n^2 \, d\psi_1.$$  

This proves the lemma. □

For later reference we remark that the last relation, with (4) and (5), implies

$$\int T_n^2 \, d\psi_1 \approx \frac{2^{x+\beta}}{n} c_n. \tag{14}$$

**Lemma 6.** Let

$$I_n^{(x+1, \beta)}(\xi) = \int_{-1}^{+1} P_n^{(x+1, \beta)}(x) \frac{1}{\xi - x} (1 - x)^{x+1}(1 + x)^\beta \, dx,$$

where $\xi > 1$, $x > -1$, $\beta > 0$. Then

(a) $I_n^{(x+1, \beta)}(1) = (2^{x+\beta+1} \Gamma(x + 1)/n^{x+1})(1 + O(1/n))$

(b) If $\xi > 1$, then

$$I_n^{(x+1, \beta)}(\xi) = \frac{1}{\phi(\xi)^{x+1/2}} \frac{1}{(\xi^2 - 1)^{1/4}} \frac{\sqrt{2\pi}}{\sqrt{n}} \left( 1 + O\left(\frac{1}{n}\right) \right),$$

where $\phi(\xi) = \xi + \sqrt{\xi^2 - 1} > 1$, $x_0 = 1/\phi(\xi)$.

**Proof.** From Rodrigues formula, (3) and integration by parts, we get

$$I_n^{(x+1, \beta)}(\xi) = \frac{(-1)^n}{2^n n!} \int_{-1}^{+1} \frac{1}{\xi - x} D^n((1 - x)^{x+1}(1 + x)^\beta) \, dx$$

$$= \frac{1}{2^n} \int_{-1}^{+1} \frac{1}{(\xi - x)^n}(1 - x)^{x+1}(1 + x)^\beta \, dx.$$

(a) For $\xi = 1$ we obtain

$$I_n^{(x+1, \beta)}(1) = \frac{1}{2^n} \int_{-1}^{+1} (1 - x)^{x+1}(1 + x)^\beta \, dx = 2^{x+\beta+1} \Gamma(x + 1, n + \beta + 1)$$

and the result follows.

(b) Let $\xi > 1$. We use the saddle point method. Write

$$I_n^{(x+1, \beta)}(\xi) = \frac{1}{2^n} \int_{-1}^{+1} e^{h(x)} g(x) \, dx$$

with

$$h(x) = \log(1 - x^2) - \log(\xi - x),$$

$$g(x) = \frac{1}{\xi - x} (1 - x)^{x+1}(1 + x)^\beta.$$
The saddle-point \( x_0 \) follows from \( h'(x_0) = 0 \), i.e. \( x_0 = 1/\phi(\xi) \). Some straightforward calculations give \( h''(x_0) = \frac{\phi''(\xi)}{\sqrt{\xi^2 - 1}} \). Then the saddle point method implies

\[
I_n^{(x+1, \beta)}(\xi) = \frac{1}{2\pi} g(x_0) e^{n h(x_0)} \frac{\sqrt{2\pi}}{\sqrt{|h''(x_0)|}} \frac{1}{\sqrt{n}} \left( 1 + O\left( \frac{1}{n} \right) \right),
\]

and the result follows after some calculations. \( \square \)

**Lemma 7.** Let \( \{c_n\}_n \) denote the sequence defined in Lemma 5. Then

(a) if \( M = 0, \xi = 1 \), then \( c_n = 1, n \geq 1 \),

(b) if \( M = 0, \xi > 1 \), then \( c_n = 1/\phi(\xi) + O(1/n) \),

(c) if \( M > 0, \xi = 1 \), then \( c_n = 1 + O(1/n) \),

(d) if \( M > 0, \xi > 1 \), then \( c_n = \phi(\xi) + O(1/n) \).

**Proof.** (a) This is just relation (13). For the other assertions we use \( \int T_n \, d\psi_t = 0 \) for \( n \geq 1 \). Using Lemma 5 we obtain

\[
\frac{n + \beta}{n + \alpha + \beta + 1} c_n I_n^{(x+1, \beta)}(\xi) + M P_n^{(x+1, \beta)}(\xi) = I_n^{(x+1, \beta)}(\xi) + M P_n^{(x+1, \beta)}(\xi).
\]

(b) The assertion follows directly from Lemma 6(b).

(c) By Lemma 6(a) the \( I \)-terms in (15) are decreasing and by (6) the \( M \)-terms are increasing. Then the result follows from (6).

(d) Now the \( I \)-terms in (15) are decreasing by Lemma 6(b) and the \( M \)-terms are increasing by Lemma 1. Then Lemma 1 gives the desired result. \( \square \)

**Remark 8.** With \( \phi(1) = 1 \) we can summarize Lemma 7 in

\[
c_n = \begin{cases} 1/\phi(\xi) + O(1/n) & \text{if } M = 0, \\ \phi(\xi) + O(1/n) & \text{if } M > 0. \end{cases}
\]

Let \( \{S_n\}_n \) denote a sequence of orthogonal polynomials with respect to the Sobolev inner product

\[
(f, g)_S = \int f(x) g(x) \, d\psi_t(x) + \lambda \int f'(x) g'(x) \, d\psi_t(x)
\]

with \( \lambda > 0 \). We normalize \( S_n \) by the condition that for \( n \geq 2 \) the leading coefficient of \( S_n \) equals the leading coefficient \( k_n^{(x-1, \beta-1)} \) of \( P_n^{(x-1, \beta-1)} \). \( S_1 \) has leading coefficient 1 and \( S_0 \equiv 1 \).

The present sequence \( \{S_n\}_n \) satisfies a relation similar to Lemma 3 in Section 3, compare with (7).

**Lemma 9.** There exist positive constants \( a_n \) such that

\[
\frac{n + \alpha + \beta - 1}{2n + \alpha + \beta - 1} P_n^{(x+1, \beta)} - \frac{n + \beta - 1}{2n + \alpha + \beta - 1} c_{n-1} P_n^{(x-1, \beta-1)} = S_n - a_{n-1} S_{n-1},
\]

for \( n \geq 3 \). Moreover

\[
a_n = O(1/n^2).
\]
Proof. From (8) and Lemma 5 we have

\[
\frac{d}{dx} \left\{ \frac{n + \alpha + \beta - 1}{2n + \alpha + \beta - 1} P_n^{(\alpha, \beta - 1)} - \frac{n + \beta - 1}{2n + \alpha + \beta - 1} c_{n-1} P_{n-1}^{(\alpha, \beta - 1)} \right\} = \frac{1}{2} (n + \alpha + \beta - 1) T_n(x).
\]

Write

\[
\frac{n + \alpha + \beta - 1}{2n + \alpha + \beta - 1} P_n^{(\alpha, \beta - 1)} - \frac{n + \beta - 1}{2n + \alpha + \beta - 1} c_{n-1} P_{n-1}^{(\alpha, \beta - 1)} = S_n + \sum_{i=0}^{n-1} \gamma_i^{(n)} S_i.
\]

Then, for \(0 \leq i \leq n - 2\), evaluating the inner product (16), we obtain

\[
\gamma_i^{(n)} (S_n, S_i)_S = \left( \frac{n + \alpha + \beta - 1}{2n + \alpha + \beta - 1} P_n^{(\alpha, \beta - 1)} - \frac{n + \beta - 1}{2n + \alpha + \beta - 1} c_{n-1} P_{n-1}^{(\alpha, \beta - 1)}, S_i \right)_S = 0
\]

For \(i = n - 1\) we have

\[
\gamma_{n-1}^{(n)} (S_{n-1}, S_{n-1})_S = -\frac{n + \beta - 1}{2n + \alpha + \beta - 1} c_{n-1} \int_{-1}^{1} P_{n-1}^{(\alpha, \beta - 1)} S_{n-1} (1 - x)^{(1+x)\beta - 1} dx
\]

\[
= -\frac{n + \beta - 1}{2n + \alpha + \beta - 1} c_{n-1} \frac{k_n^{(\alpha, \beta - 1)}}{k_{n-1}^{(\alpha, \beta - 1)}} \| P_{n-1}^{(\alpha, \beta - 1)} \|^2.
\]

Or, with \(\gamma_{n-1}^{(n)} = -a_{n-1}\), (4) and (5)

\[
a_{n-1} (S_{n-1}, S_{n-1})_S \approx \frac{2x + 3}{n} c_{n-1}.
\]

On the other hand the minimal property of the norm of \(T_{n-2}\) implies

\[
(S_{n-1}, S_{n-1})_S \geq \lambda \int (S_{n-1}')^2 d\psi_1 = \lambda (n - 1)^2 \frac{(k_{n-1}^{(\alpha, \beta - 1)})^2}{(k_{n-2}^{(\alpha, \beta)})^2} \int (T_{n-2})^2 d\psi_1
\]

\[
\approx \lambda \frac{n^2}{4} \int (T_{n-2})^2 d\psi_1.
\]

Then (14), (17) and (18) give

\[
a_n = O(1/n^2).
\]

Theorem 10. Let \(\{d\psi_0, d\psi_1\}\) denote a coherent pair where \(d\psi_0\) is the Jacobi measure \((1 - x)^\alpha (1 + x)^{\beta - 1} dx\) on \((-1, 1)\). Let \(\{S_n\}_n\) denote the sequence of polynomials orthogonal with respect to (16), where for \(n \geq 2\) the leading coefficient of \(S_n\) is equal to the leading coefficient of \(P_n^{(\alpha, \beta - 1)}\).

(a) If \(x \in C \setminus [-1, 1]\), then

\[
S_n(x) = \frac{n + \alpha + \beta - 1}{2n + \alpha + \beta - 1} P_n^{(\alpha, \beta - 1)}(x) - \frac{n + \beta - 1}{2n + \alpha + \beta - 1} c_{n-1} P_{n-1}^{(\alpha, \beta - 1)}(x) + O(\phi(x)^n n^{-5/2}),
\]

uniformly on compact subsets of \(C \setminus [-1, 1]\). Thus the relative asymptotic is \(1 + O(n^{-2})\).
(b) If \(0 < \theta < \pi\), then
\[
S_n(\cos \theta) = \frac{n + \alpha + \beta - 1}{2n + \alpha + \beta - 1} P_n^{(\alpha, \beta-1)}(\cos \theta) - \frac{n + \beta - 1}{2n + \alpha + \beta - 1} c_{n-1} P_{n-1}^{(\alpha, \beta-1)}(\cos \theta) + O(n^{-5/2})
\]
uniformly on compact subsets of \((0, \pi)\).

**Proof.** The proof is similar to the proof of Theorem 4 with \(P_n^{(\alpha, \beta-1)}\) replaced by
\[
\frac{n + \alpha + \beta - 1}{2n + \alpha + \beta - 1} P_n^{(\alpha, \beta-1)} - \frac{n + \beta - 1}{2n + \alpha + \beta - 1} c_{n-1} P_{n-1}^{(\alpha, \beta-1)}.
\]

**Remark 11.** From Lemma 1 and Remark 8 the simplified version of Theorem 10(a) follows: If \(x \in \mathbb{C} \setminus [-1, 1]\), then
\[
S_n(x) = \begin{cases} 
\frac{1}{2} P_n^{(\alpha, \beta-1)}(x) - \frac{1}{2} \phi(x) P_{n-1}^{(\alpha, \beta-1)}(x) + O(\phi(x)^n n^{-3/2}) & \text{if } M = 0, \\
\frac{1}{2} P_n^{(\alpha, \beta-1)}(x) - \frac{1}{2} \phi(x) P_{n-1}^{(\alpha, \beta-1)}(x) + O(\phi(x)^n n^{-3/2}) & \text{if } M > 0.
\end{cases}
\]
A similar simplification holds for \(0 < \theta < \pi\).

**References**


