On the square integrability of the q-Hermite functions

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Abstract

Overlap integrals over the full real line \(-\infty < x < \infty\) for a family of the q-Hermite functions \(H_n(\sin \kappa x |q|)e^{-x^2/2}\), \(0 < q = e^{-2\kappa^2} < 1\) are evaluated. In particular, an explicit form of the squared norms for these q-extensions of the Hermite functions (or the wave functions of the linear harmonic oscillator in quantum mechanics) is obtained. The classical Fourier–Gauss transform connects the q-Hermite functions with different values \(0 < q < 1\) and \(q > 1\) of the parameter q. An explicit expansion of the q-Hermite polynomials \(H_n(\sin \kappa x |q|)\) in terms of the Hermite polynomials \(H_n(x)\) emerges as a by-product. \(\copyright 1998\) Elsevier Science B.V. All rights reserved.

1. Introduction

The Hermite functions

\[
\psi_n(\bar{\xi}) := [\sqrt{n!}2^n]^{-1/2}H_n(\bar{\xi})e^{-\bar{\xi}^2/2},
\]

where \(H_n(\bar{\xi})\) are the classical Hermite polynomials, are of great mathematical interest as an explicit example of an orthonormal and complete system in the Hilbert space \(L^2(\mathbb{R})\) of square-integrable functions with respect to the full real line \(-\infty < \bar{\xi} < \infty\) [1]. In mathematical physics they are known to represent solutions of the linear harmonic oscillator problem, which plays a very important role in quantum mechanics. In what follows, we attempt to study in detail some particular q-generalization of the Hermite functions (1.1).

2. Overlap integrals and squared norms

Let us consider a family of q-Hermite functions

\[
\psi_n(\bar{\xi}|q) := c_n(q)H_n(\sin \kappa \bar{\xi}|q)e^{-\bar{\xi}^2/2}, \quad 0 < q = e^{-2\kappa^2} < 1,
\]

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where the normalization constant $c_n(q) = \sqrt{(q; q)_n^{-1}}$ and the $q$-shifted factorial $(q; q)_n$ is defined as $(z; q)_0 = 1$ and $(z; q)_n = \prod_{k=0}^{n-1}(1 - z q^k)$, $n = 1, 2, 3 \ldots$ (throughout this paper, we will employ the standard notations of $q$-special functions, see [2] or [3]). The continuous $q$-Hermite polynomials $H_n(x|q)$ in (2.1) are those $q$-extensions of the ordinary Hermite polynomials $H_n(x)$, which satisfy the three-term recurrence relation

$$H_{n+1}(x|q) = 2xH_n(x|q) - (1 - q^n)H_{n-1}(x|q), \quad n = 0, 1, 2, \ldots, \quad (2.2)$$

with the initial condition $H_0(x|q) = 1$ [4]. Their explicit form is given by the Fourier expansion

$$H_m(\sin \kappa \xi|q) = \sum_{n=0}^{m} (-1)^n \left[ \begin{array}{c} m \\ n \end{array} \right]_q e^{i(2n-m)\kappa \xi}, \quad (2.3)$$

where $\left[ \begin{array}{c} m \\ n \end{array} \right]_q$ is the $q$-binomial coefficient,

$$\left[ \begin{array}{c} m \\ n \end{array} \right]_q := \frac{(q; q)_m}{(q; q)_n(q; q)_{m-n}} = \left[ \begin{array}{c} m \\ m-n \end{array} \right]_q. \quad (2.4)$$

The $q$-Hermite polynomials (2.3) are solutions of the difference equation

$$e^{iks} \exp \left(-ik \frac{d}{ds} \right) + e^{-iks} \exp \left(ik \frac{d}{ds} \right) H_n(\sin \kappa s|q) = 2q^{-n/2} \cos \kappa s H_n(\sin \kappa s|q). \quad (2.5)$$

It is easy to verify that $\lim_{q \to 1} \kappa^{-n}(q; q)_n = 2^n n!$. Therefore it follows from the recurrence relation (2.2), that

$$\lim_{q \to 1} \kappa^{-n}H_n(\sin \kappa \xi|q) = H_n(\xi). \quad (2.6)$$

Thus, the normalization in (2.1) is chosen so that when the limit $q \to 1$ is taken they coincide with the wave functions $\psi_n(\xi)$ of the linear harmonic oscillator in quantum mechanics, i.e.

$$\psi_n(\xi|1) := \lim_{q \to 1} \psi_n(\xi|q) = \psi_n(\xi). \quad (2.7)$$

As it is well known, the wave functions $\psi_n(\xi)$ satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} \psi_m(\xi)\psi_n(\xi) \, d\xi = \delta_{mn}, \quad (2.8)$$

and may serve as a basis in the Hilbert space $L_2(\mathbb{R})$ of square-integrable functions with respect to $d\xi$.

We evaluate first the corresponding integral

$$I_{m,n}(q) := \int_{-\infty}^{\infty} \psi_m(\xi|q)\psi_n(\xi|q) \, d\xi = I_{n,m}(q) \quad (2.9)$$
for the $q$-Hermite functions (2.1). Since $H_n(-x|q) = (-1)^n H_n(x|q)$ by definition, the functions $\psi_m(\xi|q)$ and $\psi_m(\xi|q)$ of the opposite parities ($m - n = 2k + 1$, $k = 0, 1, 2, \ldots$) are orthogonal and nontrivial integrals in (2.9) are

$$I_{n,n+2k}(q) = [\pi(q; q)_n(q; q)_{n+2k}]^{-1/2} \int_{-\infty}^{\infty} H_m(\sin \kappa \xi|q) H_{n+2k}(\sin \kappa \xi|q) e^{-\xi^2} d\xi. \quad (2.10)$$

Using the Rogers linearization formula [5]

$$H_m(x|q)H_n(x|q) = \sum_{k=0}^{m} \left( \frac{q; q}{q; q} \right)_{n-k} \left[ \frac{m}{k} \right]_q H_{m-k}(x|q), \quad m \leq n, \quad (2.11)$$

for the $q$-Hermite polynomials (2.3), one can represent (2.10) as

$$I_{n,n+2k}(q) = \frac{(q^{n+1}; q)_{2k}^{1/2}}{\sqrt{\pi}} \sum_{l=0}^{n} (q; q)_{2l+1}^{-1} \left[ \frac{n}{l} \right]_q \int_{-\infty}^{\infty} H_{2k+2l}(\sin \kappa \xi|q) e^{-\xi^2} d\xi. \quad (2.12)$$

It remains only to substitute the explicit form of the $q$-Hermite polynomials (2.3) into (2.12) and to evaluate the integral with respect to the variable $\xi$ by the aid of the well-known integral transform

$$\int_{-\infty}^{\infty} dx e^{2xix-x^2} = \sqrt{\pi} e^{x^2}. \quad (2.13)$$

The result is

$$I_{n,n+2k}(q) = \frac{(q^{n+1}; q)_{2k}^{1/2}}{\sqrt{\pi}} \sum_{l=0}^{n} (q; q)_{2l+1}^{-1} \left[ \frac{n}{l} \right]_q \frac{q^{(k+l)^2/2}}{(q; q)_{2k+l}} \times \sum_{j=0}^{2(k+l)} (-1)^j \left[ \frac{2(k+l)}{j} \right]_q q^{j^2/2-j(k+l)}. \quad (2.14)$$

The sum over $j$ in (2.14) gives the factor $(q^{1/2-k-l}; q)_{2(k+l)}$ because of the Gauss identity

$$\sum_{k=0}^{n} \left[ \frac{n}{k} \right]_q q^{k(k-1)/2}(-z)^k. \quad (2.15)$$

In view of the formulas (see, for example, [2] or [3])

$$\left( z; q \right)_n + k = \left( z; q \right)_n (zq^n; q)_k, \quad (2.16)$$

$$\left( zq^{-n}; q \right)_n = q^{-n(n+1)/2}(-z)^n (q/z; q)_n, \quad (2.17)$$

this factor is equal to

$$(q^{1/2-k-l}; q)_{2(k+l)} = (-1)^{k+l} q^{-(k+l)^2/2} (q^{1/2}; q)_{2k+l}^2. \quad (2.18)$$

Now substituting (2.18) into (2.14), yields

$$I_{n,n+2k}(q) = \frac{(q^{n+1}; q)_{2k}^{1/2}}{\sqrt{\pi}} \sum_{l=0}^{n} \left[ \frac{n}{l} \right]_q \frac{q^{(1/2); q}^2_{2k+l}}{(q; q)_{2k+l}}. \quad (2.19)$$
one can express (2.19) through the basic hypergeometric series \( \phi_i \):

\[
I_{n,n+2k}(q) = \left( \frac{q^{1/2}}{q; q}_{2k} \right) (q^{n+1}; q)_{2k} 3 \phi_1(q^{-n}, q^{k+1/2}, q^{k+1/2}, q^{2k+1}, q, q^n).
\]  

(2.21)

A particular case of (2.21) with \( k = 0 \),

\[
I_{n,n}(q) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] \left( \frac{q^{1/2}}{q; q}_{k} \right) = 3 \phi_1(q^{-n}, q^{1/2}, q^{1/2}, q, q^n),
\]

(2.22)

represents the squared norm of the \( q \)-Hermite function \( \psi_n(\xi|q) \). As is evident from (2.22), \( I_{n,n}(q) \) is finite and positive for all the values of \( q \in (0, 1) \).

3. Orthogonalization

It is clear that the \( q \)-Hermite functions (2.1) are linearly independent, for they are expressed through the \( q \)-Hermite polynomials of different order (multiplied by the common exponential factor \( e^{-\xi^2/2} \)). Therefore, once the overlap integrals (2.19) for them are known, the system \( \{ \psi_n(\xi|q) \} \) can be orthogonalized by the formation of suitable linear combinations. Since the subsequences \( \{ \psi_{2k}(\xi|q) \} \) and \( \{ \psi_{2k+1}(\xi|q) \}, \quad k = 0, 1, 2, \ldots \), are mutually orthogonal by definition, one needs to form such combinations for the even and odd functions separately. In other words, if we define (see [6, p. 154])

\[
\tilde{\psi}_{2k}(\xi|q) = e^{-\xi^2/2}
\]

(3.1)

\[
\begin{pmatrix}
I_{0,0}(q) & I_{0,2}(q) & \cdots & I_{0,2k}(q) \\
I_{2,0}(q) & I_{2,2}(q) & \cdots & I_{2,2k}(q) \\
\vdots & \vdots & \ddots & \vdots \\
I_{2k-2,0}(q) & I_{2k-2,2}(q) & \cdots & I_{2k-2,2k}(q) \\
\psi_0(\xi|q) & \psi_2(\xi|q) & \cdots & \psi_{2k}(\xi|q) \\
\end{pmatrix}
\]

then \( \{ \tilde{\psi}_{2k}(\xi|q) \} \) is an orthogonal system, for (3.1) is orthogonal to \( \psi_0(\xi|q), \psi_2(\xi|q), \ldots, \psi_{2k-2}(\xi|q) \) and hence to \( \psi_{2n}(\xi|q) \) for all \( n < k \).
Similarly, for the odd \( q \)-Hermite functions the appropriate linear combinations are

\[
\tilde{\psi}_{2k+1}(\xi|q) = e^{-\xi^2/2} \begin{vmatrix}
I_{1,1}(q) & I_{1,3}(q) & \cdots & I_{1,2k+1}(q) \\
I_{3,1}(q) & I_{3,3}(q) & \cdots & I_{3,2k+1}(q) \\
\vdots & \vdots & \ddots & \vdots \\
I_{2k-1,1}(q) & I_{2k-1,3}(q) & \cdots & I_{2k-1,2k+1}(q) \\
\end{vmatrix} = e^{-\xi^2/2} \begin{vmatrix}
I_{1,1}(q) & I_{1,3}(q) & \cdots & I_{1,2k+1}(q) \\
I_{3,1}(q) & I_{3,3}(q) & \cdots & I_{3,2k+1}(q) \\
\vdots & \vdots & \ddots & \vdots \\
I_{2k-1,1}(q) & I_{2k-1,3}(q) & \cdots & I_{2k-1,2k+1}(q) \\
\end{vmatrix}.
\]

A system of the functions

\[
\tilde{\psi}_n(\xi|q) = c_n(q)\tilde{H}_n(\sin \kappa_\xi|q)e^{-\xi^2/2}, \quad n = 0, 1, 2, \ldots,
\]

is thus orthogonal over the full real line \(-\infty < \xi < \infty\) with respect to \(d\xi\). The polynomials \(\tilde{H}_n(x|q)\) in (3.3) are linear combinations of the \(q\)-Hermite polynomials (2.3) of the form

\[
\tilde{H}_n(x|q) = \sum_{k=0}^{n} \alpha_{n,k}(q)H_k(x|q).
\]

From the second determinants in (3.1) and (3.2) it follows that the connection coefficients \(\alpha_{n,k}(q)\) in (3.4) are equal to

\[
\alpha_{2k,2j}(q) = (-1)^{k+j+1} \left[ \frac{(q; q)_{2k}}{(q; q)_{2j}} \right]^{1/2} \begin{vmatrix}
I_{0,0}(q) & I_{0,2}(q) & \cdots & I_{0,2j-2}(q) & I_{0,2j}(q) & \cdots & I_{0,2k}(q) \\
I_{2,0}(q) & I_{2,2}(q) & \cdots & I_{2,2j-2}(q) & I_{2,2j}(q) & \cdots & I_{2,2k}(q) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
I_{2k-2,0}(q) & I_{2k-2,2}(q) & \cdots & I_{2k-2,2j-2}(q) & I_{2k-2,2j}(q) & \cdots & I_{2k-2,2k}(q) \\
\end{vmatrix},
\]

\[
\alpha_{2k+1,2j+1}(q) = (-1)^{k+j+1} \left[ \frac{(q; q)_{2k+1}}{(q; q)_{2j+1}} \right]^{1/2} \begin{vmatrix}
I_{1,1}(q) & I_{1,3}(q) & \cdots & I_{1,2j-1}(q) & I_{1,2j+1}(q) & \cdots & I_{1,2k+1}(q) \\
I_{3,1}(q) & I_{3,3}(q) & \cdots & I_{3,2j-1}(q) & I_{3,2j+1}(q) & \cdots & I_{3,2k+1}(q) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
I_{2k-1,1}(q) & I_{2k-1,3}(q) & \cdots & I_{2k-1,2j-1}(q) & I_{2k-1,2j+1}(q) & \cdots & I_{2k-1,2k+1}(q) \\
\end{vmatrix},
\]

for \(n = 2k\) and \(n = 2k + 1, \; k = 0, 1, 2, \ldots\), respectively.
4. Fourier expansion

Having established that the $q$-Hermite functions $\psi_n(\xi|q)$ are square integrable, it is natural to look for their expansion in terms of the Hermite functions $\psi_n(\xi)$ (or, in other words, the linear harmonic oscillator wave functions in quantum mechanics):

$$
\psi_n(\xi|q) = \sum_{k=0}^{\infty} C_{n,k}(q)\psi_k(\xi). \tag{4.1}
$$

To find Fourier coefficients $C_{n,k}(q)$ of $\psi_n(\xi|q)$ with respect to the system $\{\psi_k(\xi)\}$, multiply both sides of (4.1) by $\psi_m(\xi)$ and integrate them over the variable $\xi$ within infinite limits with the help of the orthogonality (2.8). This yields

$$
C_{n,m}(q) = \int_{-\infty}^{\infty} \psi_n(\xi|q)\psi_m(\xi)\,d\xi = [\pi 2^m m!(q;q)_n]^{-1/2} \int_{-\infty}^{\infty} H_n(\kappa\xi|q)H_m(\xi)e^{-\xi^2}\,d\xi. \tag{4.2}
$$

To evaluate the last integral in (4.2), substitute in the Fourier expansion (2.3) for $H_n(\sin \kappa\xi|q)$ and integrate it term by term by using the integral transform (see [6, p. 124, Eq. (23)])

$$
\int_{-\infty}^{\infty} H_n(x) e^{2ty-x^2}\,dx = \sqrt{\pi(2t)^n} e^{-t^2}\tag{4.3}
$$

for the Hermite polynomials $H_m(\xi)$. This results in

$$
C_{n,m}(q) = \frac{\kappa^m q^{n/2}}{\sqrt{2^m m!(q;q)_n}} \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q (2k-n)^m q^{k(k-n)/2}. \tag{4.4}
$$

Reversing the order of summation in (4.4) with respect to the index $k$ makes it evident that the Fourier coefficients $C_{n,m}(q)$ are real for $0<q<1$, namely

$$
C_{n,m}(q) = \frac{\cos(n+m)\pi/2}{\sqrt{2^m m!(q;q)_n}} \kappa^m q^{n/2} \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q (2k-n)^m q^{k(k-n)/2}. \tag{4.5}
$$

Note that since the both functions $\psi_n(\xi|q)$ and $\psi_n(\xi)$ (see (1.1) and (2.1), respectively) contain the same exponential factor $e^{-\xi^2/2}$, the relations (4.1) and (4.5) are equivalent to an explicit expansion

$$
H_n(\sin \kappa\xi|q) = \sum_{k=0}^{\infty} a_{nk}(q)H_k(\xi) \tag{4.6}
$$

of the $q$-Hermite polynomials in terms of ordinary Hermite polynomials. The coefficients of this expansion $a_{nk}(q)$ are real and equal to

$$
a_{nk}(q) = \frac{\kappa^k q^{n/2}}{k!} \cos(n+k)\pi/2 \sum_{l=0}^{n} (-1)^l \binom{n}{l}_q (l-n/2)^k q^{(l-n)/2}. \tag{4.7}
$$

As a consistency check, one may evaluate the sum over $k$ in the right-hand side of (4.6) by substituting in it the coefficients $a_{nk}(q)$ from (4.7) and using the generating function

$$
\sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(s) = e^{2st-t^2}. \tag{4.8}
$$
for the ordinary Hermite polynomials \( H_k(s) \). This gives indeed the explicit form (2.3) of the \( q \)-Hermite polynomials \( H_n(\sin \kappa \xi |q) \) in the left-hand side of (4.6).

5. Fourier integral transform

Since the \( q \)-Hermite functions (2.1) belong to \( L_2(\mathbb{R}) \), one may define their Fourier transforms with the same property of the square integrability. A remarkable fact is that the classical Fourier integral transform relates the \( q \)-Hermite functions with different values \( 0 < q < 1 \) and \( q > 1 \) of the parameter \( q \).

We remind the reader that to consider the values \( 1 < q < \infty \) of the parameter \( q \) it is convenient to introduce [7] the \( q^{-1} \)-Hermite polynomials

\[
h_n(x|q) := t^{-n}H_n(tx|q^{-1}).
\]  
(5.1)

They satisfy the three-term recurrence relation

\[
h_{n+1}(x|q) = 2xh_n(x|q) + (1 - q^{-n})h_{n-1}(x|q), \quad n = 0, 1, 2, \ldots,
\]  
(5.2)

with the initial condition \( h_0(x|q) = 1 \). As follows from the Fourier expansion (2.3) and the inversion formula

\[
\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q
\]  
(5.3)

for the \( q \)-binomial coefficient (2.4), the explicit form of \( h_n(x|q) \) is given by

\[
h_n(\sinh \kappa \xi |q) = \sum_{k=0}^{n} (-1)^k q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{(n-2k)\kappa \xi}.
\]  
(5.4)

The \( q \)-Hermite (2.3) and the \( q^{-1} \)-Hermite (5.4) polynomials are related to each other by the classical Fourier–Gauss transform [8]

\[
H_n(\sin \kappa \xi |q)e^{-\xi^2/2} = \frac{\varpi^{-3/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_n(\sinh \kappa \eta |q)e^{-\xi^2/4 - \eta^2/2} \, d\eta.
\]  
(5.5)

This means that the \( q^{-1} \)-Hermite functions

\[
\psi_n(\eta|q^{-1}) = q^{n(n+1)/4} C_n(q) h_n(\sinh \kappa \eta |q)e^{-\eta^2/2},
\]  
(5.6)

obtained from (2.1) by the change \( q \rightarrow q^{-1} \) of the parameter \( q \), are connected with the \( q \)-Hermite functions (2.1) by the classical Fourier transform

\[
\psi_n(\xi|q) = \frac{(\varpi q^{-1/4})^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi^2/2} \psi_n(\eta|q^{-1}) \, d\eta.
\]  
(5.7)

Their expansion in terms of the Hermite functions (1.1) has the form

\[
\psi_n(\eta|q^{-1}) = \sum_{k=0}^{\infty} C_{n,k}(q^{-1}) \psi_k(\eta),
\]  
(5.8)
where the Fourier coefficients $C_{n,k}(q^{-1})$ are equal to
\[ C_{n,k}(q^{-1}) = q^{n/4} \cos(n-k)\pi/2C_{n,k}(q) \]  
and the $C_{n,k}(q)$ are given in (4.5).

\section{6. Relationship with the coherent states}

In the study of a number of quantum-mechanical problems it turns out very useful to employ a system of coherent states. The wave functions of coherent states for the linear harmonic oscillator are expressed in terms of the Hermite functions (1.1) as
\[ \psi(\xi; z) = \langle \xi | z \rangle = \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \psi_n(\xi), \]  
where $z$ is the complex parameter. The $q$-Hermite functions (2.1) are in fact some linear combinations of $\psi(\xi; z)$ with particular values of the parameter $z$. Indeed, if one substitutes the explicit form of the Fourier coefficients (4.4) into (4.1), then the sum over the index $k$ in it can be evaluated by (6.1). Thus the required relationship is
\[ \psi_n(\xi; q) = \frac{n^n}{(q; q)_n^{1/2}} \sum_{k=0}^{n} (-1)^k \left[ \begin{array}{c} n \\ k \end{array} \right]_q \psi_n(\xi; \sqrt{2}k(k/2)). \]  

In a similar manner, from (5.8) and (5.9) it follows that the corresponding relationship for the $q^{-1}$-Hermite functions (5.6) is
\[ \psi_n(\eta; q^{-1}) = \frac{q^{n(n+1)/4}}{(q; q)_n^{1/2}} \sum_{k=0}^{n} (-1)^k q^k(n-k) \left[ \begin{array}{c} n \\ k \end{array} \right]_q \psi_n(\eta; \sqrt{2}k(n/2 - k)). \]

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\section{References}

