Abstract

Limit relations between classical discrete (Charlier, Meixner, Kravchuk, Hahn) to classical continuous (Jacobi, Laguerre, Hermite) orthogonal polynomials (to be called transverse limits) are nicely presented in the Askey tableau and can be described by relations of type \( \lim_{\omega \to 0} P_n(x(s, \omega)) = Q_n(s) \), where \( P_n(s) \) and \( Q_n(s) \) stand for the discrete and continuous family, respectively. Deeper information on these limiting processes can be obtained by expanding the discrete polynomial family as \( P_n(x(s, \omega)) = \sum_{k=0}^{\infty} \omega^k R_k(s; n) \). In this paper a method for computing the coefficients \( R_k(s; n) \) is designed the main tool being the consideration of some closely related connection and inversion problems. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The so-called classical continuous (Jacobi, Laguerre, Hermite) and discrete (Charlier, Meixner, Kravchuk, Hahn) orthogonal polynomials appear at different levels of the Askey tableau [8, 9] of hypergeometric polynomials. An interesting aspect of this tableau are the limit transitions between
some of these classical families [8, 13], which are relevant in several problems appearing in Mathematical Physics, Quantum Mechanics, Statistics or Coding Theory (see e.g. [4, 11, 12, 16, 20, 21] and the references therein).

These limit relations are of the form

\[
\lim_{\omega \to 0} P_n(x(s, \omega)) = Q_n(s) \tag{1}
\]

where \( \omega \) is the limiting parameter, \( \{P_n(s)\} \) and \( \{Q_n(s)\} \) are the classical families connected by the limit and \( x = x(s, \omega) \) is an affine mapping from \( x \) to \( s \). Of course, both polynomial families could depend on some extra parameters which are not involved in the limiting process.

From (1) the polynomial \( P_n(x(s, \omega)) \) can be always expanded as

\[
P_n(x(s, \omega)) = \sum_{k=0}^{\infty} \omega^k R_k(s; n). \tag{2}
\]

On this expansion the only information provided by the limit relation (1) is \( R_0(s; n) = Q_n(s) \). Knowledge of coefficients \( R_k(s; n) \) in the latter expression will give deeper information on the limit (1).

Recently [6], we have devised a recursive method which allows us the explicit computation of these \( R_k(s; n) \)-coefficients when both polynomials, \( P_n(x(s, \omega)) \) and \( Q_n(s) \), involved in the limit relation (1), belong simultaneously to a classical continuous or discrete family, i.e., when considering vertical limits (so-called because of their appearance in the Askey tableau). The main tool of the method developed in [6] is the consideration of the connection problem

\[
P_n(x(s, \omega)) = \sum_{m=0}^{n} C_m(n; \omega)Q_m(s) \tag{3}
\]

(see [6, Tables 1 and 2]) linking both families involved in (1), which can be solved recurrently by using the “Navima algorithm” [1, 5, 14, 15] developed by the authors.

It turns out that the above method for computing the coefficients \( R_k(s; n) \) in (2) cannot be applied in case of transverse limits, i.e., when the polynomials \( P_n(x(s, \omega)) \) and \( Q_m(s) \) belong to a discrete and a continuous family, respectively. This is so because, in this transverse situation, the corresponding connection problem (3) does not full fill the conditions for the “Navima algorithm” to be applicable. In particular, it requires that \( P_n(x(s, \omega)) \) and \( Q_m(s) \) in (3) satisfy differential or difference equations of the same type (see [1, 5] for details) and this is not the case when considering transverse limits, where \( P_n(x(s, \omega)) \) is solution of a difference equation, but \( Q_m(s) \) satisfies a differential one.

Main aim of this contribution is to present an alternative method (see Section 3) for solving this kind of connection problems (out of the scope of the “Navima algorithm”) giving rise to a constructive algorithm for computing the rest term coefficients \( R_k(s; n) \) in the transverse situation.

For the sake of completeness, let us mention that partial results on these problems have appeared already in the literature. For instance in [11, p. 45, Eq. (2.6.3)] it is noticed that the rest term in the expansion of Hahn polynomials, \( h_n^{(\alpha, \beta)}(s; N) \) in Jacobi polynomials, \( P_n^{(\alpha, \beta)}(s) \) can be written as

\[
\frac{1}{N^n} h_n^{(\alpha, \beta)} \left( \frac{N}{2}(1+s); N \right) - P_n^{(\alpha, \beta)}(s) = \mathcal{O} \left( \frac{1}{N} \right) \quad (N \to \infty)
\]
but also in a more precise form [11, p. 46, Eq. (2.6.5)] when $N \to \infty$:

$$\frac{1}{N^n} P_n^{(x, \beta)} \left( \frac{\tilde{N}}{2} (1 + s) - \frac{\beta + 1}{2}; N \right) - P_n^{(x, \beta)}(s) = O\left( \frac{1}{N^2} \right)$$

where $\tilde{N} = N + (x + \beta)/2$.

On the other hand, Sharapudinov [16] has established a uniform asymptotic form of the Kravchuk polynomial $K_n^{(p,1)}(s; N)$ as $n = O(N^{1/3}) \to \infty$, by using a discrete analogue of Leibniz’s theorem as well as some properties of Hermite polynomials. This paper [16] is concluded by indicating an application to coding theory. Moreover, the same author [17] has derived suitably transformed Meixner polynomials in terms of Laguerre polynomials. The asymptotic involves not only letting the degree of the orthogonal polynomial tend to $\infty$, but also allowing variation in the parameters of the Meixner polynomials. A very explicit uniform error bound containing all Laguerre polynomials $L_k(s)$, $0 \leq k \leq n$, was provided.

Our approach to the problem here considered gives more information, in the sense that it provides explicit expressions for the rest terms ($R_k(s; n)$ in Eq. (2)) and not only error bounds as in the references cited above. If we write the transverse limit in the form given by Eq. (1), then we can give explicit formulae for the $k$th rest term $R_k(s; n)$ in the expansion

$$P_n(x(s; \omega)) - Q_n(s) = \omega R_1(s; n) + \sum_{k>1} \omega^k R_k(s; n)$$

which can be deduced from the transverse connection problem

$$P_n(x(s; \omega)) = \sum_{m=0}^{n} C_m(n; \omega) Q_m(s)$$

collecting the $\omega^k$ terms in $C_m(n; \omega)$. In particular, the rest term of first order $R_1(s; n)$ is a polynomial in $s$ of degree less than or equal to $n - 1$. Computation of rest terms in the vertical limit cases [6] shows that the corresponding polynomial $R_1(s; n)$ is always a combination of only two polynomials $Q_r(s)$, $Q_j(s)$ where $r$ can be $n - 1$ or $n - 2$ and $j$ is $r$, $r - 1$ or $r - 2$. The present computations will show that in transverse limit problems this interesting property is still true in some cases.

The outline of the paper is as follows: in Section 2 we present the transverse limits and we fix the notation to be considered within this paper. Section 3 is devoted to the description of the method we use for the computation of the rest terms $R_k(s; n)$ in (4). As illustration of the method we give in Section 4 the explicit expression of the rest term of first order ($R_1(s; n)$ in (4)) for some specific limits. In doing so, the Mathematica symbolic language [22] has been systematically used in each computational step. Finally, some concluding remarks are pointed out.

2. Transverse limits and connection problems

The specific transverse limits (1) to be treated together with the corresponding connection problems (5) are listed below. From now on monic polynomials will be considered and notations are the ones commonly used in the literature (see e.g. [2, 8, 11, 19]).
Monic Meixner ($M_n^{(\gamma,\mu)}(x)$) → Monic Laguerre ($L_n^{(\gamma)}(x)$): Starting from the difference equation satisfied by the Meixner polynomials $M_n^{(\gamma,\mu)}(x)$, it is easily shown [11] that the change of variable $x = x(s, w) = s/w$ with $w = 1 - \mu$, gives at the limit $w \to 0$ the differential equation satisfied by the Laguerre polynomial $L_n^{(\gamma-1)}(s)$. So the following limit appears

$$
\lim_{w \to 0} w^n M_n^{(\gamma+1,1-w)} \left( \frac{s}{w} \right) = L_n^{(\gamma)}(s), \tag{6}
$$

where $M_n^{(\gamma,\mu)}(s)$ ($\gamma > 0$, $0 < \mu < 1$) denotes the monic $n$th degree Meixner polynomial and $L_n^{(\gamma)}(s)$ ($\gamma > -1$) stands for the monic Laguerre polynomial of degree $n$.

Concerning notations, $M_n^{(\gamma,\mu)}(s)$ denotes the monic Meixner polynomial of degree $n$, $L_n^{(\gamma)}(s)$ the monic Laguerre polynomial of degree $n$, and $H_n(s)$ the monic Hermite polynomial of degree $n$.

Monic Kravchuk ($k_n^{(p)}(x; N)$) → Monic Hermite ($H_n(x)$): The transverse limit is given by

$$
\lim_{w \to 0} w^n \left( \frac{1}{2p(1-p)} \right)^{n/2} k_n^{(p)} \left( \frac{p}{w^2} + \frac{s}{w} \sqrt{2p(1-p)}; \frac{1}{w^2} \right) = H_n(s), \tag{8}
$$

and the corresponding connection problem (5) becomes

$$
w^n \left( \frac{1}{2p(1-p)} \right)^{n/2} k_n^{(p)} \left( \frac{p}{w^2} + \frac{s}{w} \sqrt{2p(1-p)}; \frac{1}{w^2} \right) = \sum_{m=0}^{n} C_m(n; w) H_m(s). \tag{9}
$$

Here, $k_n^{(p)}(s; N)$ ($0 < p < 1$, $N \in \mathbb{Z}^+$) denotes the monic $n$th degree Kravchuk polynomial and $H_n(s)$ stands for the monic Hermite polynomial of degree $n$.

Monic Charlier ($c_n^{(\mu)}(x)$) → Monic Hermite ($H_n(x)$): The limit property is now

$$
\lim_{w \to 0} (-w)^n c_n^{(1/(2w^2))} \left( \frac{1}{2w^2} - \frac{s}{w} \right) = H_n(s), \tag{10}
$$

and the connection problem (5) becomes

$$
(-w)^n \left( \frac{1}{2w^2} - \frac{s}{w} \right) = \sum_{m=0}^{n} C_m(n; w) H_m(s). \tag{11}
$$

$c_n^{(\mu)}(x)$ ($\mu > 0$) being the monic Charlier polynomial of degree $n$ and $H_n(s)$ the monic $n$th degree Hermite polynomial.

Monic Hahn ($h_n^{(x,\beta)}(x; N)$) → Monic Jacobi ($P_n^{(x,\beta)}(x)$): The transverse limit is in this case

$$
\lim_{w \to 0} (2w)^n h_n^{(x,\beta)} \left( \frac{s + 1}{2w}; \frac{1}{w} \right) = P_n^{(x,\beta)}(s), \tag{12}
$$

and the connection problem (5) becomes

$$
(2w)^n h_n^{(x,\beta)} \left( \frac{s + 1}{2w}; \frac{1}{w} \right) = \sum_{m=0}^{n} C_m(n; w) P_m^{(x,\beta)}(s). \tag{13}
$$

Concerning notations, $h_n^{(x,\beta)}(s; N)$ ($x, \beta > -1$, $N \in \mathbb{Z}^+$) stands for the monic Hahn polynomial of degree $n$ and $P_n^{(x,\beta)}(s)$ denotes the monic $n$th degree Jacobi polynomial ($x, \beta > -1$).
Monic Gram ($G_n(s;N)$) $\rightarrow$ Monic Legendre ($P_n(s)$): This transverse limit is a particular case of the previous one. It reads as follows:

$$\lim_{w \to 0} w^n G_n \left( \frac{s}{w}; \frac{1}{w} \right) = P_n(s), \quad (14)$$

where $G_n(s;N)$ is the monic Gram polynomials of degree $n$ [7, p. 352], which will be introduced in Section 4, and $P_n(s) = P_n^{(0,0)}(s)$ is the monic $n$th degree Legendre polynomial.

The connection problem (5) can be written as

$$w^n G_n \left( \frac{s}{w}; \frac{1}{w} \right) = \sum_{m=0}^{n} C_m(n;w) P_m(s). \quad (15)$$

3. The method: computing rest terms via connection problems

As pointed out in the Introduction, the idea of our approach for the computation of rest terms $R_k(s;n)$ in Eq. (4) is to obtain the connection coefficients, $C_m(n;w)$, appearing in Eq. (5). Then, by expanding these coefficients in power series of the limiting parameter $w$ it is possible to collect all contributions to the power $w^k$ giving the searched rest term of order $k$.

Computation of the $C_m(n,w)$-coefficients in (5) can be done by solving four coupled connection problems.

Firstly, denoting by $A^{[0]} = 1, A^{[m]} = A(A-1) \cdots (A-m+1)$ the falling factorial polynomials [11, 23], we obtain the coefficients $D_m(n,w)$ in

$$P_n(x(s,w)) = \sum_{m=0}^{n} D_m(n,w)(x(s,w))^{[m]} \quad (16)$$

which can be computed from the hypergeometrical representation [8, 11] of the classical discrete polynomial $P_n(x(s,w))$.

Next, we use the well known expansion (see, e.g., [18, p. 20])

$$(x(s,w))^{[m]} = \sum_{k=0}^{m} \mathcal{S}_k(m)(x(s,w))^k \quad (17)$$

where $\mathcal{S}_k(m)$ are the Stirling numbers of first kind. Moreover, the coefficients $B_t(k,w)$ in

$$(x(s,w))^k = \sum_{t=0}^{k} B_t(k,w)s^t. \quad (18)$$

are also computed, $s = x(s;w)$ being the affine mapping appearing in each specific transverse limit.

Finally, we use the corresponding inversion formula for each classical continuous orthogonal polynomial, $Q_r(s)$,

$$s^t = \sum_{r=0}^{t} I_r(t)Q_r(s), \quad (19)$$

where the coefficients $I_r(t)$ are known (see, e.g., [3, 10, 23]).
From these four coupled connection problems and after some calculations, the following expression for the coefficients $C_m(n, w)$ in (5) is obtained:

$$C_m(n, w) = \sum_{l=m}^{n} I_m(l) \left( \sum_{i=1}^{m} B_i(l, w) \left( \sum_{j=i}^{n} D_j(n, w) S_j(i) \right) \right).$$

(20)

4. Results: explicit formulae for the rest term of first order

As illustration of the method we have just described, we give here explicit formulae for the rest term of order one ($R_1(s; n)$ in Eq. (4)) for the transverse limits described in Section 2. In doing so, we have computed the connection coefficients appearing in Eqs. (7), (9), (11), (13) and (15) using the expression (20). Then, we have collected the contributions to the first power of $w$ in the power series expansion of these coefficients in terms of this limiting parameter. The results obtained are as follows

- Coefficient $R_1(s; n)$ in the limit Monic Meixner $\rightarrow$ Monic Laguerre:

$$R_1(s; n) = \frac{n}{2} \left( (n + 2\gamma + 1)L_{n-1}^{(\gamma)}(s) - (-1)^n(\gamma + 1)(n - 1)! \left( \sum_{j=0}^{n-2} \frac{(-1)^j}{j!} L_j^{(\gamma)}(s) \right) \right).$$

(21)

Here $L_j^{(\gamma)}(s)$ stands for the monic Laguerre polynomial of degree $j$.

- Coefficient $R_1(s; n)$ in the limit Monic Kravchuk $\rightarrow$ Monic Hermite:

$$R_1(s; n) = \frac{\sqrt{2}(2p - 1)}{2\sqrt{p - p^2}} \binom{n}{2} \left( \frac{n - 2}{6} H_{n-3}(s) + H_{n-1}(s) \right), \quad n \geq 3,$$

(22)

Notice that, in this case, $R_1(s; n)$ is a linear combination of two monic Hermite polynomials ($H_{n-3}(s)$ and $H_{n-1}(s)$) which means that this rest term is a quasi-orthogonal polynomial of order two [2].

On the other hand, if $p = 1/2$ then $R_1(s; n) = 0$ and in the expansion $P_n(x(s, \omega)) - Q_n(s) = \omega^2 R_2(s; n) + \cdots$, we obtain

$$R_2(s; n) = \binom{n}{3} \left( \frac{n - 3}{8} H_{n-4}(s) + H_{n-2}(s) \right), \quad n \geq 2.$$

(23)

- Coefficient $R_1(s; n)$ in the limit Monic Charlier $\rightarrow$ Monic Hermite:

$$R_1(s; n) = \frac{n}{2} \left( \frac{n - 2}{6} H_{n-3}(s) + H_{n-1}(s) \right), \quad n \geq 3.$$

(24)

As in the latter example, here $R_1(s; n)$ is a linear combination of two Hermite polynomials which means that this coefficient is again a quasi-orthogonal polynomial of order two [2].

- Coefficient $R_1(s; n)$ in the limit Monic Gram $\rightarrow$ Monic Legendre: When considering the Hahn-Jacobi case, the polynomial $R_1(s; n)$ involves Jacobi polynomials of degree 0 to $n - 1$. Coefficients in this expansion are rather complicated expressions in terms of the three parameters appearing in the limit process. Because of this reason we restrict ourselves to the simpler case given by the
limit relation between the so-called Gram polynomials, which play a very important role in least
squares approximation over discrete sets [7], and classical Legendre polynomials.

Monic Gram polynomials have the hypergeometric representation

\[ G_n(s; M) = \frac{(n!)^2(-2M)_n}{(2n)!} \sum_{k=0}^{n} \frac{(-n)_k(n+1)_k(-M-s)_k}{(k!)^2(-2M)_k}, \]

where \(0 \leq n \leq 2M\) and \((A)_k = A(A+1) \cdots (A+k-1), (A)_0 = 1\) denotes the Pochhammer symbol [18, p. 149]. These polynomials are solution of the second order difference equation

\[ (M+1-s)(M+s) \Delta^2 G_n(s; M) - 2s \Delta G_n(s; M) + n(n+1)G_n(s; M) = 0, \]

they are orthogonal on \([-M;M]\] with respect to the weight \(\rho(s) = 1\), i.e., \(\sum_{s=-M}^{M} G_n(s; M)G_m(s; M) = k_{n,m} \delta_{n,m}\). So, they are related with the monic Hahn polynomials [11] \(h_n^{(s,p)}(x; N)\) by means of the following formula:

\[ G_n(s; M) = h_n^{(0,0)}(s+M; 2M+1). \] (25)

In this case the expression for the rest term \(R_1(s;n)\) is

\[ R_1(s;n) = \begin{cases} 
-\sum_{k=0}^{p-1} F_{2k+1}(n)P_{2k+1}(s), & \text{if } n = 2p+1, \\
-\sum_{k=0}^{p-1} F_{2k}(n)P_{2k}(s), & \text{if } n = 2p,
\end{cases} \] (26)

where

\[ F_{2k+1}(n) = \frac{p(\prod_{j=0}^{k+1}[2(k+j)+1])(\prod_{j=0}^{p-k-2}[2(k+j)+1])}{\prod_{j=0}^{p-1}[2(n-j)-1]}, \]

\[ F_{2k}(n) = \frac{p(\prod_{j=0}^{k}[2(k+j)+1])(\prod_{j=0}^{p-k-2}[2(k+j)+1])}{\prod_{j=0}^{p-1}[2(n-j)-1]}, \]

with the convention \(\prod_{j=0}^{p-1}1 = 1\).

5. Concluding remarks

It is already interesting to comment the differences in these expansions (21), (22), (24) and (26) and to explore possible extensions.

(i) The rest term \(R_1(s;n)\) contains all limit polynomials \(R_0(s;j), 0 \leq j \leq n-1\), in the Meixner–Laguerre and Hahn–Jacobi cases, but only two polynomials \(H_{n-1}(s)\) and \(H_{n-3}(s)\) in both situations.
Kravchuk–Hermite and Charlier–Hermite. In these two latter cases the expansions are the same up to a numerical factor.

(ii) Clearly, the full rest term in (4) can be written as

\[ P_n(x(s,\omega)) - Q_n(s) = \sum_{k=1}^{\infty} \omega^k R_k(s; n) \]

and all \( R_k(s; n) \) \((k > 1)\) could be obtained using our method as it is described in Section 3. It should be interesting to investigate the number of polynomials \( Q_i(s) \) contained in \( R_2(s; n), R_3(s; n), \ldots \), when \( R_1(s; n) \) contains only two terms.

(iii) As pointed out in the introduction, the method we have presented in this work gives explicit expressions for the first rest term coefficient \( (R_i(s; n) \) in (4)) and so previous results giving error bounds are improved.

(iv) The four coupled connection problems we have considered (including direct and inverse representations) for the computation of the rest terms can be also used fruitfully for transverse connection. Results in this direction have been already obtained and will be given elsewhere.

References