Orthogonal polynomials with respect to varying weights\textsuperscript{1}

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Abstract

In this paper we review a connection of orthogonal polynomials with respect to varying weights to weighted approximation, multipoint Padé approximation and to some questions of theoretical physics. © 1998 Elsevier Science B.V. All rights reserved.

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1. Review of Freud orthogonal polynomials

In the 1980s a lot of research in the theory of orthogonal polynomials was focused on Freud-type orthogonal polynomials. Let $W(x) = W_n(x) = e^{-c|x|^4}$, $x > 0$ be a so-called Freud weight on $\mathbb{R}$, and form the corresponding orthonormal polynomials $p_n(W; x) = \gamma_n(W)x^n + \cdots$ with respect to $W^2$:

$$\int p_n(W; x)p_m(W; x)W^2(x)\,dx = \delta_{n,m}.$$ 

The leading coefficient $\gamma_n(W)$ is the solution to the extremal problem

$$\frac{1}{\gamma_n(W)^2} = \inf_{p_n \text{ monic}} \int p_n^2 W^2,$$

(where integration is with respect to Lebesgue measure) and carries a lot of information on the system, that can be seen e.g. from the fact that in the three-term recurrence formula

$$xp_n(W; x) = A_{n+1} p_{n+1}(W; x) + A_n p_{n-1}(W; x)$$

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for the polynomials the recurrence coefficients are given by

\[ A_n = \gamma_{n-1}(W)/\gamma_n(W). \]

The extremal polynomial in (1) is the polynomial \( p_n(W; x)/\gamma_n(W) \), and the extremal property is completely equivalent to orthogonality: many questions regarding orthogonal polynomials can be answered by using solely this extremal property; see e.g. the monograph [22] where the orthogonality condition was never used.

The interest in the aforementioned orthogonal polynomials arose from a conjecture of G. Freud claiming that

\[ \lim_{n \to \infty} n^{-1/2} A_n = \frac{1}{2}. \]

Although this has eluded complete proof for some while, around 1985, after some partial results by others, A. Magnus gave a proof for even \( x \). The complete conjecture was settled in three papers [5, 12, 13] by Lubinsky, Saff, Knopfmacher, Nevai and Mhaskar (it is less known that already the first two papers have actually contained the solution to the full conjecture, see [26]).

Soon afterwards Lubinsky and Saff [14] came up with the amazingly precise formula

\[ \gamma_n(W) \pi^{1/2} 2^{-n} e^{-n/2} n^{(n+1)/2} x \to 1, \]

i.e. with an asymptotics for the leading coefficient, and not just for their ratios.

Let us sketch how such an asymptotics can be proven. We follow the original idea of [14].

2. How to prove strong asymptotics (after Lubinsky and Saff)

Substituting \( x \to n^{1/2} x \) changes the weight into \( \exp(-nc|x|^2) \), and we consider orthogonality with respect to \( w^n \), i.e. with respect to a varying weight. Thus, in the orthogonality relation, polynomials of the form \( w^n P_n \) appear, so let us say a few words on such weighted polynomials.

Let, more generally, \( w = e^{-Q(x)} \) be a continuous weight on \( \mathbb{R} \). We consider the so-called Gauss variational problem: minimize the energy integral

\[ I_w(\mu) = \int \int \log \left( \frac{1}{|z - t|} \right) d\mu(z) d\mu(t) + 2 \int Q d\mu \]

for all probability measures \( \mu \). If

\[ Q(x) - \log |x| \to \infty \quad \text{as} \quad |x| \to \infty, \]

then there is a unique minimizing measure \( \mu = \mu_w \), called the extremal, or equilibrium measure associated with \( w \). The above energy problem is actually the energy problem of classical electrostatics when we put a charge on a conductor (in this case on \( \mathbb{R} \)) and look for an equilibrium when an external field (in this case \( Q \)) is present. Thus, the energy has two components: one is the interaction of the particles themselves (the first double integral), and the second one is the interaction of the particles with the external field (the second single integral). The condition on \( Q \), namely \( Q(x) - \log |x| \to \infty \) as \( |x| \to \infty \) ensures that the field is sufficiently strong around infinity to confine
in equilibrium the particles to a bounded set; in other words, the support $\mathcal{S}_w = \text{supp}(\mu_w)$ of the equilibrium measure is a compact subset of $\mathbb{R}$. For all these see [21].

The paper [16] contains the fundamental observation that the “weighted polynomials live on $\mathcal{S}_w$”, i.e. independently of the actual choice of the polynomials $P_n$ of degree at most $n$, we have

$$\|w^n P_n\|_R = \|w^n P_n\|_{\mathcal{S}_w}. \tag{5}$$

Later [24] it has turned out that $\mathcal{S}_w$ is the smallest compact set with this property. Eq. (5) follows from an extension of the Bernstein–Walsh inequality for polynomials to the weighted case. Eq. (5) is a precise form of an idea of G. Freud: when dealing with weighted polynomials (in his case orthogonal polynomials) on an infinite interval we can restrict the polynomials to some explicitly given finite intervals provided the weight function is sufficiently small at infinity. Therefore, the whole machinery of the theory of orthogonal polynomials on a finite interval can be brought into play. The true importance of this (so-called “infinite–finite” range) idea was realized by Nevai [18] and was subsequently employed by many researchers.

After these let us return to Freud polynomials. As we have mentioned, we consider orthogonal polynomials with respect to $w^{2n}$ where $w(x) = \exp(-c_n|x|^2)$ is normalized so that $\mathcal{S}_w = [-1,1]$. Thus, in the present case (5) essentially means that we can restrict the orthogonal polynomials to $[-1,1]$ rather than working with them on $(-\infty, \infty)$. There is a technique of proving asymptotics for orthogonal polynomials on $[-1,1]$ that goes back to Szegő and Bernstein: for special weights that are of the form

$$\frac{1}{\sqrt{1-x^2} \rho(x)}, \quad \frac{\sqrt{1-x^2}}{\rho(x)} \quad \text{or} \quad \frac{\sqrt{1-x}}{\sqrt{1+x \rho(x)}},$$

where $\rho$ is a polynomial of degree $l$ that is positive in $[-1,1]$, the form of the orthogonal polynomials (or some related quantities) are explicitly know (at least for sufficiently large degree, say $l < 2n$). Now, the idea is to use this precise knowledge together with approximation by expressions of the above form to derive asymptotics. See [11, 23] where this technique has been applied in several situations.

For the leading coefficients the relevant formula is due to Bernstein [1, pp. 250–254] and reads as follows. Let $\phi(x) = 1 - x^2$, and $R_{2q}$ be a polynomial of degree $2q$, positive on $(-1,1)$ with possibly simple zeros at $\pm 1$. Then for $n \geq q$

$$\left( \inf_{P_n, \text{monic}} \int_{-1}^{1} \frac{\phi^{1/2}}{R_{2q}} P_n^2 \right)^{1/2} = \pi^{1/2} 2^{-n} \exp \left( \frac{1}{\pi} \int_{-1}^{1} \frac{\log(\phi^{1/2}/R_{2q}^{1/2})}{\phi^{1/2}} \right).$$

So the aforementioned idea for proving asymptotics for the leading coefficient takes the form: approximate the weight function $w^{2n}$ by expressions $\phi^{1/2}(x)/R_{2n}(x)$, i.e. we are looking for polynomials $R_{2n}$ with

$$\phi^{1/2}(x)/R_{2n}(x) \sim w^{2n}(x)$$
on $[-1,1]$, or what amounts the same, with the property

$$R_{2n}(x)w^{2n}(x) \sim \phi^{1/2}(x).$$

Thus, we arrive at an approximation problem with weighted expressions of the form $w^n P_n$, where the degree of $P_n$ matches the exponent in $w^n$. 

3. The approximation problem

The problem is the following: given a continuous weight \( w = e^{-Q} \) on the real line with property (4), what functions can be approximated by weighted polynomials \( w^n P_n \)?

A necessary condition for approximation is suggested by (5): if \( w^n P_n \to f \) uniformly on \( \mathbb{R} \), then \( f(x) = 0 \) for \( x \notin \mathcal{S}_w \), see [25, Theorem 4.1]. In many cases this simple necessary condition is sufficient, such case is covered by [25, Theorem 4.2]:

**Theorem 1.** If \( d \mu_n(t) = v(t) \, dt \) in \( \mathcal{S}_w \), where \( v \) is continuous and positive inside \( \mathcal{S}_w \), then every continuous \( f \) that vanishes outside \( \mathcal{S}_w \) is the uniform limit of some \( w^n P_n^* \).s.

What do these tell about the approximation problem for Freud weights? Let \( \Sigma = \mathbb{R} \) and \( w(x) = \exp(-c|x|^\alpha), \: \alpha > 0, \: c > 0 \). In this case the extremal support is

\[
\mathcal{S}_w = [-\gamma_x^{1/\alpha} c^{-1/\alpha}, \gamma_x^{1/\alpha} c^{-1/\alpha}]
\]

with

\[
\gamma_x = \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2 \Gamma\left(\frac{\alpha + 1}{2}\right)},
\]

(see [17]). The extremal measure is the Ullman distribution (a more appropriate name is the Ullman–Nevai–Dehesa distribution) given by the scaling to \( \mathcal{S}_w \) of the density function

\[
\frac{x}{\pi} \int_{|t|}^{1} \frac{u^{\alpha-1}}{\sqrt{u^2 - t^2}} \, du, \quad t \in (-1, 1),
\]

which is positive and continuous on the interior of \( \mathcal{S}_w \) except for the case \( \alpha \leq 1 \) and \( t = 0 \). If \( \alpha < 1 \), then at \( t = 0 \) the density has a power type singularity \( t^{\alpha-1} \), while for \( \alpha = 1 \) the singularity is of logarithmic type.

Now for \( \alpha > 1 \) the aforementioned Theorem 4.2 of [25] can be applied. Thus, in this case every \( f \in C(\mathbb{R}) \) that vanishes outside \( \mathcal{S}_w \) (and no other one) is the uniform limit of weighted polynomials \( e^{-cn\gamma_x^{1/\alpha} P_n(x)} \). The result is still valid for \( \alpha = 1 \) although it needs a different proof, see [15]. Finally, in that same paper it was shown that for \( 0 < \alpha < 1 \) functions that vanish outside \( \mathcal{S}_w \) and at the origin are the uniform limits of such weighted polynomials.

If we are interested in the approximation of a continuous \( f \) only on \( \mathcal{S}_w \), then one has to bring into the picture some results of Kuijlaars [6–8]. In this case \( f \) still must vanish at the endpoints of \( \mathcal{S}_w \) for all \( \alpha > 1 \), and also at the origin if \( \alpha < 1 \).

4. Asymptotics for orthogonal polynomials

The aforesaid positive result on the possibility of approximation for \( \alpha > 1 \) was proven by Lubinsky and Saff [13] and was a major step in settling Freud’s conjectures. In [25] a new proof was given, and the method was used to find a relatively simple proof for the strong asymptotic formula (2) of Lubinsky and Saff. In fact, the following more general theorem was proven in [25, Theorem 13.1].
Theorem 2. Let \( \{w_n\} \) be a sequence of weights on the interval \([-1, 1]\), \( u \) a fixed nonnegative function and consider the leading coefficients of the orthonormal polynomials associated with the weight \( w_n^{2n}u^2 \):

\[
\gamma_n(w_n u) := \inf_{P_n \text{ monic}} \int |P_n w_n u|^2.
\]

Suppose that the corresponding extremal measures \( \mu_{w_n} \) have support \([-1, 1]\), they are absolutely continuous there, and if we write \( d\mu_{w_n}(t) = v_n(t) dt \), then the functions \( v_n \) satisfy the conditions

\[
\begin{align*}
  v_n(t) &\leq A(1 - t^2)^\beta, \quad t \in (0, 1), \\
  v_n(t) &\geq \frac{1}{A}(1 - t^2)^{\beta_0}, \quad t \in (0, 1)
\end{align*}
\]

for some constants \( A, \beta > -1 \) and \( \beta_0 \). Suppose further that \( u \) is positive and continuous. Then for every \( k = 0, \pm 1, \ldots \)

\[
\gamma_{n+k}(w_n u) = (1 + o(1)) \frac{2^{n+k}}{\sqrt{\pi G[w_n] G[u]}}
\]

as \( n \to \infty \), where

\[
G[V] = \exp \left( \frac{1}{\pi} \int_{-1}^{1} \frac{\log V(x)}{\sqrt{1 - x^2}} dx \right)
\]

denotes the geometric mean of a function \( V \).

Actually, relation (10) is uniform in \( k \geq -K \) for every fixed \( K \). A similar result holds (see [25, Theorem 13.1]) for the extremal constant in an \( L^p \) problem (that is analogous to the above \( L^2 \) problem).

It is known that asymptotics on the leading coefficients of orthonormal polynomials is equivalent to \( L^2 \) asymptotics for the orthogonal polynomials themselves. With the above notations let \( p_{n,k} \) denote the \( k \)th orthonormal polynomial with respect to \( w_n^{2n}u^2 \):

\[
\int p_{n,k} p_{n,l} w_n^{2n} u^2 = \delta_{k,l}.
\]

Then (see [25, Theorem 14.1]) we have the following weak convergence.

Theorem 3. Let \( \{w_n\} \) be a sequence of weight functions on \([-1, 1]\) such that the corresponding extremal measures \( \mu_{w_n} \) have support \([-1, 1]\), they are absolutely continuous there, and if we write \( d\mu_{w_n}(t) = v_n(t) dt \), then the functions \( v_n \) satisfy conditions (8) and (9). Let furthermore \( u \) satisfy Szegő’s condition

\[
\int_{-1}^{1} \frac{\log u(t)}{\sqrt{1 - t^2}} dt > -\infty.
\]

Then for every fixed \( k = 0, \pm 1, \ldots \) and for every bounded and measurable \( f \)

\[
\lim_{n \to \infty} \int (p_{n+k} w_n u)^2 f = \int \frac{f}{\pi \phi^{1/2}},
\]

where

\[
\phi(t) = \frac{1}{2} \log \left( \frac{1 - t}{1 + t} \right) + \frac{1}{4} \log \left( \frac{1 - t^2}{1 + t^2} \right)
\]
where, as before, \( \phi(x) = 1 - x^2 \).

With
\[
I_\tau(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\log V(\xi) - \log V(x)}{\xi - x} \left( \frac{1 - x^2}{1 - \xi^2} \right)^{1/2} d\xi,
\]
where the integral is understood in principal value sense, we have (see [25, Theorem 14.2]).

**Theorem 4.** With the assumptions of the preceding theorem for any fixed integer \( k \) the difference
\[
p_{n,n+k}(x) w_n^\alpha(x) u(x) - \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1 - x^2}} \cos \left( \left( n + k + \frac{1}{2} \right) \arccos x + n I_{\tau_n}(x) + I_\tau(x) - \frac{\pi}{4} \right)
\]
tends to zero in \( L^2[-1,1] \) as \( n \to \infty \).

As for asymptotics outside the support set \([-1,1]\), with the so-called Szegő function
\[
D_\tau(z) = \exp \left( \sqrt{z^2 - 1} \frac{1}{2\pi} \int_{-1}^{1} \log V(x) \frac{dx}{x} \right)
\]
we have [25, Theorem 14.3]:

**Theorem 5.**
\[
p_{n,n+k}(z) = (1 + o(1)) \frac{1}{\sqrt{2\pi}} (z + \sqrt{z^2 - 1})^{n+k} (D_\tau(z))^{-1}
\]
uniformly on compact subsets of \( \mathbb{C} \setminus [-1,1] \), where
\[
V_n(t) = w_n^{\alpha}(t) u^2(t) \sqrt{1 - t^2}.
\]

Strictly speaking these formulae do not apply to Freud weights because the latter ones are not supported on \([-1,1]\). However, we have already mentioned that if their normalization is such that the corresponding extremal support is \([-1,1]\), then they are essentially localized to \([-1,1]\), and we can derive that, say, for fixed \( k = 0, \pm 1, \ldots \) the difference
\[
n^{1/2} p_{n+k}(W_{n}; n^{1/3} x) \exp(-n^{1/3}|x|^3)
\]
\[
- \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1 - x^2}} \cos \left( \left( k + \frac{1}{2} \right) \arccos x + n \int_{x}^{1} u_\tau(t) dt - \frac{\pi}{4} \right)
\]
tends to zero in \( L^2[-1,1] \), where \( u_\tau \) is the Ullman–Nevai–Dehesa distribution (6).

Above we have not told the complete truth regarding the role of the approximation problem in determining asymptotic behaviour of orthogonal polynomials with respect to varying weights. In fact,
to derive the aforementioned results one needs for example the following approximation theorem in conjunction with the monotonicity of \( \gamma_n(w) \) in \( w \) (see [25, Theorem 10.1]):

**Theorem 6.** Suppose that \( \{ v_n \} \) is a sequence of nonnegative functions on \( (0,1) \) satisfying the following conditions: \( \{ v_n \} \) is uniformly equicontinuous on every compact subinterval of \( (0,1) \),

\[
\int_0^1 v_n = 1,
\]

and for some constants \( A, \beta > -1 \) and \( \beta_0 \)

\[
v_n(t) \leq A(t(1 - t))^\beta, \quad t \in (0,1),
\]

\[
v_n(t) \geq \frac{1}{A}(t(1 - t))^\beta_0, \quad t \in (0,1).
\]

Set

\[
w_n(x) := \exp \left( \int \log \frac{1}{|x - t|} v_n(t) \, dt \right). \quad (12)
\]

If \( 1 \geq \gamma \geq 0 \) and \( u(x) \) is any positive continuous function on \([0,1]\), then there are polynomials \( H_n \)
of degree at most \( n \) such that for

\[
h_n(x) = w_n^\gamma(x) |H_n(x)|(x(1 - x))^{-\gamma} u(x)
\]

we have

\[
h_n(x) \geq 1, \quad x \in (0,1),
\]

\[
\lim_{n \to \infty} h_n(x) = 1
\]

uniformly on compact subsets of \( (0,1) \), and

\[
\lim_{n \to \infty} \int_0^1 \frac{\log h_n(x)}{\sqrt{x(1 - x)}} \, dx = 0.
\]

Furthermore, it is of utmost importance that the degree of \( H_n \) can be somewhat smaller than \( n \), namely we can have \( \deg(H_n) = n - i_n \), where \( i_n \to \infty \).

5. Multipoint Padé approximation

Historically, orthogonal polynomials originated from continued fractions, and a major part of the theory was dealing with continued fraction expansions of Markov functions, which are of the form

\[
f(z) = \int \frac{d\mu(x)}{x - z}, \quad (13)
\]
where $\mu$ is a positive measure with compact support $S(\mu) \subseteq \mathbb{R}$. For functions of type (13) A. Markov [Ma] proved that the continued fraction development

$$b_1 \over z - a_1 + (b_2/(z - a_2 + \cdots))$$

(14)

at infinity converges locally uniformly in $\mathbb{C}\setminus I(S(\mu))$, where $I(S(\mu))$ is the smallest interval containing $S(\mu)$. In what follows we shall assume that the support of $\mu$ lies in $[-1, 1]$. It is well known that the $n$th convergent is the $[n - 1/n]$ Padé approximant to function (13). Hence the convergents of (14) are rational interpolants with all interpolation points being identically infinity.

Gonchar and Lopez [4] considered rational interpolants with more general systems of interpolation points. For every $n \in \mathbb{R}$ we select a set

$$A_n = \{x_{n,0}, \ldots, x_{n,2n}\}$$

of $2n + 1$ interpolation points from $\mathbb{C}\setminus I(S(\mu))$, which are symmetric onto the real axis. The points need not to be distinct. We set

$$\omega_n(z) := \prod_{j=0}^{2n} (z - x_{n,j}).$$

(15)

The degree $d_n$ of $\omega_n$ is equal to the number of finite points in $A_n$.

Denote by $\mathcal{R}_n$ the set of all rational functions with complex coefficients with numerator and denominator degree at most $n$. By $r_n = r_n(f, A_n, \cdot) \in \mathcal{R}_n$ we denote the rational function that interpolates the function $f$ of type (13) in the $2n + 1$ points of the set $A_n = \{x_{n,0}, \ldots, x_{n,2n}\}$. If some of these points are identical, then the interpolation is understood in Hermite’s sense. It is easy to see that the definition of $r_n$ is equivalent to the assertion that the left-hand side of

$$\frac{f(z) - r_n(f, A_n; z)}{\omega_n(z)} = \mathcal{O}(|z|^{-2n+1}) \quad \text{as } |z| \to \infty,$$

is bounded at every finite point of $A_n$ and at infinity it has the indicated behavior. We note that interpolation at infinity has not been excluded.

It can be shown (see [4] or [22, Lemma 6.1.2]) that there exists a unique rational interpolant

$$r_n(z) = r_n(f, A_n; z) = \frac{q_n(z)}{p_n(z)} \in \mathcal{R}_n$$

d of the above type, and $p_n$ satisfies the weighted orthogonality relation

$$\int p_n(x)x^k \frac{d\mu(x)}{\omega_n(z)} = 0 \quad \text{for } k = 0, \ldots, n - 1,$$

i.e. they are orthogonal polynomials of (exact) degree $n$ with respect to the varying weights $\omega_n(x)^{-1} d\mu(x)$. Furthermore, the remainder term of the interpolant has the representation

$$(f - r_n(f, A_n; \cdot))(z) = \frac{\omega_n(z)}{p_n(z)} \int \frac{p_n^2(x) d\mu(x)}{\omega_n(x)(x - z)}$$

(16)
for all $z \in \mathbb{C}\setminus[-1,1]$. By homogeneity we can clearly assume that the $p_n$ is the $n$th orthonormal polynomial with respect to $\omega_n(x)^{-1} \, d\mu(x)$.

Suppose that $S(\mu) \subseteq [-1,1]$. In this case the results of Section 4 hold true, at least if the points of $A_n$ are not too close to $[-1,1]$. In fact, the function $\log 1/|\omega_n(z)|$ is the logarithmic potential of the measure $\nu_n$ that has mass 1 at every point of $A_n$. Thus, if we set

$$w_n(x) = \omega_n(x)^{1/(2n+2)},$$

then the equilibrium measure $\mu_{w_n}$ corresponding to $w_n$ is nothing else than the balayage (for this concept see for example [22, Appendix] or [21, Section II.4]) of $v_n/(2n+2)$ out of $\mathbb{C}\setminus[-1,1]$ onto $[-1,1]$ plus $1 - d_n/(2n+2)$ times the arcsine measure $(1/\pi \sqrt{1-x^2})$. It is easy to verify that if the points of $A_n$ stay away from $[-1,1]$, then the collection of all such measures has the property, that the corresponding densities are equicontinuous in compact subsets of $[-1,1]$ and they satisfy conditions (8) and (9) with $\beta = 1/2$. Hence, the results of Section 4 can be applied provided the density $u$ of $\nu_n$ is in the Szegő class (see (11)), and from the asymptotics there we can easily get strong asymptotics for the error away from $[-1,1]$ (see [25, Section 16]):

$$(f - r_n(f, A_n; \cdot))(z) = (1 + o(1))2\pi D_n u(z)(z + \sqrt{z^2 - 1})^{-1 - 2n + d_n} \prod_j \Phi(z) - \Phi(x_{n,j}) \prod_j 1 - \Phi(z) \Phi(x_{n,j}),$$

where $\Phi$ is the canonical conformal map of the complement $\mathbb{C}\setminus[-1,1]$ of $[-1,1]$ onto the exterior of the unit disk, and the product is taken for the zeros of $\omega_n$, i.e. for the finite points in the system $A_n$.

The results of Lopez [9, 10] give a different asymptotics that supersedes the above one in the sense that the points in $A_n$ can approach the interval $[-1,1]$ so long as the sum $\sum (|\Phi(x_{j,n})| - 1)$ tends to infinity.

### 6. Distribution of energy levels of quantum systems

The investigation of Freud-type orthogonal polynomials was also initiated by theoretical physicists independently of the orthogonal polynomial community, though each side had some influence on the other one. E. Wigner suggested the use of random matrices in modelling quantum systems, so without any claim for authority let us briefly describe this so-called random matrix model and its relation to orthogonal polynomials (see also [19, 27, Appendix]).

Let $\mathcal{H}_n$ be the set of all $n \times n$ Hermitian matrices $M = (m_{i,j})^n_{i,j=1}$, and let there be given a probability distribution on $\mathcal{H}_n$ of the form

$$p_n(M) \, dM = D_n^{-1} \exp(-n \text{Tr}\{V(M)\}) \, dM,$$

where $V(\lambda), \lambda \in \mathbb{R}$, is real-valued function that increases sufficiently fast at infinity (typically a polynomial), $\text{Tr}\{H\}$ denotes the trace of the matrix $H$,

$$dM = \prod_{k=1}^n \, dm_{kk} \prod_{k<j} \, d\Re m_{k,j} \, d\Im m_{k,j}$$

is the “Lebesgue” measure for the Hermitian matrices, and $D_n$ is just a normalizing constant so that the total integral of $p_n(M) \, dM$ is one. To have a concrete example in mind, set $V(x) = x^2$ where $x$
is an even natural number. The case \( \alpha = 2 \) is just the Gaussian case, in which the system of matrices with the given distribution is called the Gaussian Unitary Ensemble studied in detail by E. Wigner, M.L. Mehta, F.J. Dyson and many other theoretical physicists since 1960s.

Every matrix \( M \in \mathcal{H}_n \) has \( n \) real eigenvalues which carry the physical information on the system when it is in the state described by \( M \). The joint probability distribution of the eigenvalues of the random matrices \( M \in \mathcal{H}_n \) is given by the density

\[
q_n(\lambda_1, \ldots, \lambda_n) = d_n^{-1} \prod_{i 
eq j} |\lambda_i - \lambda_j| \prod_{i=1}^n e^{-nV(\lambda_i)},
\]

where \( d_n \) is again a normalizing constant. The quantity

\[
N_n(\lambda) = \frac{\# \{ \text{eigenvalues in } \lambda \}}{n}
\]

is of special importance, and it is the random variable that equals the normalized number of eigenvalues in the interval \( \lambda \). The expected value \( EN_n(\lambda) \) of \( N_n(\lambda) \) is obtained by integrating (17) for all \( \lambda_i \geq 2 \) over \((-\infty, \infty)\), and for \( \lambda_i \) over \( \lambda \). As \( n \to \infty \), the normalized expected number of eigenvalues \( EN_n(\lambda) \) has a limit \( \mu(\lambda) \), and it turns out that \( \mu \) is the equilibrium distribution for the external field \( Q = V/2 \) mentioned after (3). In statistical physics \( \mu \) is known as the integrated density of states, and its density as the density of states.

In the Gaussian case \( V(x) = 2x^2 \) the density of states is given by

\[
d(\mu) = \frac{2}{\pi} \sqrt{1 - t^2},
\]

which is the celebrated Wigner’s semi-circle law. This is a special case of general Freud weights \( V(x) = 2\gamma x|x|^\alpha \), where the normalizing constant is

\[
\gamma = \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{1}{2}\right) / 2\Gamma\left(\frac{\alpha + 1}{2}\right).
\]

For the general Freud case we have the Ullman–Nevai–Dehesa density

\[
\frac{\alpha}{\pi} \int_{|t|}^{1} \frac{u^{\alpha-1}}{\sqrt{\mu^2 - t^2}} \, du, \quad t \in [-1, 1]
\]

for the density of states.

The distribution (17) has the following form:

\[
d_n^{-1} \begin{bmatrix}
1 & 1 & \ldots & 1 \\
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1}
\end{bmatrix}^2 \prod_{i=1}^n e^{-nV(\lambda_i)},
\]
and here we can add to any row any linear combination of the other rows. In particular, if $p_j(w^n, x)$ are the orthonormal polynomials with respect to $w^n(x), w(x) = \exp(-Q(x)), Q(x) = V(x)/2$, defined as

$$\int p_j(w^n, x) p_k(w^n, x) w^{2n}(x) \, dx = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

then the preceding expression equals

$$(d_n^*)^{-1} |p_{j-1}(w^n, \hat{\lambda}_j) w^n(\hat{\lambda}_j)|_{1 \leq i, j \leq n},$$

where $d_n^*$ is another normalizing constant built from $d_n$ and the leading coefficients of the $p_j(w^n, \cdot)$. By computing the square of the determinants we obtain that the joint probability distribution of the eigenvalues (states) has the form

$$(d_n^*)^{-1} |K_n(\hat{\lambda}_i, \hat{\lambda}_j)|_{1 \leq i, j \leq n},$$

where

$$K_n(t, s) = \sum_{j=0}^{n-1} p_j(w^n, t) w^n(t) p_j(w^n, s) w^n(s)$$

is the so-called reproducing kernel for the weight $w^n$. It turns out that the normalizing constant $d_n^*$ is $n!$, which follows from the formulae

$$\int K_n(t, \tau) K_n(\tau, s) \, d\tau = K_n(t, s), \quad \int K_n^2(t, s) \, dt \, ds = n, \quad \int K_n(t, t) \, dt = n.$$

These same formulae can be used to find the joint probability distribution of $l \leq n$ eigenvalues $\hat{\lambda}_1, \ldots, \hat{\lambda}_l$, which is given by

$$\frac{1}{n(n-1) \cdots (n-l+1)} |K_n(\hat{\lambda}_j, \hat{\lambda}_k)|_{1 \leq j, k \leq l}.$$

In particular, when $l = 1$ we get

$$\text{EN}_n(A) = \int_A \frac{K_n(\hat{\lambda}_i, \hat{\lambda}_i)}{n} \, d\hat{\lambda}_i,$$

where the integrand is nothing else than the reciprocal of the $n$th (weighted) normalized Christoffel function associated with the weight $w^n$.

These formulae indicate the role of orthogonal polynomials in statistical physics, and also show why orthogonal polynomials with varying weights $w^n$ are important there.

For all these results see [19] and the references there, where other methods to treat random matrices were also outlined.

Since the expected number of eigenvalues in any fixed interval contained in the support of the extremal measure is a constant times $n$, we expect that the distance of the nearest eigenvalues is proportional to $1/n$ with large probability. The so-called universality conjecture claims that if we are looking on the distribution of eigenvalues in intervals of length of order $1/n$, then this distribution
is asymptotically independent of the original distribution as \( n \to \infty \), i.e. of \( V \). This can be translated in terms of orthogonal polynomials, as

\[
\frac{K_n(x + t/K_n(x,x), x + s/K_n(x,x))}{K_n(x,x)} \to \frac{\sin(t - s)}{t - s}
\]

for points lying in the support of the extremal measure. Thus, proving the universality conjecture amounts to the same as proving (18), which is a problem purely in the theory of orthogonal polynomials.

Of the numerous papers on the universality conjecture let us mention the paper [20] by Pastur and Shcherbina, who verified the universality conjecture with a Stieltjes transform technique for \( C^3 \) functions; and the announcement of Deift et al. [3], who verified it for the case when \( Q \) is real analytic. This latter paper also outlines a new approach to asymptotics for orthogonal polynomials with respect to varying weights that is based on a matrix version of the Riemann–Hilbert problem.

References


