Some properties of the Sobolev–Laguerre polynomials

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Received 5 November 1997; received in revised form 8 June 1998

Abstract

In this paper, we study the case \( \alpha = 0 \) of the Sobolev–Laguerre polynomials. We determine a generating function for the polynomials and an expansion formula. © 1998 Elsevier Science B.V. All rights reserved.

AMS classification: 33C49; 33C99

Keywords: Sobolev orthogonal polynomials; Laguerre polynomials; Sobolev–Laguerre polynomials; Bessel polynomials; Reproducing kernels; Generating functions; Laplace transforms

1. Introduction

The so-called Sobolev–Laguerre polynomials have been discussed by a number of different writers. The (general) Sobolev–Laguerre polynomials are orthogonal with respect to the inner product

\[
(f,g) = \int_0^\infty x^\lambda e^{-x} f(x)g(x) \, dx + \lambda \int_0^\infty x^\lambda e^{-x} f'(x)g'(x) \, dx, \quad \lambda \geq 0.
\]

The polynomials were exhaustively studied by Brenner in the case \( \alpha = 0 \) who found a (not very useful) explicit formula for the polynomials [1]. For the more recently investigated case of general \( \alpha \), the article by Marcellán et al. [4] provides an excellent and timely bibliography, as well as a number of original findings.

In this paper, we study the case \( \alpha = 0 \). The notation we use for these polynomials is \( L_n^\lambda(x) \). We determine an elegant generating function for the polynomials and some expansions involving the polynomials. These results seem to be new.

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2. Results

We will employ the notation: if $A$ is a matrix with elements $a_{i,j}$, then $A$ will denote the determinant whose elements are $a_{i,j}$, and vice versa.

In our determinants and matrices, $i$ will denote row position, $j$ column position.

We can write $L_n^2(x)$ as an $(n+1) \times (n+1)$ Gram-type determinant of moments and powers of $x$,

\[
L_n^2(x) = \begin{vmatrix}
1 & x & x^2 & \cdots & x^n \\
1 & 1 & 1 & \cdots & 1 \\
\lambda + 2 & \lambda + 3 & \cdots & \lambda + n \\
\lambda + 3 & 2\lambda + 6 & \cdots & \lambda + n \\
\vdots & \vdots & \ddots & \vdots \\
x^{n-1} & \cdots & x & 1 \\
x^n & \cdots & \vdots & 1 \\
x^n/n! & \cdots & \vdots & \cdots & \cdots & x^n/n!
\end{vmatrix},
\]

(1)

where

\[
c_{i,j} = \langle x^i, x^j \rangle = \begin{cases} (i+j)! & \text{if } i = 0 \text{ or } j = 0, \\
(i+j)! + \lambda i j (i+j-2)! & \text{otherwise.} \end{cases}
\]

(2)

(It is easily verified that (1) provides a legitimate expression for $L_n^2(x)$ by taking the inner product of the determinant with $x^i$, $i = 0, 1, \ldots, n-1$. The last column of the resulting determinant is equal to another column, so the result is 0.)

With these polynomials are associated two normalization constants, $k_n^{(i)}$, the coefficient of the highest power of $x$ in $L_n^{(i)}$, and

\[
h_n^{(i)} = \langle L_n^{(i)}, L_n^{(i)} \rangle.
\]

(3)

Taking inner products of the determinant (1) with $L_n^{(i)}$ shows

\[
h_n^{(i)} = k_n^{(i)} k_{n+1}^{(i)},
\]

(4)

a useful formula.

It is convenient to factor $(i-1)!$ out of the $i$th row, $i = 1, 2, \ldots, n+1$, and $(j-1)!$ out of the $j$th column, $j = 1, 2, \ldots, n$ of $L_n^{(i)}$ and to consider instead the polynomial

\[
V_n = \begin{vmatrix}
1 & x & x^2/2 & \cdots & x^n/n! \\
1 & 1 & 1 & \cdots & 1 \\
1 & \lambda + 2 & \lambda + 3 & \cdots & \lambda + n \\
1 & \lambda + 3 & 2\lambda + 6 & \cdots & \lambda + n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \cdots & \cdots & \cdots & 1 \\
x^{n-1} & \cdots & x & 1 \\
x^n & \cdots & \vdots & \cdots & \cdots & x^n/n! \\
x^{n-1}/n! & \cdots & \cdots & \cdots & \cdots & \cdots & x^n/n!
\end{vmatrix} = \rho_n T_n^{(i)},
\]

(5)

where

\[
d_{i,j} = \begin{cases} 1 & i = 0 \text{ or } j = 0, \\
[(i+j)! + \lambda i j (i+j-2)!]/i! j! & \text{otherwise}, \end{cases}
\]

(6)
and

\[ \rho_n = \left( \prod_{j=1}^{n} j!(j-1)! \right)^{-1}. \]  

(7)

We will investigate three \((n + 1) \times (n + 1)\) determinants, which are auxiliary polynomials in \(x\) of degree \(n\):

\[
P_n(x) = \begin{vmatrix}
1 & 1 & 1 & \ldots & 1 \\
x & \lambda + 2 & \lambda + 3 & \ldots & x \\
x^2 & \lambda + 3 & 2\lambda + 6 & \ldots & x^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x^n & 1 & \ldots & \ldots & x^n
\end{vmatrix}.
\]

(8)

\[
Q_n(x) = \begin{vmatrix}
1 & \lambda + 2 & \lambda + 3 & \lambda + 4 & \ldots & 1 \\
x & \lambda + 3 & 2\lambda + 6 & 3\lambda + 10 & \ldots & x \\
x^2 & \lambda + 4 & 3\lambda + 10 & 6\lambda + 20 & \ldots & x^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x^n & \ldots & \ldots & \ldots & \ldots & x^n
\end{vmatrix},
\]

(9)

\[
R_n(x) = \begin{vmatrix}
1 & 1 & 1 & \ldots & 1 \\
x & \lambda + 2 & \lambda + 3 & \lambda + 4 & \ldots & x \\
x^2 & \lambda + 3 & 2\lambda + 6 & 3\lambda + 10 & \ldots & x^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x^n & 1 & \ldots & \ldots & \ldots & x^n
\end{vmatrix}.
\]

(10)

Note that \(L_n(x)\) is essentially the inverse Laplace transform of \((1/p)P_n(1/p)\). This crucial fact will be used later.

We will also investigate the three following \(n \times n\) constant (independent of \(x\)) determinants:

\[
A_n = \begin{vmatrix}
1 & 1 & 1 & \ldots \\
1 & \lambda + 2 & \lambda + 3 & \ldots \\
1 & \lambda + 3 & 2\lambda + 6 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{vmatrix},
\]

(11)

\[
B_n = \begin{vmatrix}
\lambda + 2 & \lambda + 3 & \lambda + 4 & \ldots \\
\lambda + 3 & 2\lambda + 6 & 3\lambda + 10 & \ldots \\
\lambda + 4 & 3\lambda + 10 & 6\lambda + 20 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{vmatrix}.
\]

(12)
\[ C_n = \begin{bmatrix} d_{i,j} \end{bmatrix}_{i=0 \to n-1, j=1 \to n} = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ \lambda + 2 & \lambda + 3 & \lambda + 4 & \cdots \\ \lambda + 3 & 2\lambda + 6 & 3\lambda + 10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \] \quad (13)

Our proofs of results pertaining to these quantities will utilize the three \( n \times n \) elementary matrices

\[ E_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}, \quad F_n = E_n^T, \]

\[ G_n = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \]

Note that \( G_n \) differs from \( F_n \) by a single element, having 0 in the \((n-1, n)\) position, rather than \(-1\). Note \( E_n = F_n = G_n = 1 \).

**Theorem 1.** Let

\[ \lambda = 4 \sinh^2 \theta. \] \quad (15)

Then

\[ A_n = \frac{\cosh(2n - 1)\theta}{\cosh \theta}, \]

\[ B_n = \frac{\sinh(2n + 2)\theta}{\sinh 2\theta}, \]

\[ C_n = 1. \] \quad (18)

**Proof.** A simple computation shows that

\[ E_n C_n F_n = \begin{bmatrix} 1 & \lambda + 1 & 1 & 1 & \cdots \\ 0 & 1 & \lambda + 2 & \lambda + 3 & \cdots \\ 0 & 1 & \lambda + 3 & 2\lambda + 6 & \cdots \\ 0 & 1 & \lambda + 4 & 3\lambda + 10 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}^T \]

\[ (19) \]

Taking determinants gives \( C_n = C_{n-1} \) and induction immediately gives the third statement.
Next, we have
\[
E_n A_n F_n = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 1 & \cdots \\
0 & 1 & 3 & 3 & \cdots \\
0 & 1 & 6 & 6 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Taking determinants, expanding the determinant on the right by minors of the first column, and then expanding that determinant by minors of its first column give
\[
A_n = A_{n-1} + \lambda B_{n-2}. \tag{21}
\]

Applying Sylvester’s theorem (see [7, p. 251]) to \(A_n\) and utilizing the known value of \(C_n\) give
\[
A_n B_{n-2} = A_{n-1} B_{n-1} - 1. \tag{22}
\]

Solving this equation for \(B_{n-2}\) and substituting it into the previous equation yields
\[
A_n^2 = A_{n+1} A_{n-1} - \lambda. \tag{23}
\]

Induction on this equation gives the \(A_n\) result.

Finally, substituting the known value of \(A_n\) into (21) gives the \(B_n\) result.

\textbf{Theorem 2.} Let

\[D = x^2 - \lambda(1 - x).\]

Then

(i) \[P_n = \frac{(-1)^n \lambda(x - 1)}{D} + \frac{x(x - 1)^n}{D} [(x - 1)A_n + A_{n+1}],\]

(ii) \[Q_n = \frac{(-1)^n}{D} + \frac{(x - 1)^{n+1}}{\lambda D} [(\lambda + x)A_{n+2} - xA_{n+1}],\]

(iii) \[R_n = (-1)^n \left[ \frac{A_{n+1} - (1 - x)A_{n+2} - x(1 - x)^{n+1}}{D} \right].\]

\textbf{Proof.} Considering the determinants of the matrices \(E_{n+1} P_n G_{n+1}\) and \(E_{n+1} R_n G_{n+1}\) in much the same way as in the proof of Theorem 1 shows that

\[
P_n = (x - 1)P_{n-1} + \lambda x(x - 1)Q_{n-2}, \tag{24}
\]

\[
R_n = (x - 1)R_{n-1} + (-1)^n A_{n+1}. \tag{25}
\]
Applying Sylvester’s theorem to \( P_n \) gives
\[
P_nB_{n-1} = A_nxQ_{n-1} - R_{n-1}. \tag{26}
\]

We write (25) as
\[
\frac{R_n}{(x-1)^n} - \frac{R_{n-1}}{(x-1)^{n-1}} = \frac{(-1)^n \cosh(2n+1)\theta}{\cosh \theta(x-1)^n}. \tag{27}
\]
Expressing the hyperbolic functions as exponentials and iterating this expression, i.e., replacing \( n \) by \( k \) and summing from \( k = 1 \) to \( n \) (note (25) requires interpreting \( R_0 = 1 \)), give, after some numbing algebra, (iii).

We now solve (24) for \( Q_{n-2} \), replace \( n \) by \( n + 1 \), substitute the result into (26), substitute the just obtained value of \( R_n \) into (26), and finally solve for \( B_{n-1} \) from (21) and substitute that into the equation. We obtain the following first-order difference equation for \( P_n \):
\[
\frac{P_n A_{n+1}}{(x-1)^n} - \frac{P_{n+1} A_n}{(x-1)^{n+1}} = \frac{\lambda}{D} \left[ (-1)^n \frac{A_n}{(x-1)^n} + (-1)^n \frac{A_{n+1}}{(x-1)^{n+1}} \right] - \frac{\lambda x}{D}. \tag{28}
\]

It is easily verified, using the known properties of \( A_n \), that particular solutions corresponding to the first and to the second term on the right are, respectively,
\[
P_n^{(1)} = \frac{\lambda (-1)^n (x-1)}{D}, \quad P_n^{(2)} = \frac{A_{n+1} x (x-1)^n}{D}. \tag{29}
\]
A homogeneous solution is
\[
P_n^{(3)} = A_n (x-1)^n. \tag{30}
\]
We write
\[
P_n = P_n^{(1)} + P_n^{(2)} + CP_n^{(3)}. \tag{31}
\]
\( C \) is determined by setting \( n = 1 \) and using \( P_1 = x - 1 \). The result is (i). Finally, using the above value of \( P_n \) we solve for \( Q_n \) from (24). The result is then simplified by expressing \( A_{n+3} \) in terms of \( A_{n+2} \) and \( A_{n+1} \) and the result is (ii).

In what follows, we let
\[
a = \frac{-\lambda - 2 + \sqrt{\lambda^2 + 4\lambda}}{2}, \quad \rho_n = \left( \prod_{j=1}^{n} j!(j-1)! \right)^{-1}. \tag{32}
\]

**Theorem 3.** For \( |t| < \min\{1, |a|/|x-1|\} \),
\[
h(x,t) = \sum_{n=0}^{\infty} P_n(x)(-t)^n, \tag{33}
\]
where
\[
h(x,t) = \frac{[(1-t^2) + \lambda t(x-1) + tx(1-t)]}{(1-t)U}, \tag{34}
\]
and
\[ U = (t(x - 1) + 1)^2 + \lambda t(x - 1). \] (35)

**Proof.** Easy algebra gives
\[ \sum_{n=0}^{\infty} A_n (-t)^n(x-1)^n = \frac{1 + t(x-1)(\lambda + 1)}{U}, \]
\[ \sum_{n=0}^{\infty} A_{n+1} (-t)^n(x-1)^n = \frac{1 + t(x-1)}{U}. \] (36)

(Note \( A_0 = 1 \).) Multiplying Theorem 2(i) by \((-t)^n\) and summing gives the result. (It requires a rather formidable amount of algebra to show the factor \( D \) cancels out of the right-hand side.) The region of convergence of the series is easily determined by examining the location of the poles of the left-hand side. These occur at \( t = 1, \ t = a/(x - 1), \ t = 1/[a(x - 1)] \). Since \( |a| < 1 < |1/a| \), the region of convergence is as indicated.

**Theorem 4.** For \( |t| < |a| \),
\[ \frac{a e^{x(t+\lambda)} - e^{x(t+1/a)}}{(a - 1)(1 - t)} = \sum_{n=0}^{\infty} p_n L_n^{(2)}(x)(-t)^n. \] (37)

**Proof.** We observe that
\[ \frac{1}{p} P_n \left( \frac{1}{p} \right) = p_n \mathcal{L} \{ L_n^{(2)}(x) \}, \] (38)
where \( \mathcal{L} \) is the Laplace transform operator,
\[ \mathcal{L} \{ f(x) \} = \int_{0}^{\infty} e^{-px} f(x) \, dx. \] (39)

Replacing \( x \) by \( 1/p \) in (33) gives
\[ \sum_{n=0}^{\infty} \frac{1}{p} P_n \left( \frac{1}{p} \right) (-t)^n = \frac{p(1-t)^2 + \lambda t (1-p) + t(1-t)}{(1-t)(p^2 + (2 + \lambda)t p (1-p) + t^2(1-p)^2)}. \] (40)

The right-hand side as a function of \( p \) has two poles which are, generally, simple.
\[ p^\pm = t \left( \frac{2t - \lambda - 2 \pm \sqrt{4\lambda + \lambda^2}}{2(1-t)^2 - t\lambda} \right). \] (41)

Multiplying by the factor \( e^{p\lambda} \) and inverting the Laplace transform by integrating along the usual complex contour gives the result. By an examination of the poles of the denominator as a function of \( t \), we see that the series (40) converges for all \( |t| < |a| \) \((p/(p-1)) \) and thus the complex integration along any \( p \)-contour of the type \( c - i\infty \) to \( c+i\infty, \ c > 1 \), is justified. The result is an analytic function of \( t \). Since \( |a| < 1 < |1/a| \) the series (37) converges for \(|t| < |a| \).
Note when $\lambda = 0$, (37) yields the standard generating function for the Laguerre polynomials, [3, vol. 2, p. 189, formula (21)]. To see this, observe that

\[ (-1)^n \rho_n L_n^{(\lambda)}(x) \big|_{x=0} = C_n = 1. \]  

(42)

The Laguerre polynomials are normalized to have the value 1 at $x = 0$. Thus,

\[ (-1)^n \rho_n L_n^{(0)}(x) = L_n(x). \]  

(43)

An interesting inequality for $L_n^{(\lambda)}$ follows from the generating function (37). The singularities of the function on the left are at $t = 1$, $t = -a$, $t = -1/a$. Expressed in terms of $\theta$ these are $t = 1$, $t = e^{-2\theta}$, $t = e^{2\theta}$. The smallest of these is $e^{-2\theta}$. Applying Cauchy’s inequality shows that for any $\varepsilon$, $0 < \varepsilon < e^{-2\theta}$

\[ |\rho_n L_n^{(\lambda)}| \leq \frac{M_\varepsilon}{(e^{-2\theta} - \varepsilon)^n}, \]  

(44)

where $M_\varepsilon$ is some constant independent of $x$.

**Theorem 5.**

\[ e^{-cx} = \sum_{n=0}^{\infty} \pi_n L_n^{(\lambda)}(x), \]  

(45)

where

\[ \pi_n = \rho_n (1 - a) \left\{ \left( \frac{c}{c + 1} \right)^n \frac{a^{n-1}}{(a^{2n-1} - 1)} + \left( \frac{c}{c + 1} \right)^{n+1} \frac{a^n}{(a^{2n+1} - 1)} \right\}, \]  

(46)

and

\[ a = \frac{-\lambda - 2 + \sqrt{\lambda^2 + 4\lambda}}{2}, \quad \rho_n = \left( \prod_{j=1}^{n} j!(j - 1)! \right)^{-1}. \]  

(47)

The series converges uniformly on compact subsets of the complex $x$-plane for all $\lambda > 0$, $c > 0$, as can be seen from the estimate (44).

**Proof.** Denote the generating on the left of (37) by $g(x,t)$. We have

\[ \langle e^{-cx}, g(x,t) \rangle = \sum_{n=0}^{\infty} \rho_n \langle e^{-cx}, L_n^{(\lambda)}(x) \rangle (-t)^n \]

\[ = -\frac{a(2c + a + 1)}{(a - 1)(ct + ac + a)} + \frac{(ac + a + c)}{(a - 1)(cat + c + 1)}. \]  

(48)
Selecting the coefficient of \(t^n\) from the last expression above gives the value of \(\langle e^{-cx}, L_n^{(j)}(x) \rangle\). We now need the value of \(h_n^{(j)}\). However, this is easy to obtain. We have

\[
\langle g(x,t), g(x,u) \rangle = \sum_{n=0}^{\infty} \rho_n^2 L_n^{(j)}(x), L_n^{(j)}(x)) (ut)^n = \sum_{n=0}^{\infty} \rho_n^2 h_n^{(j)}(ut)^n
\]

\[
= \frac{a}{(a-1)^2(1-ut/a^2)} + \frac{a^2 + 1}{(a-1)^2(1-ut)} - \frac{a}{(a-1)^2(1-a^2 ut)}.
\]

The last expression above is obtained by the evaluation of a Laplace transform integral and partial fractions. Selecting the coefficient of \((ut)^n\) from this expression gives \(h_n^{(j)}\),

\[
h_n^{(j)} = \frac{-(a^{2n-1} - 1)(a^{2n+1} - 1)}{a^{2n-1}(a-1)^2 \rho_n^2}.
\]

We then divide \(\langle e^{-cx}, L_n^{(j)}(x) \rangle\) by this expression and do a little algebra to get the coefficients given in formula (46).

Many other expansions of special functions in the polynomials \(L_n^{(j)}\) can be obtained by similarly manipulating the generating function (37), including expansions for the confluent hypergeometric functions \(\Phi\) and \(\Psi\) analogous to those expansions in Laguerre polynomials given in [3, vol. 2, p. 215].

Marcellán et al. have shown that the Laguerre–Sobolev polynomials for general \(z\) satisfy a third-order (four-term) recurrence relation. One starts with the equation [4] (3.6), which relates the general Sobolev–Laguerre polynomials \(Q_n^{(z)}(x)\) (orthogonal with respect to the inner product \((\ast)\)) and the monic Laguerre polynomials \(\tilde{L}_n^{(z)}(x)\). In the notation of [4],

\[
\tilde{L}_n^{(z)}(x) + n\tilde{L}_{n+1}^{(z)}(x) = Q_{n}^{(z)}(x) + d_{n-1}(\lambda)Q_{n-1}^{(z)}(x).
\]

\(d_n(\lambda)\) is a ratio of two Pollaczek polynomials. Now replace \(n\) by \(n-1\) then by \(n-2\) and each time use the recurrence for \(\tilde{L}_n^{(z)}(x)\) to express \(\tilde{L}_{n-2}^{(z)}(x)\) in terms of \(\tilde{L}_n^{(z)}(x)\) and \(\tilde{L}_{n-1}^{(z)}(x)\). Eliminating the latter two quantities from the set of three equations yields the four-term recursion relation given in [4, p. 258]:

\[
Q_n^{(z)}(x) + \left[2n + x - 2 - x + \frac{q_{n-3}(n + x - 1)}{q_{n-1}}\right]Q_{n-1}^{(z)}(x)
+ (n - 1)(n + x - 2) \left[1 + \frac{q_{n-3}(2n + x - 2 - x)}{q_{n-2}}\right]Q_{n-2}^{(z)}(x)
+ \frac{q_{n-3}(n - 1)(n - 2)}{q_{n-3}}(n + x - 2)(n + x - 3)Q_{n-3}^{(z)}(x) = 0.
\]

Using the recurrence in Proposition 3.4 of [4], one finds that \(q_n(\lambda)\) may be expressed a linear combination of Pollaczek polynomials
\begin{align*}
q_n(\lambda) &= (x + 1) \left[p_n^{(1-\alpha/2)} \left(\frac{x}{2} + 1; -\alpha/2, x/2, x\right) - \frac{1}{(x + 1)} p_{n-1}^{(1-\alpha/2)} \left(\frac{x}{2} + 1; -\alpha/2, x/2, x + 1\right)\right], \\
&= (x + 1) \left[p_n^{(0)}(1) \left(\frac{x}{2} + 1; -\alpha/2, x/2, x\right) - \frac{1}{(x + 1)} p_{n-1}^{(0)}(1) \left(\frac{x}{2} + 1; -\alpha/2, x/2, x + 1\right)\right].
\end{align*}
see also [2, p. 185; 3, vol. 2, p. 220]. There are at least two values of $\lambda$ for which the above expression simplifies. When $\lambda = 0$ the explicit value of the Pollaczek polynomials above can be found from the formulas [5, (12); 6, (4.4)]. The former provides a formula

$$3F_2 \left( \begin{array}{c} -n, a, \delta \\ a + 1, \delta + 1 \end{array} ; 1 \right) = \frac{n!a^n}{(\delta - a)} \left[ \frac{1}{(a)_{n+1}^\delta} - \frac{1}{(\delta)_{n+1}^a} \right],$$

(54)

that can be used in the explicit formula for the Pollaczek polynomial given in [6, (4.4)]. We find that for the special case $b = -a$,

$$P_n^{(\lambda)}(1; a, -a; c) = \frac{c}{(2\lambda - 1)} \left[ \frac{(c + 2\lambda - 1)_{n+1}}{(c)_{n+1}} - 1 \right].$$

(55)

Using this formula in (53) shows

$$q_n(0) = (x + 1)_n,$$

(56)

and for this case (52) is easily found to be an iteration of the recurrence of the classical monic Laguerre polynomial.

A simplification of $q_n(\lambda)$ also occurs in the special case $x = 0$, since

$$P_n^{(1)} \left( \frac{\lambda}{2} + 1; 0, 0; c \right) = U_n \left( \frac{\lambda}{2} + 1 \right),$$

(57)

where $U_n$ denotes the Chebyshev polynomial of the second kind. As is well known, these polynomials are elementary functions,

$$U_n \left( \frac{\lambda}{2} + 1 \right) = (-1)^n \frac{a^{n+1} - a^{-n-1}}{a - a^{-1}}, \quad a = \frac{-\lambda - 2 + \sqrt{\lambda^2 + 4\lambda}}{2}.$$

(58)

We find that when $x = 0$,

$$q_n(\lambda) = n!(-1)^n \frac{a^{n+1} - a^{-n}}{a - 1}, \quad a = \frac{-\lambda - 2 + \sqrt{\lambda^2 + 4\lambda}}{2}.$$

(59)

Eq. (15) shows

$$a = -e^{-\theta},$$

(60)

so

$$q_n(\lambda) = n!A_{n+1},$$

(61)

$$A_n = \frac{\cosh(2n - 1)\theta}{\cosh \theta} = \frac{(\sqrt{1 + \lambda/4} + \sqrt{\lambda/2})^{2n-1} + (\sqrt{1 + \lambda/4} - \sqrt{\lambda/2})^{2n-1}}{2\sqrt{1 + \lambda/4}}.$$


The resulting four-term recurrence relation for the monic ordinary Laguerre–Sobolev polynomials is quite simple:

\[
Q_n^{(0)}(x) + \left(2n - 2 - x + \frac{nA_n - 1}{A_n}\right)Q_{n-1}^{(0)}(x)
\]
\[
+ (n - 1) \left((n - 2) + \frac{(2n - 2 - x)A_{n-2}}{A_{n-1}}\right)Q_{n-2}^{(0)}(x)
\]
\[
+ \frac{(n - 1)(n - 2)^2A_{n-3}}{A_{n-2}}Q_{n-3}^{(0)}(x) = 0.
\]

A referee has pointed out that this recurrence can be obtained directly from the generating function (37) without recourse to Pollaczek polynomials. Rewriting that equation, we have

\[
(1 - t) \sum_{n=0}^{\infty} \rho_n L_n^{(i)}(x) (-t)^n
\]
\[
= a e^{t/(t+a)} \left(\frac{e^{t/(t+1/a)}}{(a - 1)} - \frac{e^{t/(t+1/a)}}{(a - 1)}\right)
\]
\[
= \frac{a}{a - 1} \left(1 + \frac{t}{a}\right) \sum_{n=0}^{\infty} L_n(x) \left(-\frac{t}{a}\right)^n - \frac{1}{a - 1}(1 + at) \sum_{n=0}^{\infty} L_n(x)(-at)^n.
\]

Comparing coefficients of \((-t)^n\) gives

\[
\rho_nL_n^{(i)}(x) + \rho_{n-1}L_{n-1}^{(i)}(x) = \frac{1}{a - 1} \left(\frac{1}{a^{n-1}} - a^n\right) (L_n(x) - L_{n-1}(x)).
\]

Now,

\[
\frac{1}{a - 1} \left(\frac{1}{a^{n-1}} - a^n\right) = (-1)^n A_n, \quad \rho_nL_n^{(i)}(x) = \frac{A_n}{n!} Q_n^{(0)}(x).
\]

When (65) is used and the Laguerre polynomials are eliminated from the relation (64), one recovers the recurrence (62).

Acknowledgements

The authors are greatly indebted to the referees, whose painstaking and attentive readings allowed us to correct many misprints. In addition, they showed us how to correct one false result and how to substantially simplify the recurrence (62).

References


