A structure theorem for reproducing kernel Pontryagin spaces

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Received 29 January 1998; received in revised form 1 July 1998

Abstract

We illustrate a relationship between reproducing kernel spaces and orthogonal polynomials via a general structure theorem. The Christoffel–Darboux formula emerges as a limit case. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Reproducing kernel spaces; Structured kernels

1. Introduction

Finite-dimensional reproducing kernel spaces and orthogonal polynomials are two closely related topics. In this paper we review some of the links between these two subjects by proving a general structure theorem. For further relationships, in particular with the Schur algorithm and with the theory of linear systems, we refer to [1]. Recall [12] that a vector space $P$ endowed with an (in general indefinite) inner product $h\cdot,i$ is called a Krein space if one can write $P = P_+ + P_-$ where

1. The space $P_+$ endowed with the inner product $h\cdot,i$ is a Hilbert space.
2. The space $P_-$ endowed with the inner product $-h\cdot,i$ is a Hilbert space.
3. For each $p_+ \in P_+$ and $p_- \in P_-$, it holds that
   
   $h p_+, p_- i = 0$.

4. Each element of $P$ admits a unique decomposition $p = p_+ + p_-$ with $p_+ \in P_+$ and $p_- \in P_-$. The space is called a Pontryagin space if $\dim P_- < \infty$. A Pontryagin space $\mathcal{P}$ whose elements are functions defined on a set $\Omega$ and with values in $\mathbb{C}^p$ is called a reproducing kernel Pontryagin space if there exists a $\mathbb{C}^{p \times p}$-valued function $K(z, \omega)$ with the following two properties:
1. For every choice of $\omega \in \Omega$ and $c \in \mathbb{C}^p$, the function
   
   $K_{\omega}c : z \mapsto K(z, \omega)c$

   belongs to $\mathcal{P}$.
2. For every choice of \( \omega \in \Omega \), \( c \in \mathbb{C}^p \) and of \( f \in \mathcal{P} \), it holds that
\[
\langle f, K_\omega c \rangle_\mathcal{P} = c^* f(\omega).
\]

It is well to recall that the function \( K(z, \omega) \) has \( \kappa \) negative squares (with \( \kappa = \dim \mathcal{P}_- \)) in the following sense: for every choice of integer \( n \), of points \( \omega_1, \ldots, \omega_n \in \Omega \) and vectors \( c_1, \ldots, c_n \in \mathbb{C}^p \)
the \( n \times n \) hermitian matrix with \( ij \) entry equal to \( c_i^* K(\omega_i, \omega_j) c_j \) has at most \( \kappa \) strictly negative eigenvalues, and exactly \( \kappa \) such eigenvalues for some choice of \( n \) of points \( \omega_1, \ldots, \omega_n \) and \( c_1, \ldots, c_n \).

There is a one-to-one correspondence between such functions and reproducing kernel Pontryagin spaces. We refer the reader to [10] for the Hilbert space case and to [22, 21, 5] for the case of Pontryagin spaces. The case of Krein spaces is more involved and will not be considered here (see [21, 2]).

In the present paper we review work on finite-dimensional reproducing kernel spaces (a particular instance of which will be the case of polynomials). In particular, following a strategy due to de Branges (and pursued later by his student Li [18]), we characterize the case where the reproducing kernel is of the form
\[
U(z) J U(\omega)^* - V(z) J V(\omega)^* \over 1 - z \omega^*.
\]

In this expression, \( U \) and \( V \) are matrix-valued functions (say \( \mathbb{C}^{p \times p} \)-valued) analytic in an open subset of the open unit disk and with nonidentically vanishing determinant, and \( J \) is a signature matrix, i.e., a matrix which is both unitary and hermitian. More generally, one could consider denominators of the form
\[
a(z) a(\omega)^* - b(z) b(\omega)^*
\]
where the functions \( a \) and \( b \) are analytic in a connected open subset \( \Omega \subset \mathbb{C} \) and such that the three sets
\[
\Omega_+ = \{ z \in \Omega; \; |a(z)| > |b(z)| \}, \\
\Omega_- = \{ z \in \Omega; \; |a(z)| < |b(z)| \}, \\
\Omega_0 = \{ z \in \Omega; \; |a(z)| = |b(z)| \}
\]
are all nonempty, see [7]. This allows to treat in a unified way Toeplitz and Hankel matrices.

2. Finite-dimensional reproducing kernel Hilbert spaces

When the space is finite-dimensional, the reproducing kernel is given by a simple and well-known formula:

**Theorem 2.1.** Let \( P \) be a finite-dimensional Pontryagin space, whose elements are functions \( \Omega \to \mathbb{C}^p \). Let \( \{ f_1, \ldots, f_N \} \) be a basis of \( P \). Let \( P \) be the matrix with \( i, j \) entry given by \( P_{ij} = \langle f_i, f_j \rangle_\mathcal{P} \). Then \( P \) is hermitian and nonsingular and \( P \) is a reproducing kernel Pontryagin space with reproducing kernel
\[
K(z, \omega) = (f_1(z) \cdots f_N(z)) P^{-1} (f_1(\omega) \cdots f_N(\omega))^*.
\]
Proof. Indeed, let \( P^{-1} = (\gamma_{ij}) \). One can write

\[
K(z, \omega) = \sum_{i,j} f_i(z) \gamma_{ij} f_j(\omega)^*
\]

and thus, with \( c \in \mathbb{C}^P \),

\[
K(z, \omega)c = \sum_{i,j} f_i(z) \gamma_{ij} f_j(\omega)^* c
\]

and the function \( z \mapsto K(z, \omega)c \in P \). Let \( f = \sum_l \alpha_l f_l \in P \). We have

\[
\langle f, K(\cdot, \omega)c \rangle_p = \sum_l \alpha_l \langle f_l, K(\cdot, \omega)c \rangle_p = \sum_l \sum_{i,j} \alpha_i \gamma_{ij} c^* f_j(\omega) (f_l, f_i)_p = \sum_{l,i,j} \alpha_i \gamma_{ij} c^* f_j(\omega) P_{il},
\]

from which one easily concludes since \( \gamma_{ij}^* = \gamma_{ji} \) and \( \sum_{i} \gamma_{ij} P_{il} \) is equal to 0 or 1, depending on \( j \neq l \) or \( j = l \). \( \square \)

The matrix \( P \) is called the gramian of the basis \( f_i, i = 1, \ldots, n \); see [20, p. 2]. It is of interest to relate the structure of the reproducing kernel and of the Gram matrix. The celebrated Christoffel–Darboux formula can be viewed as an instance of such a link, as will be made clear in the sequel. The spaces of polynomials of degree less or equal to a given integer motivate the assumption that the space \( P \) is invariant under the backward shift operators \( R_z \) defined by

\[
(R_z f)(z) = \frac{f(z) - f(\omega)}{z - \omega}.
\]

In fact, a less stringent hypothesis will be made: let \( \omega \in \mathbb{C} \). If \( f \in \mathcal{P} \) is analytic at \( \omega \) and \( f(\omega) = 0 \), then we will require that the function

\[
z \mapsto f(z) \frac{1 - z\omega^*}{z - \omega}
\]

belongs to \( \mathcal{P} \). Such hypothesis originates with the work of de Branges and are motivated by the theory of (in general nondensely defined) relations in Hilbert space; see [13], and especially Théorème 23, p. 59. In view of the equality

\[
\frac{1 - z\omega^*}{z - \omega} = \frac{1 - |\omega|^2}{z - \omega} - \omega^*
\]

we have in particular that the function \( z \mapsto f(z)/(z - \omega) \) belongs to \( \mathcal{P} \) for \( \omega \) off the unit circle. The invariance condition (2.3) forces the structure of the space:

**Lemma 2.2.** Let \( \mathcal{P} \) be a finite-dimensional vector space of functions analytic in some open set \( \Omega \subset \mathbb{C} \) with values in \( \mathbb{C}^p \) and assume that there is a point \( z \in \Omega \) such that the span of the vectors
\( f(z) \), as \( f \) runs through \( \mathcal{P} \), is equal to \( \mathbb{C}^p \). Assume furthermore that the invariance condition (2.3) holds. Then, there is an analytic \( \mathbb{C}^{p \times N} \)-valued function \( X(z) \) and a matrix \( A \in \mathbb{C}^{N \times N} \) such that the columns of a basis of \( \mathcal{P} \) are given by
\[
F(z) = X(z)(I + zA - zA)^{-1}.
\]

**Proof.** By hypothesis, there is a \( p \times p \) matrix \( E \) whose columns are functions of \( \mathcal{P} \) and whose determinant at some point is nonzero. (This will then happen at all points of \( \Omega \), at the exception of a zero set, i.e., the set of zeros of an analytic function.) Let now \( N = \dim \mathcal{P} \) and \( F \) be a \( p \times N \) matrix whose columns form a basis of \( \mathcal{P} \). The function
\[
F(z) - E(z)E(x)^{-1}F(x)
\]
begins to \( \mathcal{P} \) and vanishes at the point \( x \). So by hypothesis the columns of
\[
F(z) - E(z)E(x)^{-1}F(x)
\]
are in \( \mathcal{P} \). There is thus a matrix \( A \in \mathbb{C}^{N \times N} \) such that
\[
F(z) - E(z)E(x)^{-1}F(x) = F(z)A.
\]
Hence,
\[
F(z) = E(z)E(x)^{-1}F(x)(A(z - x) - I)^{-1},
\]
which allows to conclude. \( \square \)

### 3. Some reproducing kernel spaces

We now review some properties of reproducing kernels of the form (1.1). Most of the material can be found in [4, 5]. First let us assume that \( U = J = I \). If the function \( V \) is rational and inner (i.e. in analytic in the open unit disk and takes unitary values on the unit circle) the reproducing kernel space with reproducing kernel (1.1) is equal to \( H^p_2 \oplus VH^p_2 \), where \( H^p_2 \) is the Hardy space
\[
\left\{ f(z) = \sum_{n=0}^{\infty} f_n z^n; \; f_n \in \mathbb{C}^p \text{ and } \sum_{n=0}^{\infty} \|f_n\|_{i,p}^2 < \infty \right\}.
\]
The dimension of the space \( H^p_2 \oplus VH^p_2 \) is the McMillan degree of \( V \). We refer the reader to [11] for the definition of the McMillan degree. If the function \( V \) is not anymore inner but is still rational and takes unitary values on the unit circle, a theorem in [17] asserts that \( V = V_2^{-1}V_1 \), where both \( V_1 \) and \( V_2 \) are inner and may be chosen such that
\[
(H^p_2 \oplus V_1 H^p_2) \cap (H^p_2 \oplus V_2 H^p_2) = 0.
\]
Then the equality
\[
\frac{I - V(z)V(\omega)^*}{1 - z\omega^*} = V_2(z)^{-1}\left(\frac{I - V_1(z)V_1(\omega)^*}{1 - z\omega^*} - \frac{I - V_2(z)V_2(\omega)^*}{1 - z\omega^*}\right) V_2(\omega)^{-*}
\]
allows to conclude that $\mathcal{P}(V)$ consists of the functions of the form
\[
f(z) = V(z)^{-1}(f_1(z) + f_2(z)), \quad f_i \in (H^2 \ominus V_i H^2), \quad i = 1, 2,
\]
with the indefinite inner product
\[
\|f\|^2 = \|f_1\|^2_{H^2 \ominus V_1 H^2} - \|f_2\|^2_{H^2 \ominus V_2 H^2},
\]
see [5, Theorem 6.6, p. 133]. In particular, the space $\mathcal{P}(V)$ possesses the property (2) of Theorem 4.1 since the same property trivially holds for the spaces $H^2 \ominus V_i H^2$ associated to inner functions.

Let us still assume that $U = I$ in (1.1) but consider an arbitrary signature matrix $J$. Since $J = J^* = J^{-1}$, the matrix $J$ is unitarily equivalent to
\[
J_0 = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix},
\]
where $r$ (resp. $s$) is the multiplicity of the eigenvalue $+1$ (resp. $-1$). Without loss of generality, we may assume that $J = J_0$. Set
\[
P = \frac{I + J_0}{2}, \quad Q = \frac{I - J_0}{2}.
\]
The matrix
\[
\sigma = (PV + Q)(P + QV)^{-1}
\]
is well defined and is called the Potapov–Ginzburg transform of $V$. Because of the equality
\[
\frac{I - \sigma(z)\sigma(\omega)^*}{1 - z\omega^*} = (P - V(z)Q)^{-1}J_0 - V(z)J_0V(\omega)^*(P - V(\omega)Q)^{-1}
\]
(see [5, Theorem 6.8, p. 136]) one reduces the case of arbitrary $J$ to the case $J = I$ and shows that property (2) of Theorem 4.1 still holds. We refer to [14, 4].

We mention that an analogue of the Krein–Langer factorization theorem for the case of an arbitrary signature matrix $J$ does not hold. The multiplicative structure of $J$-inner functions is well understood [19]. The case of $J$-unitary functions is mostly open, even in the rational case, see [5, 9].

4. Structure theorems

We now characterize finite-dimensional reproducing kernel Pontryagin spaces with reproducing kernel of the form (1.1), supposing moreover that a full rank condition (explicit in the theorem) is met. First one definition: an open subset of $\mathbb{C}$ is said to be symmetric with respect to the unit circle if $1/z^* \in \Omega$ for every nonzero $z \in \Omega$.

**Theorem 4.1.** Let $\mathcal{P} \neq \{0\}$ be a finite-dimensional reproducing kernel Pontryagin space of $\mathbb{C}^p$-valued functions analytic in an open set $\Omega \subset \mathbb{C}$ which is symmetric with respect to the unit circle, and whose intersection with the unit circle is not empty. Then, the reproducing kernel of
\( \mathcal{P} \) is of the form (1.1) for some signature matrix \( J \) and \( \mathbb{C}^{p \times p} \)-valued rational functions \( U \) and \( V \) with non identically vanishing determinant if and only if the following conditions hold:

1. There is a point \( z \in \Omega \) such that the span of the vectors \( f(z) \), as \( f \) runs through \( \mathcal{P} \), is equal to \( \mathbb{C}^p \).
2. If \( f \in \mathcal{P} \) is analytic at \( \omega \in \Omega \) and \( f(\omega) = 0 \), the function defined by (2.3) belongs to \( \mathcal{P} \) and has same norm as \( f \).

The first condition in fact holds for all points in \( \Omega \), with the possible exception of a zero set owing to the connectedness and analyticity hypothesis. In the scalar case and for spaces of polynomials this theorem appears in the work of Li [18]. The proof of this result for the Hilbert space case appears in [3], and is an adaptation of arguments of de Branges.

**Proof of Theorem 4.1.** We first assume that the two conditions of the theorem hold, and show that the reproducing kernel is of the required form. We set \( K \) to be the reproducing kernel of \( \mathcal{P} \) and proceed in a number of steps:

**Step 1:** There is \( \beta \neq 0 \in \Omega \) such that \( K(\beta, \beta) \) and \( K(1/\beta^*, 1/\beta^*) \) are nonsingular.

**Proof of step 1.** Let \( \omega \) be a point at which \( K(\omega, \omega) \) is singular; there exists a nonzero vector \( \xi \in \mathbb{C}^p \) such that \( F(\omega)P^{-1}F(\omega)^*\xi = 0 \). The same argument as the one of [6, Theorem 4.2] forces then \( F(\mu)P^{-1}F(\nu)^*\xi = 0 \) for all \( \mu, \nu \in \Omega \). It follows then that for every \( \nu \in \Omega \) and every \( f \in \mathcal{P} \),

\[
\xi^* f(\nu) = \langle f, K(\cdot, \nu)\xi \rangle_{\mathcal{P}} = 0,
\]

which contradicts the full range hypothesis on the values \( f(\nu), f \in \mathcal{P} \).

We note that the argument of [6] takes full advantage of the connectedness of the set of analyticity of the elements of \( \mathcal{P} \). For a counterexample when this hypothesis is not in force, see [6, p. 51].

**Step 2:** The reproducing kernel is of the form (1.1).

**Proof of step 2.** Let \( c \in \mathbb{C}^p \) and let \( \omega \in \mathbb{C} \) be a point where the elements of \( \mathcal{P} \) are analytic; the function

\[
z \mapsto (K(z, \omega) - K(z, \beta)K(\beta, \beta)^{-1}K(\beta, \omega))c
\]

belongs to \( \mathcal{P} \) and vanishes at \( \omega \). From the second assumption, the function

\[
z \mapsto \frac{1 - z\beta^*}{z - \beta} (K(z, \omega) - K(z, \beta)K(\beta, \beta)^{-1}K(\beta, \omega))c
\]

still belongs to \( \mathcal{P} \). Let \( F \) be an element of \( \mathcal{P} \) which vanishes at \( 1/\beta^* \). In view of the hypothesis on the inner product of \( \mathcal{P} \) we have

\[
\left[ F, \frac{1 - z\beta^*}{z - \beta} (K(z, \omega) - K(z, \beta)K(\beta, \beta)^{-1}K(\beta, \omega))c \right]_{\mathcal{P}} = \left[ F(z) \frac{1 - z\beta}{z - 1/\beta^*}, \frac{1 - z\beta^*}{z - \beta} \frac{1 - z/\beta}{z - 1/\beta^*} (K(z, \omega) - K(z, \beta)K(\beta, \beta)^{-1}K(\beta, \omega))c \right]_{\mathcal{P}}
\]
\[
\begin{align*}
F(z) &= \frac{z - \beta}{1 - z^{\beta^*}}, (K(z, \omega) - K(z, \beta)K(\beta, \beta)^{-1}K(\beta, \omega))c \\
F(\omega) &= \frac{\omega - \beta}{1 - \omega^{\beta^*}} \\
F, (K(z, \omega) - K(z, 1/\beta^*)K(1/\beta^*, 1/\beta^*)^{-1}K(1/\beta^*, \omega))c \left( \frac{\omega^* - \beta^*}{1 - \omega^* \beta^*} \right) \\
\end{align*}
\]

The equality between the first and last line of this chain of equalities is valid for every function in \(\mathcal{P}\) which vanishes at 1/\(\beta^*\). Therefore,

\[
(K(z, \omega) - K(z, \beta)K(\beta, \beta)^{-1}K(\beta, \omega)) \left( \frac{1 - z^{\beta^*}}{z - \beta} \right) = (K(z, \omega) - K(z, 1/\beta^*)K(1/\beta^*, 1/\beta^*)^{-1}K(1/\beta^*, \omega)) \left( \frac{\omega^* - \beta^*}{1 - \omega^* \beta^*} \right).
\]

Solving for \(K(z, \omega)\), we obtain

\[
K(z, \omega) \left( \frac{1 - z^{\beta^*}}{z - \beta} - \frac{\omega^* - \beta^*}{1 - \omega^* \beta^*} \right) = K(z, \beta)K(\beta, \beta)^{-1}K(\beta, \omega) \left( \frac{1 - z^{\beta^*}}{z - \beta} \right) - (K(z, 1/\beta^*)K(1/\beta^*, 1/\beta^*)^{-1}K(1/\beta^*, \omega)) \left( \frac{\omega^* - \beta^*}{1 - \omega^* \beta^*} \right)
\]

and hence

\[
K(z, \omega)(1 - z\omega^*)(1 - |\beta|^2) = K(z, \beta)(1 - z^{\beta^*})K(\beta, \beta)^{-1}K(\beta, \omega)(1 - \omega^* \beta) - K(z, 1/\beta^*)K(1/\beta^*, 1/\beta^*)^{-1}K(1/\beta^*, \omega)(\omega^* - \beta^*).
\]

The matrices \(K(\beta, \beta)^{-1}\) and \(K(1/\beta^*, 1/\beta^*)^{-1}\) need not be positive; let us write them as \(MJ_1M^*\) and \(NJ_2N^*\), where \(J_1\) and \(J_2\) are (a priori different) signature matrices and \(M, N\) are invertible matrices. It follows that \(K\) is of the form

\[
\frac{U(z)J_1U(\omega)^* - V(z)J_2V(\omega)^*}{1 - z\omega^*}, 
\]

with

\[
U(z) = \frac{1 - z^{\beta^*}}{\sqrt{1 - |\beta|^2}} K(z, \beta)M, \\
V(z) = \frac{z - \beta}{\sqrt{1 - |\beta|^2}} K(z, 1/\beta^*)N.
\]

The matrices \(U(\beta)\) and \(V(1/\beta^*)\) are invertible. Furthermore, setting \(z\) to be a point on the unit circle, we see that

\[
U(z)J_1U(z)^* = V(z)J_2V(z)^*
\]
Hence, $J_1$ and $J_2$ can be taken to be equal to a common signature matrix, which we will denote by $J$.

We now study the converse of the theorem. The function $B = U^{-1}V$ is $J$-unitary and rational, and the reproducing kernel Pontryagin space $\mathcal{P}(B)$ with reproducing kernel $(J - B(z)B(w))/(1 - zw^*)$ is $R_z$-invariant (see [5]). Let $g = Uf \in \mathcal{P}$, with $f \in \mathcal{P}(B)$. Assume that $\omega$ is off the unit circle, such that $g(\omega) = 0$ and $\det U(\omega) \neq 0$. Then, $f(\omega) = 0$. Thus, $z \mapsto f(z)/(z - \omega) \in \mathcal{P}(B)$. It follows that $z \mapsto F(z)/(z - \omega)$ belongs to $\mathcal{P}$ and so does

$$g(z) \frac{1 - z\omega^*}{z - \omega} = -\omega^* g(z) + (1 - |\omega|^2) \frac{g(z)}{z - \omega}.$$

The norm condition is verified using the properties of $\mathcal{P}(B)$ spaces.

**Corollary 4.2.** In the preceding theorem, assume that $\mathcal{P}$ is a Hilbert space and that $J = I$. Then there is a positive measure $d\mu$ on the unit circle such that $\mathcal{P}$ is isometrically included in the Lebesgue space $L^p(d\mu)$.

Indeed, we then have $\mathcal{P}(B) = H^p_2 \ominus BH^p_2$ and one can take

$$d\mu(t) = U(e^{it})^*U(e^{it}).$$

The formula

$$K(z, \omega) = \frac{K(z, \beta)(1 - z\beta^*)K(\beta, \beta)^{-1}K(\beta, \omega)(1 - \omega^* \beta)}{(1 - z\omega^*)(1 - |\beta|^2)} - \frac{K(z, 1/\beta^*)(z - \beta)K(1/\beta^*, 1/\beta^*)^{-1}K(1/\beta^*, \omega)(\omega^* - \beta^*)}{(1 - z\omega^*)(1 - |\beta|^2)}$$

really means that to compute the kernel function, it is enough to know it on two symmetric values, namely at $\beta$ and $1/\beta^*$.

The Toeplitz case is of special interest: let $P_n$ be the space of polynomials of the form $\sum_0^n A_j z^j$, where the $A_j \in \mathbb{C}^{p \times p}$ endowed with the inner product defined by an nonsingular block-hermitian Toeplitz matrix $P$. The conditions of the theorem are easily seen to be satisfied and in order to apply the result one looks first for a nonzero number $\beta$ such that

$$(I_p \beta I_p \ldots \beta^n I_p)P^{-1}(I_p \beta I_p \ldots \beta^n I_p)^*$$

and

$$(I_p 1/\beta I_p \ldots 1/\beta^n I_p)P^{-1}(I_p 1/\beta I_p \ldots 1/\beta^n I_p)^*$$

are both nonsingular. The limiting process $\beta \to \infty$ leads to the Christoffel–Darboux formula, which is another way of expressing the Gohberg–Heinig formula [15]. This formula expresses the inverse of a (say hermitian invertible) Toeplitz matrix in terms of the first and last columns of its inverse, provided not only the matrix is invertible but also the first main minor; this is equivalent to require that the matrices $\gamma_{00}^{(n)}$ and $\gamma_{in}^{(n)}$ in the block decomposition $P^{-1} = (\gamma_{ij}^{(n)})$ are invertible; see [9].
Setting

\[ E_n(z) = (I_p, zI_p, \ldots, z^nI_p)P^{-1} \begin{pmatrix} I_p \\ 0 \\ \vdots \\ z^nI_p \end{pmatrix} = \frac{\gamma_{n0}^{(n)} + z\gamma_{n1}^{(n)} + \cdots + \gamma_{nn}^{(n)}}{\gamma_{00}^{(n)}} \]

\[ F_n(z) = (I_p, zI_p, \ldots, z^nI_p)P^{-1} \begin{pmatrix} 0 \\ \vdots \\ I_p \end{pmatrix} = \frac{\gamma_{n0}^{(n)} + z\gamma_{n1}^{(n)} + \cdots + \gamma_{nn}^{(n)}}{\gamma_{00}^{(n)}} \]

one has

\[ \lim_{\beta \to \infty} \frac{K(z, 1/\beta^*)(z - \beta)K(1/\beta^*, 1/\beta^*)^{-1}K(1/\beta^*, \omega)(\omega^* - \beta^*)}{1 - |\beta|^2} = - E_n(z)(\gamma_{00}^{(n)})^{-1}E_n(\omega)^* \]

and

\[ \lim_{\beta \to \infty} \frac{K(z, \beta)(1 - z\beta^*)K(\beta, \beta)^{-1}K(\beta, \omega)(1 - \omega^*\beta)}{1 - |\beta|^2} = - zF_n(z)(\gamma_{nn}^{(n)})^{-1}\omega^*F_n(\omega)^* \]

and the kernel takes the form

\[ \frac{E_n(z)(\gamma_{00}^{(n)})^{-1}E_n(\omega)^* - zF_n(z)(\gamma_{nn}^{(n)})^{-1}\omega^*F_n(\omega)^*}{1 - z\omega^*}, \]

i.e. we get to the Christoffell–Darboux formula (see for instance [8, Theorem 4.1, p. 38]).

One recognizes in \( E_n \) and \( F_n \) orthogonal polynomials; a result of Krein [16] characterizes the number of their zeros in the non positive case and the scalar case; for an extension to the matrix case, see [8].

In connection with the previous discussion, let us mention:

**Problem 4.3.** Can one express the kernel (1.1) in terms of two non symmetric values \( K(z, \beta_1) \) and \( K(z, \beta_2) \)?

**Acknowledgements**

It is a pleasure to thank Martine Olivi (INRIA Sophia-Antipolis) for a careful reading of the manuscript.

**References**


