Conditionally independent private information in OCS wildcat auctions

Tong Li\textsuperscript{a}, Isabelle Perrigne\textsuperscript{b,*,} Quang Vuong\textsuperscript{b,c}

\textsuperscript{a}Indiana University, Los Angeles, USA
\textsuperscript{b}Department of Economics, University of Southern California, Los Angeles 90089-0253, USA
\textsuperscript{c}INRA, France

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Abstract

In this paper, we consider the conditionally independent private information (CIPI) model which includes the conditionally independent private value (CIPV) model and the pure common value (CV) model as polar cases. Specifically, we model each bidder’s private information as the product of two unobserved independent components, one specific to the auctioned object and common to all bidders, the other specific to each bidder. The structural elements of the model include the distributions of the common component and the idiosyncratic component. Noting that the above decomposition is related to a measurement error problem with multiple indicators, we show that both distributions are identified from observed bids in the CIPV case. On the other hand, identification of the pure CV model is achieved under additional restrictions. We then propose a computationally simple two-step nonparametric estimation procedure using kernel estimators in the first step and empirical characteristic functions in the second step. The consistency of the two density estimators is established. An application to the OCS wildcat auctions shows that the distribution of the common component is much more concentrated than the distribution of the idiosyncratic component. This suggests that idiosyncratic components are more likely to explain the variability of private information and hence of bids than the common component. © 2000 Elsevier Science S.A. All rights reserved.

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*Corresponding author. Tel.: (213) 740 3528; fax: (213) 740 8543.
E-mail address: perrigne@usc.edu (I. Perrigne).
1. Introduction

Over the last ten years, the analysis of auction data has attracted much interest through the development of the structural approach. This approach relies on econometric models closely derived from game-theoretic auction models that emphasize strategic behavior and asymmetric information among participants. The structural econometrics of auction models then consists in the identification and estimation of the structural elements of the theoretical model from available data. The structural elements usually include the latent distribution of private information while observations are usually bids. Previous studies have mainly adopted the independent private values (IPV) paradigm, where each bidder knows his own private value for the auctioned object but does not know others' valuations which are independent from his. An alternative paradigm is the pure common value (CV) paradigm, where the value of the auctioned object is assumed to be common ex post but unknown ex ante to all bidders who have private signals about this value. It has been used in Paarsch (1992). More recently, Li et al. (1999) have extended the structural approach to the more general affiliated private value (APV) model where bidders' private values are affiliated in the sense of Milgrom and Weber (1982).

In this paper, we consider a model where affiliation among private information arises through an unknown common component. Specifically, we assume that each private information (either private values or signals) can be decomposed as the product of two unknown independent random components, one common to all bidders and the other specific to each bidder. Because private information are independent conditionally upon the common component, the resulting model is called the conditionally independent private information (CIPI) model. As we shall see, the CIPI model is quite general and includes as special cases the conditionally independent private value (CIPV) model and the pure common value (CV) model.

The structural elements of the CIPI model include the distributions of the common and idiosyncratic components of private information. Considering a quite general model and decomposing private information render the identification and estimation of the structural elements from observed bids more

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1 Donald and Paarsch (1996), Paarsch (1992) and Laffont et al. (1995) have proposed some parametric estimation methods while Guerre et al. (2000) and Elyakime et al. (1994, 1997) have developed some nonparametric ones.
complicated than in previous studies such as Guerre et al. (2000) for the IPV model and Li et al. (1999) for the APV model. It turns out that identification and estimation of the CIPI model is related to the measurement error model when many indicators are available as studied by Li and Vuong (1998). We show that the CIPV model is fully identified nonparametrically. On the other hand, the identification of the CV model requires some restrictions as the CV model is not identifiable in general from observed bids, see LaFont and Vuong (1996). In either case, combining Li et al. (1999) and Li and Vuong (1998) we propose a two-step nonparametric procedure for estimating the density of the idiosyncratic component and the density of the common component. In particular, our procedure uses kernel estimators in the first step and empirical characteristic functions in the second step. We then establish the consistency of our estimators.

As an application, we study the Outer Continental Shelf (OCS) wildcat auctions. These auctions present some specific features which render the CIPI model relevant. On one hand, the common component can be viewed as the unknown ex post value. On the other hand, the idiosyncratic component can be considered as arising from a firm’s specific drilling, prospecting and development strategies, capital and financial constraints, opportunity costs as well as the precision of its own estimate of the value of the tract derived from geological surveys.

The structure of the CIPI model enables us to assess the roles played by both the common and idiosyncratic components in firms’ bidding strategies. Consequently, our approach complements previous studies done by Hendricks and Porter (see Porter, 1995 for a survey) who adopted the pure common value paradigm within a reduced form approach. In particular, comparisons of the distributions of the common and idiosyncratic components indicate that the idiosyncratic component explains a large part of the variability of bidders’ signals, and hence of bids.

Our paper is original for several reasons. First, it contributes to the structural analysis of auction data as it shows that the structural approach can be extended to the CIPV and pure CV models. Second, from an econometric point of view, we propose an original method for estimating nonparametrically the latent distributions of each model that combines kernel methods and empirical characteristic functions. To our knowledge, the latter have been seldom used in empirical work. Third, we provide a new analysis of OCS wildcat auctions through a specified structure of affiliation among valuations.

The paper is organized as follows. Section 2 presents the CIPI model and its special cases, the CIPV and pure CV models. We consider identification of both models and propose a two-step nonparametric procedure to estimate the underlying structural elements, namely the density of the idiosyncratic component and the density of the common component. Moreover, we show that our estimators are consistent. In Section 3, after a brief description of the OCS wildcat auctions and the corresponding data, we illustrate our procedure to
auctions with two bidders and present our empirical findings. Section 4 concludes the paper. Proofs are contained in an appendix.

2. The CIPI model and the structural approach

In this section, we first present the CIPI model as well as the related CIPV and pure CV models. We then address their identification from observed bids. Finally, we propose a two-step nonparametric procedure for estimating the underlying structural elements.

2.1. The CIPI, CIPV and CV models

We begin with the general affiliated value (AV) model introduced by Wilson (1977) and Milgrom and Weber (1982).

A single and indivisible object is auctioned to \( n \geq 2 \) bidders. The utility of each potential bidder \( i, i = 1, \ldots, n \), for the object is \( U_i = U(\sigma_i, v) \) where \( U(\cdot) \) is a nonnegative function strictly increasing in both arguments, \( \sigma_i \) denotes the \( i \)th player’s private signal or information and \( v \) represents a common component or value affecting all utilities. The vector \((\sigma_1, \ldots, \sigma_n, v)\) is viewed as a realization of a random vector whose \((n + 1)\)-dimensional cumulative distribution function \( F(\cdot) \) is affiliated and exchangeable in its first \( n \) arguments.\(^2\) The distribution \( F(\cdot) \) is assumed to have a support \([\sigma^*, v^*] = [\sigma, v^*] \times [v^*, v^*] \) with \( \sigma^* > 0 \) and \( v^* > 0 \) and a density \( f(\cdot) \) with respect to Lebesgue measure. Each player \( i \) knows the value of his signal \( \sigma_i \) as well as \( F(\cdot) \). However, he does not know the other bidders’ private signals and the common component \( v \).

In the CIPI model, it is assumed that bidders’ private signals \( \sigma_i \) are conditionally independent given the common component \( v \). Let \( F_v(\cdot) \) and \( F_{\sigma|v}(\cdot|\cdot) \) denote the cumulative distribution functions of \( v \) and \( \sigma \) given \( v \), respectively, with corresponding densities \( f_v(\cdot) \) and \( f_{\sigma|v}(\cdot|\cdot) \) and \([\sigma, \tilde{\sigma}]\). Hence in the CIPI model, the joint distribution \( F(\cdot) \) of \((\sigma_1, \ldots, \sigma_n, v)\) is entirely determined by the pair \([F_v(\cdot), F_{\sigma|v}(\cdot|\cdot)]\) as

\[
f(\sigma_1, \ldots, \sigma_n, v) = f_v(v) \prod_{i=1}^{n} f_{\sigma|v}(\sigma_i|v) = f_v(v) \prod_{i=1}^{n} f_{\sigma|v}(\sigma_i|v) \tag{1}
\]

It can be easily shown from (1) that \( F(\cdot) \) is symmetric or exchangeable in its first \( n \) arguments and affiliated. Consequently, the CIPI model is a special case of the general affiliated value model introduced by Wilson (1977) and Milgrom and Weber (1982).

\(^2\) See Milgrom and Weber (1982) for a definition of affiliation. Intuitively, affiliation implies that a bidder who evaluates the object highly will expect others to evaluate the object highly too.
We focus on the first-price sealed-bid auction, which is the mechanism used in the OCS wildcat auctions analyzed in this paper. As usual, we restrict our attention to strictly increasing differentiable symmetric Bayesian Nash equilibrium strategies. At such an equilibrium, player $i$ chooses his bid $b_i$ by maximizing $E[(U_i - b_i) I(B_i \leq b_i) \sigma_i]$ where $B_i = s(y_i)$, $y_i = \max_{j \neq i} \sigma_j$, $s(\cdot)$ is the equilibrium strategy, and $E[\cdot | \sigma_i]$ denotes the expectation with respect to all random elements conditional on $\sigma_i$. As is well-known, the equilibrium strategy satisfies the first-order differential equation

$$s'(\sigma_i) = [V(\sigma_i, \sigma_i) - s(\sigma_i)] f_{y_1|\sigma_i}(\sigma_i|\sigma_i) F_{y_1|\sigma_i}(\sigma_i|\sigma_i),$$

(2)

for all $\sigma_i \in [\sigma, \bar{\sigma}]$ subject to the boundary condition $s(\bar{\sigma}) = V(\bar{\sigma}, \bar{\sigma})$, where $V(\sigma_i, y_i) = E[U(\sigma_i, v)|\sigma_i, y_i]$, $F_{y_1|\sigma_i}(\cdot | \cdot)$ denotes the conditional distribution of $y_1$ given $\sigma_i$, and $f_{y_1|\sigma_i}(\cdot | \cdot)$ is its density. The index ‘1’ refers to any bidder among the $n$ bidders because the game is symmetric. The distribution $F_{y_1|\sigma_i}(\cdot | \cdot)$ is the one corresponding to the probability structure defined in (1). As shown by Milgrom and Weber (1982), when the reservation price is nonbinding, the solution is

$$b_i = s(\sigma_i) = V(\sigma_i, \sigma_i) - \int_{\sigma}^{\bar{\sigma}} L(x|\sigma_i) dV(x, \sigma),$$

(3)

where $L(x|\sigma_i) = \exp[- \int_{x}^{\bar{\sigma}} f_{y_1|\sigma_i}(u|\sigma_i) F_{y_1|\sigma_i}(u|\sigma_i) du]$. It is easy to verify that $b_i = s(\sigma_i)$ is strictly increasing in $\sigma_i$ on $[\sigma, \bar{\sigma}]$.

Two important models are derived from the CIPF model. These are the CIPV model and the pure CV model.

2.1.1. The CIPV model

In this model, it is assumed that $U(\sigma_i, v) = \sigma_i$ so that each bidder’s private information is his own utility, which he knows fully at the time of the auction. We are thus in the private value paradigm and $V(\sigma_i, y_i) = \sigma_i$.

An economic interpretation of the CIPV model is as follows. Bidders’ valuations are independently drawn from a common distribution $F_{\sigma_i|v}(\cdot | v)$ that depends on an unknown parameter $v$. Thus $F_v(\cdot)$ can be interpreted as bidders’ common prior distribution on $v$. In particular, $v$ can be interpreted as the ex post value of the auctioned object for the average bidder while the discrepancy between $\sigma_i$ and $v$ results from bidder’s specific characteristics such as his productive efficiency, opportunity costs, idiosyncratic preferences, etc. The CIPV model extends the IPV model by allowing for affiliation among bidders’ private values through the unknown common component $v$. Because it specifies a special structure on affiliation, it is a special case of the APV model.

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3 In fact, if one assumes that $(\sigma_1, \ldots, \sigma_n)$ are exchangeable for every $n$, then $(\sigma_1, \ldots, \sigma_n)$ are necessarily conditionally independent by de Finetti’s Theorem. See, e.g. Kingman (1978). Consequently, any APV model is a CIPV model under such an assumption.
The CIPV model formalizes the idea that a bidder who evaluates the object highly will expect others to evaluate it highly too. It is well suited to auction situations, where there is some ‘prestige’ value in owning the auctioned object which is admired by other bidders and there is a possibility of resale at some currently undetermined price. These include auction of works of art, memorabilia, collectibles, etc. In the case of OCS auctions, the auctioned tract adds to the capital of the winning oil company while the mineral content of the tract is uncertain to the firm.

2.1.2. The pure CV model

In this model, it is assumed that $U(\sigma_i, v) = v$ so that each bidder derives the same but unknown utility from the auctioned object while $\sigma_i$ is bidder $i$'s private estimate of the common value. In this case, $V(\sigma_i, y_i) = E[v|\sigma_i, y_i = \sigma_i]$.

The economic interpretation of the pure CV model is well known. Differences in bidders' preferences are neglected. On the other hand, bidders differ through their private information about the value of the auctioned object. This model has been frequently used to analyze auctions of drilling rights as the mineral content of the tract is subject to important uncertainty (see, e.g. Rothkopf, 1969, Wilson, 1977). As a result the model is sometimes called the mineral rights model (see Milgrom and Weber, 1982). In particular, oil companies are assumed to have different estimates of the value of the tract, which are derived from geological surveys, but are assumed to have identical productive efficiency, opportunity costs, etc.

2.2. Nonparametric identification

The structural approach relies on the hypothesis that the observed bids are the equilibrium bids of the auction model under consideration. Hence, the structural econometric model associated with the CIPV model is defined as

$$b_i = s(\sigma_i, U, F_v, F_{\sigma|v}) \quad \text{for } i = 1, \ldots, n, \quad n \geq 2,$$

where $s(\cdot)$ is the equilibrium strategy (3). In particular, because private signals are random and unobserved, bids are naturally random with an $n$-dimensional joint distribution $G(\cdot)$ determined by the structural elements of the model $[U(\cdot), F_v(\cdot), F_{\sigma|v}(\cdot|\cdot)]$.

To implement the structural approach, a fundamental issue is whether the structural elements of the economic model are identified from available observations. In general, all firms’ private information as well as the common component are unknown to the econometrician, while only bids are observed. Therefore, the identification of the CIPV model reduces to whether the utility function $U(\cdot)$ and the two underlying structural distributions $F_v(\cdot)$ and $F_{\sigma|v}(\cdot|\cdot)$ can be determined uniquely from observed bids. An important feature of (4) is that the joint equilibrium bid distribution $G(\cdot)$ depends on the underlying distributions
$F_v(\cdot)$ and $F_{\sigma|v}(\cdot|\cdot)$ in two ways: (i) through the unobserved $\sigma_i$, which is drawn from $F_{\sigma|v}(\cdot|v)$ while $v$ is drawn from $F_v(\cdot)$, and (ii) through the equilibrium strategy, which is a complex function of $F(\cdot)$ and therefore of $F_v(\cdot)$ and $F_{\sigma|v}(\cdot|\cdot)$ through (1) (see (3)). This feature complicates the analysis of identifiability.

Following a similar argument as in Li et al. (1999), denote the conditional distribution of $B_1$ given $b_1$ by $G_{B_1|b_1}(\cdot|\cdot)$ and its density by $g_{B_1|b_1}(\cdot|\cdot)$. Then

$$G_{B_1|b_1}(X_1|x_1) = \Pr(B_1 \leq X_1 | b_1 = x_1)$$

$$= \Pr(y_1 \leq s^{-1}(X_1)|\sigma_1 = s^{-1}(x_1))$$

$$= F_{y_1|\sigma_1}(s^{-1}(X_1)|s^{-1}(x_1)).$$

It follows that

$$g_{B_1|b_1}(X_1|x_1) = \frac{f_{y_1|\sigma_1}(s^{-1}(X_1)|s^{-1}(x_1))}{s(s^{-1}(X_1))}.$$  

Using the last two equations and $\sigma = s^{-1}(b)$, the first-order differential equation (2) can be written as

$$V(\sigma, \sigma) = b + \frac{G_{B_1|b_1}(b|b)}{g_{B_1|b_1}(b|b)} \equiv \zeta(b, G). \tag{5}$$

Recall that $V(\sigma, \sigma) = E[U(\sigma_1, v)|\sigma_1 = \sigma, y_1 = \sigma]$. Thus a distinguishing feature of (5) is that it expresses such an expected value as an explicit function of the corresponding observed bid, the distribution $G_{B_1|b_1}(\cdot|\cdot)$ and density $g_{B_1|b_1}(\cdot|\cdot)$ of bids without solving the differential equation (2). The above equation forms the basis upon which the identifiability of the CIPI, CIPV and CV models can be studied.\footnote{Because the conditional independence of signals was not used, (5) actually holds for the general AV model. It also forms the basis upon which the identification of the AV model can be studied.}

The next proposition relates the identifiability of these three models. Hereafter, we use the standard notion of observational equivalence of competing models from observables, which are here the observed bids. See Laffont and Vuong (1996) for a formal definition.

**Proposition 1.** Any CIPI model is observationally equivalent to a CIPV model. Hence, any pure CV model is observationally equivalent to a CIPV model.

The first part of Proposition 1 says that when explaining bids with conditionally independent signals, one can restrict oneself to a CIPV model without loss of explanatory power. The second part says that any interpretation in terms of the pure CV model can be equally given in terms of a CIPV model. This
proposition parallels Proposition 1 in Laffont and Vuong (1996), which relates AV to APV models. The difference here is that private signals are now assumed to be conditionally independent, which leads naturally to the CIPI and CIPV models. Intuitively, to establish that any CIPI model is observationally equivalent to a CIPV model, the argument is that $U(\sigma, v)$ can be replaced by $\tilde{U}_i = \tilde{\sigma}_i$, where $\tilde{\sigma}_i = V(\sigma_i, \sigma_i)$ are conditionally independent given $v$. In other words, $U(\cdot)$ is not identified. Thus, we focus below on the CIPV and pure CV models where $U(\sigma, v) = \sigma$ and $U(\sigma, v) = v$, respectively.$^5$

As noted by Laffont and Vuong (1996, Proposition 4), however, the pure CV model is not identified. Similarly, it can be shown that the CIPV model is not identified either. Intuitively, the conditioning variable $v$ in a CIPV model can be replaced by any strictly increasing transformation of $v$ while retaining the same probabilistic structure on the utilities $(\sigma_1, \ldots, \sigma_n)$. This implies that additional restrictions are needed for identification. Assuming that signals are unbiased is not sufficient by itself. In this paper, we assume the multiplicative decomposition $\sigma_i = v\eta_i$, where $v$ is the common component and $\eta_i$ is specific to the $i$th bidder. Moreover, the following assumptions are made.

A1: The $\eta_i$’s are identically distributed with a mean equal to one.
A2: $v$ and the $\eta_i$’s are mutually independent.

The mean requirement in Assumption A1 is a natural normalization and ensures that the signals are unbiased estimates of the common component as in Wilson (1977). Together Assumptions A1 and A2 imply that private signals $\sigma_i$’s are independent and identically distributed conditionally upon $v$. The structural elements of either model are now the pair $[F_v(\cdot), F_\eta(\cdot)]$, where $F_\eta(\cdot)$ is the cumulative distribution of $\eta$ with support $[\bar{v}, \bar{v})$ so that $[\bar{\sigma}, \bar{\eta}] = [v\bar{\eta}, \bar{v}\bar{\eta}]$.$^6$

2.2.1. Identification of the CIPV model

As noted earlier, the CIPV model is a special case of the APV model. From the nonparametric identification of the APV model (see Li et al., 1999, Proposition 2.1), it follows that the joint distribution of private values in the CIPV model is identified from observed bids. The remaining question is whether the structural elements $[F_v(\cdot), F_\eta(\cdot)]$ of the CIPV model are uniquely determined

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$^5$Given the similar probabilistic structure of the CIPV and pure CV models, an interesting question is whether any CIPI model is observationally equivalent to a pure CV model. A positive answer would imply the converse of the second part of Proposition 1. At this stage, however, it is not known whether such a conjecture is true.

$^6$See also Wilson (1998). An alternative decomposition for private signals is $\sigma_i = v + \eta_i$ with the same Assumptions A1 and A2 with the exception that the $\eta_i$’s have a zero mean. We have chosen the multiplicative decomposition instead because it is more natural with nonnegative variables. Below, we indicate how to adapt our results under the additive decomposition.
by such a joint distribution. To address this question, we note that the multiplicative decomposition leads to \( \log \sigma_i = \log c + \log \varepsilon_i \), where

\[
\log c \equiv [\log v + \mathbb{E}(\log \eta)], \quad \log \varepsilon_i \equiv [\log \eta_i - \mathbb{E}(\log \eta)],
\]

\( i = 1, \ldots, n \geq 2 \), and \( \log \varepsilon_i \) has zero mean. Hence, under Assumptions A1–A2, this problem is related to an error-in-variable model with multiple indicators. Indeed, \( \log \varepsilon_i \) can be considered as the error term in the classical measurement error model where \( \log c \) is unobserved. Because the \( \sigma_i \)'s can be recovered from observed bids through (5) where \( V(\sigma, \sigma) = \sigma \), their logs can be viewed as indicators for \( \log c \).

When the densities for \( \log c \) and \( \log \varepsilon \) are both unknown, the error-in-variable model with multiple indicators is nonparametrically identified under a mild additional condition (see Li and Vuong, 1998, Lemma 2.1). In our context, such a condition is satisfied under the following assumption.

A3: The characteristic functions \( \phi_c(\cdot) \) and \( \phi_\eta(\cdot) \) of \( \log v \) and \( \log \eta \) are nonvanishing everywhere.

Such an assumption is standard in the related deconvolution problem with \( \phi_\eta(\cdot) \) known and only one indicator (see, e.g. Fan, 1991; Diggle and Hall, 1993 for recent contributions).\(^7\) From Li and Vuong’s (1998) identification result, which relies upon Kotlarski’s result (see Rao, 1992, p. 21), we have immediately the following lemma, which will be also useful in the estimation part. Throughout, we use \( h_x(\cdot) \) to denote the density of \( \log x \), keeping \( f_x(\cdot) \) for the density of \( x \).

**Lemma 1.** Given A1–A3, the densities \( h_c(\cdot) \) and \( h_\varepsilon(\cdot) \) are uniquely determined by the joint distribution of an arbitrary pair \( (\log \sigma_1, \log \sigma_2) \). Their characteristic functions are

\[
\phi_c(t) = \exp \int_0^t \frac{\hat{\psi}(0, u_2) \partial u_1}{\psi(0, u_2)} \, du_2, \tag{7}
\]

\[
\phi_\varepsilon(t) = \frac{\hat{\psi}(0, t) \partial \phi_c(t)}{\phi_c(t)}, \tag{8}
\]

where \( \psi(t_1, t_2) \) is the characteristic function of \( (\log \sigma_1, \log \sigma_2) \).

The identification result in Lemma 1 is useful because not only are the densities \( h_c(\cdot) \) and \( h_\varepsilon(\cdot) \) of \( \log c \) and \( \log \varepsilon \) identified by the joint distribution of

\(^7\)To our knowledge, the mutual independence of \( c \) and \( \varepsilon \) and hence of \( v \) and \( \eta \) in A2 is crucial for the nonparametric identification of the measurement error model with multiple indicators. A stronger assumption than A1 is that the density of \( \log \eta \) is symmetric about its mean, i.e. that the density of \( \log \varepsilon \) is symmetric about zero. For a recent contribution using the latter assumption within a panel data framework, see Horowitz and Markatou (1996).
(log \(\sigma_1\), log \(\sigma_2\)), but explicit formulae for the characteristic functions of these densities are also available.\(^8\)

The next proposition establishes the identification of the CIPV model and characterizes the restrictions on the distribution of observed bids that are imposed by the CIPV model. In particular, it uses Lemma 1 and the equality \(E(\log \eta) = - \log E(\varepsilon)\), which results from the normalization condition \(E(\eta) = 1\).

**Proposition 2.** Given A1–A3, the CIPV model is identified. Moreover, a distribution of observed bids can be rationalized by a CIPV model if and only if (i) bids are symmetric and conditionally independent, and (ii) the function \(\xi(\cdot, G)\) is strictly increasing on \([b, \overline{b}]\).

2.2.2. Identification of the pure CV model

The next proposition establishes the partial identification of the pure CV model under the following assumption.

A4. \(V(\sigma, \sigma)\) is loglinear in log \(\sigma\), i.e. \(\log E[v|\sigma_1 = \sigma, y_1 = \sigma] = C + D \log \sigma\), where \((C, D) \in \mathbb{R} \times \mathbb{R}^+\).\(^9\)

The usefulness of A4 under the multiplicative decomposition can be seen by noting that \(\log E[v|\sigma_i, y_i = \sigma_i] = \log c + \log \varepsilon_i\), where \(\log c\) and \(\log \varepsilon_i\) are now defined as

\[
\log c = C + D \log E(\eta) + D \log v, \quad \log \varepsilon_i = D[\log \eta_i - E(\log \eta)],
\]

where \(E(\log \eta) = - \log E(\varepsilon^{1/D})\) using the normalization \(E(\eta) = 1\).

**Proposition 3.** Given A1–A4, the distributions of \(v\) and \(\eta\) are given by

\[
\log v = \log E[\varepsilon^{1/D}] - \frac{C}{D} + \frac{1}{D} \log c, \quad \log \eta = - \log E[\varepsilon^{1/D}] + \frac{1}{D} \log \varepsilon,
\]

where the distributions of \(c\) and \(\varepsilon\) are identified from observed bids through Lemma 1 with \(\psi(t_1, t_2)\) being now the characteristic function of \((\log V(\sigma_1, \sigma_1), \log V(\sigma_2, \sigma_2))\) (say).

Proposition 3 says that, up to \((C, D)\) which determine the location and scale, the distributions of \(\log v\) and \(\log \eta\) are uniquely determined from observed bids. Moreover, because the scale is common, the ratio \(\text{Var}(\log \eta)/\text{Var}(\log v)\) is independent of \((C, D)\) and equal to the ratio \(\text{Var}(\log \varepsilon)/\text{Var}(\log c)\). Since the latter is

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\(^8\)As indicated in Li and Vuong (1998), an alternative way to establish the identifiability of \((h_i(\cdot), h_i(\cdot))\) consists in showing that all moments of \(\log c\) and \(\log \varepsilon\) are identified from the moments of the joint density of \(\log \sigma_1, \log \sigma_2\). For instance, \(E[\log c] = E[\log \sigma_1] = E[\log \sigma_2], E[\log^2 c] = E[\log \sigma_1 \log \sigma_2], E[\log^2 \varepsilon] = E[\log^2 \sigma_1] - E[\log \sigma_1 \log \sigma_2]\), etc.

\(^9\)Because of affiliation, \(E[v|\sigma_1 = \sigma, y_1 = \sigma]\) is strictly increasing in \(\sigma\). Hence \(D > 0\).
identified, it can be used to assess the relative variability of bidders' idiosyncratic component and the common value.

It remains to discuss when A4 is satisfied, which will provide additional information on (C, D). A leading case when A4 holds is when the prior on the common value is inversely proportional to \( v^2 \), i.e. \( f_c(v) \propto 1/v^2, \gamma \in \mathbb{R} \). In this case, it can be shown (see Appendix) that

\[
\log \mathbb{E}[v | \sigma_1 = \sigma, y_1 = \sigma] = \log \left( \frac{\mathbb{E}[\eta_1^{-1} | \eta_1 = \max_{j \neq 1} \eta_j]}{\mathbb{E}[\eta_1 | \eta_1 = \max_{j \neq 1} \eta_j]} \right) + \log \sigma, \tag{10}
\]

so that \( D = 1 \). When \( \gamma = 2 \) and a homogeneity assumption on the distribution of signals holds, which is satisfied here by the multiplicative decomposition, Smiley (1979) has shown that the bidding strategy is proportional to the signal. Using Smiley’s result, Paarsch (1992) proposes a parametric estimation of the pure CV model. Assumption A4 is much more general and allows for nonlinear bidding strategies.

Another case when A4 holds occurs with two bidders \( (n = 2) \) when \( (v, \eta_1, \eta_2) \) is jointly log-normally distributed with parameters \( (\mu_v, \sigma^2_v, \mu_\eta, \sigma^2_\eta) \) and \( \exp(\mu_\eta + \sigma^2_\eta/2) = 1 \) because of the normalization \( \mathbb{E}[\eta] = 1 \). It can be shown (see Appendix) that

\[
\log \mathbb{E}[v | \sigma_1 = \sigma, \sigma_2 = \sigma] = \frac{(\mu_v + \sigma^2_v/2)\sigma^2_\eta - 2\sigma^2_\eta \mu_\eta}{\sigma^2_\eta + \sigma^2_v} + \frac{2\sigma^2_v}{\sigma^2_\eta + 2\sigma^2_v} \log \sigma. \tag{11}
\]

Moreover, from (9) the distributions of \( c \) and \( e \) are log-normal with parameters \( (\mu_c, \sigma^2_c, \sigma^2_e) \) and \( \mu_e = 0 \), and we can identify all the structural parameters \( (\mu_v, \sigma^2_v, \mu_\eta, \sigma^2_\eta) \) using

\[
\mu_v = \mu_c - \frac{\sigma^2_e}{8} \left( \frac{\sigma^2_c}{\sigma^2_e} + 2 \right), \quad \sigma^2_v = \frac{\sigma^2_c}{4} \left( \frac{\sigma^2_e}{\sigma^2_c} + 2 \right)^2, \quad \sigma^2_\eta = \frac{\sigma^2_c}{4} \left( \frac{\sigma^2_e}{\sigma^2_c} + 2 \right)^2, \tag{12}
\]

and \( \mu_\eta = -\sigma^2_\eta/2. \)

The preceding results are important for several reasons. First, for the CIPV model, we have achieved full nonparametric identification result of the underlying structural distributions. Second, for the pure CV model, we have proposed partial nonparametric identification results, which are the most general to date. Third, in both cases, we can assess the relative variability of the common

\[\text{under the additive decomposition } \sigma_i = v + \eta_i, \ A4 \text{ is replaced by the linearity in } \sigma \text{ of } \mathbb{E}[v | \sigma_1 = \sigma, y_1 = \sigma]. \text{ By a similar argument used for establishing (10), it can be easily shown that such an assumption A4' is satisfied with a flat prior on } v \text{ in which case } \mathbb{E}[v | \sigma_1 = \sigma, y_1 = \sigma] = -\mathbb{E}[\eta_1 | \eta_1 = \max_{j \neq 1} \eta_j] + \sigma. \text{ Alternatively, when } n = 2, \ A4' \text{ is satisfied if } \mathbb{E}[v | \sigma_1, \sigma_2] \text{ is linear in } \sigma_1 \text{ and } \sigma_2, \text{ in which case the posterior mean is a weighted average of the prior mean and the average of the signals. For instance, this is the case when } (v, \eta_1, \eta_2) \text{ is multivariate normal.} \]
component and the idiosyncratic component in bidders’ private information. Fourth, it is interesting to note that provided one knows $G$, one has neither to solve the differential equation (2) nor to apply numerical integration in (3) so as to determine $V(\sigma, \sigma)$. For knowledge of $G(\cdot)$ and hence of $\zeta(\cdot, G)$ determines either the private value $\sigma$ in the CIPV model or $E(v|\sigma_1 = \sigma, y_1 = \sigma)$ in the pure CV model for any given bid through (5) and, hence the distributions of the common and private components, respectively. Eqs. (5), (7) and (8) form the basis upon which the nonparametric procedure proposed in the next subsection rests.

2.3. Nonparametric structural estimation

We now propose a two-step nonparametric procedure for estimating the densities $h_c(\cdot)$ and $h_e(\cdot)$. Note that these determine uniquely the structural densities $h_v(\cdot)$ and $h_g(\cdot)$ in the CIPV model through (6) since $E(\log g) = -\log E(e)$ from the normalization $E(\eta) = 1$. On the other hand, up to $C$ and $D$ which determine the location and common scale, the structural densities $h_v(\cdot)$ and $h_g(\cdot)$ in the pure CV model can be recovered from Proposition 3. Also, because we do not impose any parametric restriction on the underlying densities, our nonparametric estimation procedure is equivalent to estimating them separately for each number of bidders.\footnote{That is, our procedure estimates $h_{v|n}(\cdot|n)$ and $h_{g|n}(\cdot|n)$ for each value of $n$. In practice, auctioned objects may display some observed heterogeneity captured by some exogenous variables $X$. If this is the case, then Steps 1 and 2 below need to be modified accordingly through some smoothing over $X$. Though we exclude observed heterogeneity across auctions here and in Section 3, note that we do allow for unobserved heterogeneity across these auctions through the common component.}

The basic idea of our two-step estimation procedure is to use (5) followed by Lemma 1. Specifically, if one knew $G_{b_i|b_1}(\cdot|\cdot)$ and $g_{b_i|b_1}(\cdot|\cdot)$, then one could use (5) to recover $V_i \equiv V(\sigma_i, \sigma_i)$ for bidder $i$, $i = 1, \ldots, n$, which is the private value $\sigma_i$ in the CIPV model and the expectation $E[v|\sigma_i, y_i = \sigma_i]$ in the pure CV model. These can be used to estimate the joint characteristic function of $(\log V_1, \log V_2)$ (say). Hence nonparametric estimates of densities of interest can be obtained from Lemma 1 through (7)–(8). Hence, our estimation procedure is as follows:

**Step 1:** Construct a sample based on (5) using nonparametric estimates of $G_{b_i|b_1}(\cdot|\cdot)$ and $g_{b_i|b_1}(\cdot|\cdot)$ from observed bids.

**Step 2:** Use the pseudo sample in logarithm constructed in Step 1 to estimate nonparametrically $h_c(\cdot)$ and $h_e(\cdot)$ via their estimated characteristic functions. These are then used to estimate $h_v(\cdot)$ and $h_g(\cdot)$ for either model.

To be more specific, let $n$ be a given number of bidders. Let $L$ be the number of auctions corresponding to the chosen $n$, and let $\ell$ index the $\ell$th auction,
In Step 1, using the observed bids \( \{b_{i\ell} ; i = 1, \ldots, n; \ell = 1, \ldots, L \} \), we estimate nonparametrically the ratio \( G_{B_{\ell} | b_{i\ell}}(\cdot | \cdot) / g_{B_{\ell} | b_{i\ell}}(\cdot | \cdot) \) by \( \hat{G}_{B_{\ell}, b_{i\ell}}(\cdot, \cdot) / \hat{g}_{B_{\ell}, b_{i\ell}}(\cdot, \cdot) \), where

\[
\hat{G}_{B_{\ell}, b_{i\ell}}(B, b) = \frac{1}{Lh_1} \sum_{\ell} \sum_{i=1}^{n} 1(B_{i\ell} \leq B)K_{G}\left(\frac{b - b_{i\ell}}{h_G}\right),
\]

\[
\hat{g}_{B_{\ell}, b_{i\ell}}(B, b) = \frac{1}{Lh_2} \sum_{\ell} \sum_{i=1}^{n} K_{g}\left(\frac{B - B_{i\ell}}{h_g}, \frac{b - b_{i\ell}}{h_g}\right).
\]

for any value \((B, b)\) where \(h_G\) and \(h_g\) are some bandwidths, and \(K_G\) and \(K_g\) are kernels. Using (5) we obtain estimates of the unobserved \( V_{i\ell} \) as

\[
\hat{V}_{i\ell} = \hat{\xi}(b_{i\ell}) \equiv b_{i\ell} + \frac{\hat{G}_{B_{\ell}, b_{i\ell}}(b_{i\ell}, b_{i\ell})}{\hat{g}_{B_{\ell}, b_{i\ell}}(b_{i\ell}, b_{i\ell})}.
\]

Step 1 is similar to the first step in the two-step nonparametric estimation procedure of the APV model proposed in Li et al. (1999). As mentioned by Guerre et al. (1999), some trimming is necessary in order to correct for the boundary effects caused by the density estimate in the denominator of (15). Such a trimming is presented in Section 3.2. A consequence of the trimming is that it reduces the number of estimates and hence the number of auctions that can be used in Step 2. Let \( L_T \) be the number of auctions after trimming.

We now turn to Step 2, which is composed of three substeps.

- **Substep 1.** Estimate the joint characteristic function of any two bidders’ \( \log V_{i\ell} \) among \( n \) bidders by\(^{12}\)

\[
\hat{\psi}(u_1, u_2) = \frac{1}{n(n-1)} \sum_{i \neq j \leq n} \sum_{\ell = 1}^{L_T} \exp(iu_1 \log \hat{V}_{i\ell} + iu_2 \log \hat{V}_{j\ell}).
\]

- **Substep 2.** Estimate \( h_c(\cdot) \) and \( h_{\tilde{c}}(\cdot) \) by

\[
\hat{h}_c(x) = \frac{1}{2\pi} \int_{-T}^{T} e^{-iux} \hat{\phi}_c(t) \, dt
\]

\[
\hat{h}_{\tilde{c}}(y) = \frac{1}{2\pi} \int_{-T}^{T} e^{-iy\tilde{c}} \hat{\phi}_{\tilde{c}}(t) \, dt
\]

for \( x \in [\log c, \log \tilde{c}] \) and \( y \in [\log \tilde{c}, \log \tilde{c}] \), where \( T \) is some smoothing parameter, and

\[
\hat{\phi}_c(t) = \exp \int_0^t \frac{\partial \hat{\psi}(0, u_2)/\partial u_1}{\hat{\psi}(0, u_2)} \, du_2,
\]

\[
\hat{\phi}_{\tilde{c}}(t) = \hat{\psi}(t, 0) / \hat{\phi}_c(t).
\]

\(^{12}\) Note that, as required by the theoretical model, symmetry is imposed to improve efficiency.
Substep 3. Estimates of the structural densities are obtained in the CIPV model by
\[ \hat{h}_c(x) = \hat{h}_c[x + \hat{E}(\log \eta)], \quad \hat{h}_e(y) = \hat{h}_e[y - \hat{E}(\log \eta)], \]
where \( \hat{E}(\log \eta) = - \log \hat{E}(e) \). In the pure CV model, they are obtained as
\[ \hat{h}_c(x) = D\hat{h}_c[D(x + (C/D) - \log \hat{E}(e^{1/D})]], \]
\[ \hat{h}_e(y) = D\hat{h}_e[D(y + \log \hat{E}(e^{1/D})]], \]
given \((C,D)\).

In Li and Vuong (1998), the uniform consistency of the nonparametric estimators proposed in Step 2 is established when the indicators are observed. This is done by assuming that the densities of interest are either ordinary or super-smooth through the tail behavior of their characteristic functions. Following Fan (1991), we have

**Definition 1.** The distribution of a random variable \( Z \) is ordinary smooth of order \( \beta \) if its characteristic function \( \phi_Z(t) \) satisfies
\[ d_0|t|^{-\beta} \leq |\phi_Z(t)| \leq d_1|t|^{-\beta} \]
as \( t \to \infty \) for some positive constants \( d_0, d_1, \beta \).

On the other hand, it is super-smooth of order \( \beta \) if \( \phi_Z(t) \) satisfies
\[ d_0|t|^{\beta_0} \exp(-|t|^{\beta_0}/\gamma) \leq |\phi_Z(t)| \leq d_1|t|^{\beta_1} \exp(-|t|^{\beta_1}/\gamma) \]
as \( t \to \infty \) for some positive constants \( d_0, d_1, \beta, \gamma \) and constants \( \beta_0 \) and \( \beta_1 \).

Specifically, we make the following assumption.

A5. The characteristic functions \( \phi_c(\cdot) \) and \( \phi_e(\cdot) \) are ordinary smooth with \( \beta > 1 \) or super-smooth.

Note that the characteristic functions \( \phi_c(\cdot) \) and \( \phi_e(\cdot) \) are necessarily both integrable. In addition, Li and Vuong (1998) make the next assumption, which we maintain here.

A6. The supports of \( h_c(\cdot) \) and \( h_e(\cdot) \) are bounded intervals of \( \mathbb{R} \).

Unlike in Li and Vuong (1998), the indicators \( V_i \) are unobserved. Instead, we use their estimates \( \hat{V}_i \) obtained in Step 1. Moreover, the pseudo values used in Step 2 are trimmed to correct for boundary effects. Nonetheless, we can still establish the following result.

**Theorem 1.** Under A1–A6, \( \hat{h}_c(\cdot) \) and \( \hat{h}_e(\cdot) \) are uniformly consistent estimators for \( h_c(\cdot) \) and \( h_e(\cdot) \) on their respective supports, provided \( T \) diverges appropriately to
infinity as $L \to \infty$. Hence, if $C$ and $D$ are known, $\hat{h}_c(\cdot)$ and $\hat{h}_g(\cdot)$ are uniformly consistent estimators for $h_c(\cdot)$ and $h_g(\cdot)$ in either the CIPV or pure CV model. \(^{13}\)

The proof of Theorem 1 is given in Appendix. It relies on an important lemma, which establishes the uniform convergence of the estimator (19) on $[-T, T]$. This is proved using the log-log law and von Mises differentials.

3. Application to the OCS wildcat auctions

In this section, we illustrate our estimation results on OCS wildcat auctions with two bidders.\(^{14}\) A first subsection briefly discusses the data. A second subsection deals with some practical issues for implementing our structural estimation procedure. Our empirical findings within either the CIPV or the pure CV model are discussed in a third subsection.

3.1. Data

The U.S. federal government began auctioning its mineral rights on oil and gas on offshore lands off the coasts of Texas and Louisiana in the gulf of Mexico in 1954. In this application, we focus on wildcat tracts sold through sales held between 1954 and 1969. This gives us a total of 217 auctioned wildcat tracts with two bidders.

Before each sale, the government announces to oil companies that an area is available for exploration. This area is divided into a number of tracts, each of which is usually a block of 5000 or 5760 acres. Firms are allowed to get limited information about tracts using seismic surveys and off-site drilling. However, no drilling is allowed on each tract before the auction. Because bidders have equal access to the same information about the tract, they can be considered as identical ex ante so the game is symmetric (see McAfee and Vincent, 1992 for more details).

\(^{13}\)In the CIPV model, Assumption A4 is automatically satisfied with $C$ and $D$ known (and equal to zero and one, respectively) since $\log V(\sigma, \sigma) = \log \sigma$. In the pure CV model, the assumption that $C$ and $D$ are known is restrictive. If $C$ and $D$ are unknown, however, $h_c(\cdot)$ and $h_g(\cdot)$ are estimated consistently up to location and common scale by $\hat{h}_c(\cdot)$ and $\hat{h}_g(\cdot)$, as mentioned earlier. The appropriate divergence rate of $T$ is given in the appendix, Lemma A.1.

\(^{14}\)Recall that our procedure can provide estimates of $h_{\eta|\sigma}(\cdot|n)$ and $h_{\eta|\sigma}(\cdot|n)$ for each value of $n = 2, 3, \ldots$. Thus, in this case we estimate $h_{\eta|\sigma}(\cdot|2)$ and $h_{\eta|\sigma}(\cdot|2)$ as auctions with two bidders provide us with the largest number of bids. Because the first step involves estimating a bivariate density (see (14)), the curse of dimensionality and data availability prevent us to consider $n > 3$. If one is willing to accept the hypothesis that $n$ is exogenous so that $h_{\eta|\sigma}(\cdot|n)$ and $h_{\eta|\sigma}(\cdot|n)$ are independent of $n$, one can pool all the data. Our previous work, however, indicates that this is not the case (see Li et al., 1999).
The U.S. federal government organizes a first-price sealed-bid auction for the lease of drilling rights. Firms submit their bids, the highest bid wins the tract and the winner pays his bid provided his bid is higher than the reserve price announced prior to the auction. In principle, the presence of a reserve price induces a discrepancy between the number of actual bidders and the number \( n \) of potential bidders considered in auction theory. However, in OCS auctions the announced reserve price of $15 per acre has been acknowledged to be very low by all researchers in the field (see, McAfee and Vincent, 1992 and the work by Hendricks and Porter). Hence the reserve price does not act as an effective screening device for the participation of bidders. Likewise, the federal government has the right to reject the highest bid in any auction. Only 1.8% out of the 217 auctions were rejected, which indicates that this rejection rule does not have much effect on bidding strategies. Lastly, the federal government allows the practice of joint bidding, which could introduce some ex ante asymmetry among bidders. Only 23 tracts out of the 217 tracts received one joint bid and 9 of them were won by a joint bid. The ratio 9/23 is not statistically different from 1/2. Hence joint bidding does not seem to have introduced asymmetries. Consequently, we can consider that the framework presented in Section 2.1 constitutes a reasonable first approximation for our application.

We now turn to the data set used (see Hendricks et al., 1987 for a more detailed description of the complete data set). For each auctioned wildcat tract, we know in particular its acreage, the number of bidders, and their bids in constant 1972 dollars. As Li et al. (1999) have shown through a regression of the log of bids per acre on a complete vector of tract specific dummies, tract heterogeneity becomes insignificant conditionally on the number of bidders. That is, for auctions with a given number of bidders, one can consider that bids variability is mainly due to differences among firms and not to differences among tracts. Table 1 provides some summary statistics on the log of bids analyzed hereafter.

Table 1 shows that the overall standard deviation (STD) is largely due to the within standard deviation. This indicates a large variability of bids within

### Table 1: Summary statistics for log bids

<table>
<thead>
<tr>
<th># tracts</th>
<th>Mean</th>
<th>Minimum</th>
<th>Maximum</th>
<th>STD</th>
<th>Within STD</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 2 )</td>
<td>217</td>
<td>4.383</td>
<td>2.980</td>
<td>7.705</td>
<td>0.966</td>
</tr>
</tbody>
</table>

\[ \text{Within STD} = \sqrt{\frac{1}{L(n-1)} \sum_{i=1}^{L} \sum_{\nu=1}^{n} (\log b_{\nu i} - \log b_i)^2} \], where \( L \) is the number of such tracts.
tracts as can be seen from Fig. 1, which plots all the pairs \((b_1, b_2)\) with \(b_1 \geq b_2\). A regression of the 434 log of bids on 217 tract dummies gives an \(F\)-test of the equality of all tract dummies of 1.55 with (216, 217) degrees of freedom, which is barely significant at the 1% level.\(^\text{16}\)

3.2. Some practical issues

To implement our estimation procedure, a number of practical issues have to be addressed. First, while the range of bids for tracts with two bidders is \([19.70, 2220.28]\) in \$/acre, we observe a high concentration (about 70%) of observations in the interval \([20, 200]\), i.e. a highly skewed distribution of bids. To avoid trimming out many observations, we transform the data using the logarithmic function. With such a transformation, (5) becomes

\[
V(\sigma, \sigma) = \exp(a) \left( 1 + \frac{G_{A|a}(a|a)}{g_{A|a}(a|a)} \right) - 1 \equiv \tau(a) \tag{21}
\]

where \(a = \log(1 + b)\), \(G_{A|a}(\cdot|\cdot)\) is the conditional distribution of \(\max_{i=2,\ldots,n} \log(1 + b_i)\) given \(\log(1 + b_1)\), and \(g_{A|a}(\cdot|\cdot)\) is its density. Using the

\(^{16}\)In fact the \(F\)-test is somewhat inaccurate here because bids within tracts are not independent, as shown in Li et al. (1999) through a Blum et al. (1961) nonparametric independence test. Consequently, the \(F\)-statistic has a tendency to over-reject the null hypothesis of homogeneity.
trimming introduced in Guerre et al. (1999), the pseudo values \( \hat{V}_{i\ell} \) are defined as

\[
\hat{V}_{i\ell} = \begin{cases} 
\exp(a_{i\ell}) \left( 1 + \frac{\hat{G}_{A,a}(a_{i\ell}, a_{i\ell})}{\hat{g}_{A,a}(a_{i\ell}, a_{i\ell})} \right) - 1 & \text{if } 2\max\{h_G, h_y\} \leq a_{i\ell} \\
+ \infty & \text{otherwise}
\end{cases}
\]

(22)

for \( i = 1, \ldots, n \) and \( \ell = 1, \ldots, L \). Here, \( a_{\max} \) is the maximum value of all log-bids, i.e. \( a_{\max} = \max_{i,\ell} a_{i\ell} \). This trimming is necessary in view of boundary effects in kernel estimation.\(^{17}\) As in Section 2.3, the nonparametric estimates \( \hat{G}_{A,a}(\cdot, \cdot) \) and \( \hat{g}_{A,a}(\cdot, \cdot) \) are obtained from (13) and (14), with the exception that all bids are now in \( \log(1 + b) \). The second step of our estimation procedure only uses the pseudo values that are finite as defined in (22). We now discuss the practical issues specific to the first and second steps.

In the first step, we need to address the choice of the kernel functions \( K_G \) and \( K_g \) and their corresponding bandwidths \( h_G \) and \( h_g \). Though the choice of kernels does not have much effect in practice, we choose a kernel with compact support that is continuously differentiable on its support including the boundaries so as to satisfy the assumptions in Guerre et al. (1999). Numerous kernel functions satisfy these properties (see Hardle, 1991). This is the case for the triweight kernel defined as

\[
K(u) = \frac{35}{32} (1 - u^2)^3 1(|u| \leq 1).
\]

(23)

Thus \( K_g \) is defined as the product of two univariate triweight kernels.

In contrast, the choice of the bandwidths requires more attention. From the rates given in Li et al. (1999), we use bandwidths of the form \( h_G = c_G(nL)^{-1/5} \) and \( h_y = c_g(nL)^{-1/6} \). Note that these rates correspond to the usual ones so that \( c_G \) and \( c_g \) can be obtained by the so-called rule of thumb. Specifically, we use \( h_G = 2.978 \times 1.06\hat{\sigma}_a(nL)^{-1/5} \) and \( h_y = 2.978 \times 1.06\hat{\sigma}_a(nL)^{-1/6} \), where \( \hat{\sigma}_a \) is the standard deviation of the logarithm of \( (1 + \text{bids}) \), and the factor 2.978 follows from the use of the triweight kernel instead of the Gaussian kernel (see Hardle, 1991). Thus \( h_G \) and \( h_y \) are equal to 0.9049 and 1.1079, respectively. After performing the first step estimation and the trimming on the pseudo values \( \hat{V}_{i\ell} \), we find that the trimmed values have a mean equal to $253.41 per acre, while the

\(^{17}\) We assume that \( \sigma = 0 \). Hence, \( \hat{\sigma} = 0 \) in view of (3) and A4. The transformation \( \log(1 + \cdot) \) then ensures that \( \sigma = 0 \) and that the support \( [a, \hat{\sigma}] \) is compact as soon as \( \hat{b} < \infty \), i.e. as soon as \( \hat{\sigma} < \infty \) because \( \hat{b} = \hat{\sigma}(\hat{\sigma}) \). Thus we do not need to estimate the lower bound \( a \). Moreover, despite the log transformation, observations remain relatively sparse in the right tail. Thus, it is more cautious to use a larger trimming (such as twice the bandwidth as in (22)) than the one used in the asymptotic theory of Guerre et al. (1999).
minimum and maximum are 19.73 and 1181.38, respectively. After trimming, 174 auctioned tracts remain out of 217.

In the second step, some new practical problems are encountered. First, the use of empirical characteristic functions for estimating their corresponding densities typically produces many oscillations because of the large range of values estimated in Step 1. To mitigate this problem, we divide the logarithm of all pseudo values $\tilde{V}_i$ by 7 so as to get an interval close to $[0, 1]$. Second, as noted by Diggle and Hall (1993), the estimators (17) and (18), which are obtained by truncated inverse Fourier transformation, may have some sharp fluctuating tails. Such an unattractive feature can be alleviated by adding a damping factor to the integrals in (17) and (18). Following Diggle and Hall (1993), we introduce a damping factor defined as

$$d(t) = \begin{cases} 1 - |t|/T & \text{if } |t| \leq T, \\ 0 & \text{otherwise}. \end{cases}$$

(24)

Thus, estimators (17)–(18) are generalized to

$$\hat{h}_c(x) = \frac{1}{2\pi} \int_{-T}^{T} d(t)e^{-ix\tilde{\phi}_c(t)} \, dt,$$

(25)

$$\hat{h}_e(y) = \frac{1}{2\pi} \int_{-T}^{T} d(t)e^{-iy\tilde{\phi}_e(t)} \, dt.$$  

(26)

In practice, this damping factor will ‘smooth’ the tails of our density estimators.

Third, though the smoothing parameter $T$ can be chosen to diverge slowly as $L \to \infty$ so as to ensure the uniform consistency of our estimators, the actual choice of $T$ in finite samples has not yet been addressed in the literature. Indeed, $T$ plays here the role of a smoothing parameter, as large (small) $T$ will produce undersmoothing (oversmoothing) of the density estimate. In our case, $T$ is chosen through some empirical or data-driven criterion. As mentioned in footnote 8, we can obtain estimates for all moments of log $c$ and log $e$ from the moments of (log $V_1$, log $V_2$). Hence, using the trimmed pseudo sample of values obtained in Step 1, we can obtain estimates of the means and variances of log $c$ and log $e$ as $\hat{\mu}_c = 4.937$, $\hat{\mu}_e = 0$, $\hat{\sigma}_c^2 = 0.00222$ and $\hat{\sigma}_e^2 = 0.02195$. These two estimates can then be used to choose a value of $T$. Specifically, we try different $T$’s and obtain corresponding estimates of $h_c(\cdot)$ and $h_e(\cdot)$ through (25)–(26). From each estimated density, we compute the corresponding means and variances $\hat{\mu}_c$, $\hat{\mu}_e$, $\hat{\sigma}_c^2$ and $\hat{\sigma}_e^2$, respectively. This gives the goodness-of-fit criterion $|\hat{\sigma}_c^2 - \hat{\sigma}_c^2| + (\hat{\mu}_c - \hat{\mu}_c)^2$ for $c$, and similarly for $e$. The value of $T$ we choose minimizes the sum of these errors in percentage of $\hat{\sigma}_c^2$ and $\hat{\sigma}_e^2$. This gives $T$ equal to 50.
3.3. Empirical findings

The structural approach adopted in this paper allows us to recover the two crucial densities, namely, \( h_c(\cdot) \) and \( h_e(\cdot) \). These in turn allow us to determine the density of the common component \( v \) and the density of the private component \( \eta_i \) in both the CIPV and pure CV models, provided \( C \) and \( D \) are known in the latter case. As noted earlier, if \( C \) and \( D \) are unknown, the densities of \( \log v \) and \( \log \eta \) are recovered up to location and common scale only. In the OCS case, the economic interpretation is as follows. While \( v \) can be viewed as the ex post net value of the tract for the average "firm," \( \eta_i \) is due to firm’s private information about the tract, which include firm’s productive efficiency, opportunity costs in the CIPV model, or firm’s idiosyncratic estimate in the pure CV model. Thus, a positive \( \eta_i \) indicates that firm \( i \) has a higher (expected) value for the tract given his private information due to either higher efficiency and lower opportunity costs in the CIPV model, or higher estimate in the pure CV model.

Fig. 2 displays the estimated function \( \hat{\xi}(\cdot) \), which is the inverse of the equilibrium strategy. The vertical line corresponds to the value \( \exp(d_{\text{max}} - 2 \max\{h_G, h_g\}) \), which defines our upper trimming (see (22)). About 85% of the observed bids are below this upper trimming value. Though the function \( \hat{\xi}(\cdot) \) is not always increasing, a striking feature of Fig. 2 is that it is strictly increasing on \([0, \exp(d_{\text{max}} - 2 \max\{h_G, h_g\})]\), which corresponds to the region where \( \hat{\xi}(\cdot) \) is well estimated. In view of Proposition 2, it follows that the CIPV model is not rejected by the data. The monotonicity of \( \hat{\xi}(\cdot) \) is also in
agreement with the monotonicity of the bidding strategy \( b = s(\sigma) \) in the pure CV model.\(^{18}\)

In continuous line, Fig. 3 displays the nonparametric estimate of the density of the trimmed log \( \hat{V}_i \)'s using the triweight kernel (23) with a bandwidth given by \( h = 2.978 \times 1.06\hat{\sigma}_V(nL_T)^{-1/6} \), where \( \hat{\sigma}_V = 1.088 \) is the standard deviation of the trimmed log \( \hat{V}_i \)'s.\(^{19}\) For comparison, in dotted line we also display the normal density with mean equal to 0.494 (the empirical mean of the trimmed log \( \hat{V}_i \)'s) and standard deviation \( \hat{\sigma}_V = 1.088 \). Fig. 3 indicates that the estimated density of log \( V_i \) is not normal with two modes. Since log \( V_i = \log c + \log \varepsilon \), it follows that the distributions of \( c \) and \( \varepsilon \) (and hence of \( v \) and \( \eta \)) cannot be both log-normal. As a matter of fact, from Cramèr (1936), none of these distributions can be log-normal.

Fig. 4 displays the estimates of \( h_c(\cdot) \) and \( h_\varepsilon(\cdot) \) in continuous lines. Though the mean of log \( \varepsilon \) is zero by definition, its estimated density \( \hat{h}_\varepsilon(\cdot) \) has been centered around the estimated mean \( \hat{\mu}_\varepsilon = 4.937 \) of log \( c \) to facilitate comparison. Similarly, Fig. 4 displays the normal densities with common mean \( \hat{\mu}_\varepsilon = 4.937 \) and variances \( \hat{\sigma}_c^2 = 0.00222 \) and \( \hat{\sigma}_\varepsilon^2 = 0.02195 \). Fig. 4 confirms our earlier finding that neither the density of \( c \) nor the density of \( \varepsilon \) is log-normal. Indeed, the density \( \hat{h}_c(\cdot) \) displays a bump in its right tail, while \( \hat{h}_\varepsilon(\cdot) \) has two modes. Moreover, the

\(^{18}\)See Hendricks et al. (1999) for a recent contribution to testing the pure CV model.

\(^{19}\)The rate of the bandwidth can be obtained from the asymptotic results given in Li et al. (1999).
Fig. 4. Estimated densities $\hat{h}_v(\cdot)$ and $\hat{h}_g(\cdot)$. The former is much less spread out than the latter with a variance about ten times smaller.

As $\log v$ and $\log \eta$ are related to $\log c$ and $\log \varepsilon$ through linear transformations with a common scale factor (see (6) and (9)), the preceding findings apply to the structural densities $\hat{h}_v(\cdot)$ and $\hat{h}_g(\cdot)$ as well. In particular, whether the CIPV or pure CV model is adopted, we find that the firms’ prior distribution of the common component $v$ is ten times less spread out (as measured by the variance of the density of the logarithm) than the distribution of firms’ idiosyncratic component $\eta$. Such a result agrees with previous empirical studies stressing the variability of firms’ private information as the main source of bids’ variability in OCS auctions (see, e.g. Hendricks et al. (1987) in the pure CV model, and Li et al. (1999) in the APV model).

Fig. 4 also indicates that the estimated density of $\log v$ is essentially single-peaked, while the estimated density of $\log \eta$ has two modes. This suggests that, whatever the paradigm, firms can be broadly classified into two groups according to their private information. Such a result is confirmed by looking at the distribution of private signals $\sigma_i$. Since $\log \sigma_i = \log v + \log \eta_i = (1/D)(\log V_i - C)$, the convolution of $\hat{h}_v(\cdot)$ and $\hat{h}_g(\cdot)$ actually leads to a density $\hat{h}_\sigma(\cdot)$ with two modes similar to that of $\log V$ displayed in Fig. 3. In the CIPV model, $\sigma_i$ is firm $i$’s utility, which is directly related to its productive efficiency (including opportunity costs), while in the pure CV model $\sigma_i$ is firm $i$’s estimate of the tract. Our results thus suggest that firms can be broadly classified into two groups in terms of either cost efficiency or estimate of the tract.
Without information about the constants $C$ and $D$, nothing further can be said in the pure CV model. On the other hand, $C = 0$ and $D = 1$ in the CIPV model so that the densities of $v$ and $\eta$ are completely identified as indicated in Proposition 2. Specifically, from (6) the densities of $\log v$ and $\log \eta$ are shifted versions of the densities of $\log c$ and $\log \varepsilon$, respectively. An estimate of $E(\log \eta)$ can be obtained from $E(\log \eta) = -\log E(\varepsilon)$ with $E(\varepsilon)$ computed numerically from the density estimate $\hat{h}_\varepsilon(\cdot)$. We obtain $\hat{E}(\log \eta) = -1.816$. In particular, we find that the estimated mean of $\log v$ is $\hat{\mu}_v = 6.753$. The variances of $\log v$ and $\log \eta$ are equal to the variances of $\log c$ and $\log \varepsilon$, and thus their estimates are $\hat{\sigma}_v^2 = 0.00222$ and $\hat{\sigma}_\eta^2 = 0.02195$, respectively. Hence, as noted earlier, the density of the firm’s idiosyncratic component $\eta_i$ is much more spread out than the density of the common component $v$.

It is interesting to assess the estimated densities $\hat{h}_v(\cdot)$ and $\hat{h}_\eta(\cdot)$ in the framework of the CIPV model. As firm $i$’s private value $\sigma_i$ can be decomposed as $\log \sigma_i = \log v + \log \eta_i$ with $v$ and $\eta_i$ independent, the variance of $\log \sigma_i$ is equal to the sum of the variances of $\log v$ and $\log \eta$. The ratio $\text{Var}(\log v)/\text{Var}(\sigma)$ gives the percentage of variability of $\log \sigma_i$ explained by the variability of $\log v$. This ratio is $9.16\%$, which means that only $9.16\%$ of the variability of private values (in logarithm) is explained by the variability of the common component $v$. Alternatively, we can conclude that the variability of private values can be attributed for $90.84\%$ to the variability of firms’ specific factors. The ratio $9.16\%$ is also the linear correlation coefficient between any two private values in logarithm since $\log \sigma_i = \log v + \log \eta_i$. Hence linear correlation is low though affiliation is not negligible, as shown by Li et al. (1999) through a Blum et al. (1961) non-parametric test of independence.

4. Conclusion

In this paper, we consider the CIPI model, which is derived from the general affiliated value model by assuming that bidders’ private information are conditionally independent given some unknown common component. The CIPI model is interesting as it nests two important polar cases, which are the CIPV model and the pure CV model. We show that the CIPI model is unidentified from observed bids without additional restrictions. This leads us to assume that each bidder’s private information $\sigma_i$ can be decomposed as the product of a common component $v$ and a bidder’s idiosyncratic component $\eta_i$ that are mutually independent.

Under this additional assumption and some regularity conditions, we show that the CIPV model is fully identified. We also establish that under log-linearity of $E[v|\sigma_1 = \sigma, y_1 = \sigma]$ in $\log \sigma$, the pure CV model is essentially identified up to location and scale. We then propose a computationally convenient two-step nonparametric procedure for estimating the underlying densities of the common
component \( v \) and the idiosyncratic component \( \eta \). Our estimation procedure uses kernel estimators in a first step and empirical characteristic functions in a second step. Consistency of the resulting estimators is established by extending Li and Vuong (1998) results on the nonparametric identification and estimation of the measurement error problem with multiple indicators to the case where the indicators are estimated.

We illustrate our method by analyzing OCS wildcat auctions. Whether the CIPV model or the pure CV model is adopted, our empirical findings indicate that firms’ private information play a major role in explaining the variability of observed bids. In particular, we find that the firms’ prior distribution of the common component \( v \) is much less spread out than the distribution of firms’ idiosyncratic component \( \eta \). We also find that, whatever the paradigm, the distribution of firms’s private information is not normal with two modes. This suggests that firms can be broadly classified into two groups according to either cost efficiencies and opportunity costs in the CIPV model or estimates of the tract in the pure CV model.

Lastly, our paper has provided the first step towards the nonparametric identification and estimation of the pure CV model. An important line of research is to expand these results through other restrictions or/and under additional information. A more general goal will be to develop a complete theory of identification and estimation of the general AV models. This would allow us to discriminate among competing models such as the CIPV versus the pure CV model from auction data.

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Appendix

Proof of Proposition 1. Let \( M \) be an arbitrary CIPI model defined by the structure \([U(\cdot), F_v(\cdot), F_{\sigma v}(\cdot | \cdot)]\) for the utility function, the common component \( v \), and signals \( \sigma_i, i = 1, \ldots, n \). Define a new utility function \( \tilde{U}(\cdot) \), a new common component \( \tilde{v} \), and new signals \( \tilde{\sigma}_i, i = 1, \ldots, n \) such that
\(\tilde{U}(\tilde{\sigma}_i, v) = \tilde{\sigma}_i, \tilde{v} = v,\) and \(\tilde{\sigma}_i = E[U(v, \sigma_i)|\sigma_i, y_i = \sigma_i] \equiv \lambda(\sigma_i),\) which is strictly increasing in \(\sigma_i.\) Note that the \(\tilde{\sigma}_i\)'s are conditionally independent given \(\tilde{v}.\) Hence the new model \(\tilde{M}\) with structure \([\tilde{U}(\cdot), \tilde{F}_i(\cdot), \tilde{F}_{\sigma_i}(\cdot | \cdot)]\) for the utility function, the common component \(\tilde{v},\) and signals \(\tilde{\sigma}_i, i = 1, \ldots, n\) is a CIPV model. Moreover, it can be verified that

\[
\frac{\tilde{F}_{\sigma_i} \tilde{F}_{\sigma_i}(\cdot | \cdot)}{\tilde{F}_{\sigma_i}(\cdot | \cdot)} = \frac{1}{\lambda^{-1}(\cdot)} \frac{1}{\lambda^{-1}(\cdot)}.
\]  

(A.1)

Therefore, comparing the differential equations (2) for the CIPI model \(M\) and the CIPV model \(\tilde{M}\) subject to their respective boundary conditions, and using \(V(\sigma, \sigma) = \lambda(\sigma),\) it follows that the equilibrium strategies in \(M\) and \(\tilde{M}\) are related by \(\tilde{s}(\cdot) = s[\tilde{\lambda}^{-1}(\cdot)].\) Hence \(\tilde{b} = \tilde{s}(\tilde{\sigma}) = s(\sigma)\) Thus the equilibrium bid distribution in \(M\) is equal to that in \(\tilde{M},\) i.e. \(M\) is observationally equivalent to \(\tilde{M}.\)

**Proof of Lemma 1.** The characteristic functions \(\phi_c(\cdot)\) and \(\phi_\varepsilon(\cdot)\) of \(\log c\) and \(\log \varepsilon\) are related to those of \(\log v\) and \(\log \eta\) by

\[
\phi_c(t) = \phi_c(t)e^{itE[\log \eta]}, \quad \phi_\varepsilon(t) = \phi_\varepsilon(t)e^{-itE[\log \eta]}.
\]  

(A.2)

Hence A3 implies that \(\phi_c(\cdot)\) and \(\phi_\varepsilon(\cdot)\) are also nonvanishing everywhere. Thus, given A1–A3, \(\log c\) and \(\log \varepsilon\) satisfy the assumptions of Lemma 2.1 in Li and Vuong (1998). The desired result follows.

**Proof of Proposition 2.** The identification of \([F_c(\cdot), F_\varepsilon(\cdot)]\) in the CIPV model follows from (i) the identification of the joint distribution of \((\sigma_1, \ldots, \sigma_n)\) from observed bids because a CIPV model is an APV model, which is identified by Li et al. (1999, Proposition 2.1), (ii) the identification of the distributions of \(\log c\) and \(\log \varepsilon\) from the joint distribution of \((\log \sigma_1, \ldots, \log \sigma_n)\) by Lemma 1, and (iii) the equalities \(\log v = \log c - E[\log \eta]\) and \(\log \eta = \log \varepsilon + E[\log \eta],\) where \(E[\log \eta] = -\log E[\varepsilon]\) from the normalization \(E[\eta] = 1.\)

The proof of the second part is similar to the second part of the proof of Proposition 2.1 in Li et al. (1999) with the exception that the utilities \(\sigma_i\) are now conditionally independent. Note that variables that are conditionally independent given some other variable are necessarily affiliated.

**Proof of Proposition 3.** The proof is similar to the first part of the proof of Proposition 2. Specifically, in (i) the joint distribution of observed bids now determines uniquely the joint distribution of \((V(\sigma_1, \sigma_1), \ldots, V(\sigma_n, \sigma_n))\) from (5), (ii) stays the same with \(\sigma_i\) replaced by \(V(\sigma_i, \sigma_i)\) and (A.2) replaced by

\[
\phi_c(t) = \phi_c(Dt)e^{itc + DE[\log \eta]}, \quad \phi_\varepsilon(t) = \phi_\varepsilon(Dt)e^{-itDE[\log \eta]},
\]  

(A.3)

while (iii) now uses (9). In particular, solving (9) for \(\log v\) and \(\log \eta,\) and using \(E[\log \eta] = -\log E(e^{1/s})\) from the normalization \(E(\eta) = 1\) give the desired result.
Proof of Eq. (10). Under the multiplicative decomposition, we have \( \sigma_1 = v \eta_1 \) and \( y_1 = v \max_{j \neq 1} \eta_j \). Hence, using the Jacobian of the transformation, the joint density of \((v, \sigma_1, y_1)\) is

\[
f(v, \sigma_1, y_1) = [(n-1)/v^2] f_\eta(\sigma_1/v) f_\eta(y_1/v) F_n^{-2}(y_1/v) f_v(v).
\]

Hence

\[
E[v | \sigma_1 = \sigma, y_1 = \sigma] = \frac{\int_0^{\infty} (1/v) f_\eta^2(\sigma/v) F_n^{-2}(\sigma/v) f_v(v) \, dv}{\int_0^{\infty} (1/v^2) f_\eta(\sigma/v) F_n^{-2}(\sigma/v) f_v(v) \, dv}.
\]

(A.4)

Suppose now that \( f_v(v) \propto 1/v^2 \). Thus we obtain

\[
E[v | \sigma_1 = \sigma, y_1 = \sigma] = \frac{\int_0^{\infty} u^{-1} f_\eta^2(u) F_n^{-2}(u) \, du}{\int_0^{\infty} u^{-2} f_\eta^2(u) F_n^{-2}(u) \, du},
\]

where the second equality follows from the change of variable \( u = \sigma/v \). The desired Eq. (10) follows from interpreting the proportionality factor of \( \sigma \) and taking the logarithm. \( \Box \)

Proof of Eq. (11). Given A1–A2, the multiplicative decomposition, and the log-normality of \((v, \eta_1, \eta_2)\), we have

\[
\begin{pmatrix}
\log v \\
\log \sigma_1 \\
\log \sigma_2
\end{pmatrix} 
\sim \mathcal{N}
\begin{pmatrix}
\begin{pmatrix}
\mu_v \\
\mu_v + \mu_\eta
\end{pmatrix} & \begin{pmatrix}
\sigma_v^2 & \sigma_v^2 \\
* & \sigma_\eta^2 + \sigma_v^2
\end{pmatrix} & \begin{pmatrix}
\sigma_v^2 \\
* & \sigma_\eta^2 + \sigma_v^2
\end{pmatrix}
\end{pmatrix}.
\]

Hence, \( v | \sigma_1, \sigma_2 \sim \text{LN}(\mu_v, \sigma_v^2) \), where

\[
\mu_v = \mu_v + (\sigma_v^2, \sigma_v^2) \Sigma^{-1} \begin{pmatrix}
\log \sigma_1 - \mu_v - \mu_\eta \\
\log \sigma_1 - \mu_v - \mu_\eta
\end{pmatrix},
\]

\[
\sigma_v^2 = \sigma_v^2 \left( 1 - (\sigma_v^2, \sigma_v^2) \Sigma^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right),
\]

with \( \Sigma \) being the covariance matrix of \((\log \sigma_1, \log \sigma_2)\). Using \( \log E[V | \sigma_1, \sigma_2] = \mu_v + \sigma_v^2/2 \) and \( \sigma_1 = \sigma_2 = \sigma \) gives (11) after some elementary algebra. \( \Box \)

Proof of Eq. (12). From (9) and (11), we have \( \sigma_\eta^2 = D^2 \sigma_v^2 \) and \( \sigma_\eta^2 = D^2 \sigma_v^2 \) with \( D = 2 \sigma_v^2/(\sigma_\eta^2 + 2 \sigma_v^2) \). Elementary algebra gives (12), which expresses \( \sigma_v^2 \) and \( \sigma_\eta^2 \) as function of \( \sigma_v^2 \) and \( \sigma_\eta^2 \). Using (9) and (11) again, we have \( \mu_v = \mu_v + [\sigma_v^2 \sigma_\eta^2 / 2(\sigma_\eta^2 + 2 \sigma_v^2)] \). By replacing \( \sigma_\eta^2 \) and \( \sigma_v^2 \) by their expressions in
Proof of Theorem 1. We need to prove the first part only as the second part is straightforward. We note that

$$\log V_{i'} = \log c + \log \varepsilon_{i'}, \quad i = 1, \ldots, n, \ell = 1, \ldots, L,$$

where $V_{i'} = \sigma_{i'}$ in the CIPV model and $V_{i'} = \mathbb{E}[\epsilon_{i'}|\sigma_{i'}, y_{i'} = \sigma_{i'}]$ in the pure CV model. If the $V_{i'}$'s were observed, one would have a measurement model with $n$ indicators. Moreover, given $A1$–$A6$, $\log c$ and $\log \varepsilon$ satisfy all the assumptions in Li and Vuong (1998) in view of (6), (9), (A.2) and (A.3). Hence Theorem 1 would directly follow from Li and Vuong (1998) results.

The $V_{i'}$'s are, however, unobserved but they can be estimated by $\hat{V}_{i'}$s from (15). Now, in Li and Vuong (1998), the crucial result upon which the uniform convergence of the density estimators (17) and (18) is established is given by Lemma 4.1 in that paper. Here, we prove the following Lemma A.1 which plays an analogous role. The difference between Lemma A.1 and Lemma 4.1 in Li and Vuong (1998) is that here we deal with indicators that are (trimmed) estimates in Li et al. (1999, Proposition A2) as $L \to \infty$, while Li and Vuong (1998) deal with indicators $V_{i'}$ that are observed. Once Lemma A.1 is established, Theorem 1 can be proved by following the proofs of Theorems 3.1–3.4 in Li and Vuong (1998).

Hereafter, we let $v_{i'} \equiv \log V_{i'}$ and $\hat{v}_{i'} \equiv \log \hat{V}_{i'}$. Let $d_L = \sup_{i'} |\hat{v}_{i'} - v_{i'}|$ and $c_L = \sup F_{L_{i'}}(\mu_i') - F_i(\mu_i')$, where $F_i(\mu_i')$ is the joint distribution of any two bidders’ $v_{i'}$ and $\hat{V}_{i'}$ is its (infeasible) empirical counterpart obtained by using the true but unobserved $v_{i'}$ of any two bidders. Note that $d_L$ converges to zero from Li et al. (1999, Proposition A2) as $L \to \infty$, while $c_L$ converges to zero from the log–law (see Serfling, 1980) as $L_T \to \infty$.


(i) Suppose $|v_i(0, t)| \geq d_1 |t|^{-\beta}$ as $t \to \infty$ for some positive constants $d_1$ and $\beta$. Then

$$\sup_{t \in [-T, T]} |\hat{\phi}_c(t) - \phi_c(t)| = e_L^{1-x},$$

where $0 < x < 1$ and $T = e_L^{x(2(1+\beta))}$.

(ii) Suppose $|v_i(0, t)| \geq d_0 |t|^{\beta_0} \exp( - |t|^{\beta/\gamma})$ as $t \to \infty$ for some positive constants $d_0$, $\beta$, $\gamma$ and constant $\beta_0$. Then

$$\sup_{t \in [-T, T]} |\hat{\phi}_c(t) - \phi_c(t)| = e_L^{1-x} \left( \log \frac{1}{e_L} \right)^{\frac{2}{1-\beta_0}/\beta},$$

where $0 < x < 1$, and $T = [-(\beta/2)\log e_L]^{1/\beta}$. 

Proof. From (16) we only need to prove that this lemma holds when

$$
\hat{\psi}(u_1, u_2) = \frac{1}{L_T} \sum_{\ell=1}^{L_T} \exp(iu_1 \hat{v}_{1\ell} + iu_2 \hat{v}_{2\ell}),
$$

(A.5)

where $\hat{v}_{1\ell}$ and $\hat{v}_{2\ell}$, $\ell = 1, \ldots, L_T$ are estimated $v_{i\ell}$ for any two bidders, say bidder 1 and bidder 2. Thus the result for $\hat{\psi}(\cdot, \cdot)$ defined by (16) can be readily obtained as it is an average of (A.5) among $n$ bidders imposing symmetry. Hereafter, we redefine (19) in terms of $\hat{\psi}(\cdot, \cdot)$ given by (A.5) instead of (16).

Using (7) and (19), a Taylor series expansion gives

$$
\frac{d}{d\lambda} \left[ \int_0^t \frac{\partial \hat{\psi}(0, u_2) / \partial u_1}{\hat{\psi}(0, u_2)} \, du_2 \right] = \int_0^t \frac{\partial \hat{\psi}(0, u_2) / \partial u_1}{\hat{\psi}(0, u_2)} \, du_2.
$$

(A.6)

Note that $A(u_2)$ does not depend on $\lambda$. Moreover, the derivative with respect to $\lambda$ of the term in brackets in the denominator of (A.6) is

$$
B(u_2) = \hat{\psi}(0, u_2) - \psi(0, u_2),
$$
which is independent of \( \lambda \). Thus successive differentiation of (A.8) gives for any \( k \geq 1 \)

\[
d_k T(\psi; \hat{\psi} - \psi) = (-1)^{k-1} k! \int_0^T A(u_2) B(u_2)^{k-1} du_2.
\]  
(A.9)

Now let us consider \( A(u_2) \) and \( B(u_2) \), respectively. Define

\[
\bar{\psi}(u_1, u_2) = \frac{1}{L_T} \sum_{\ell = 1}^{L_T} \exp(iu_1 v_{1\ell} + iu_2 v_{2\ell}),
\]  
(A.10)

where \( v_{1\ell} \) and \( v_{2\ell} \), \( \ell = 1, \ldots, L_T \), are true but unobserved \( v_{i\ell} \) for any two bidders, say bidder 1 and bidder 2. Thus

\[
B(u_2) = \hat{\psi}(0, u_2) - \bar{\psi}(0, u_2) + \bar{\psi}(0, u_2) - \psi(0, u_2)
\]  

\[
= \frac{1}{L_T} \sum_{\ell = 1}^{L_T} \left( e^{iu_1 v_{1\ell}} - e^{iu_1 v_{1\ell}} + e^{iu_2 v_{2\ell}} \right) + \int e^{iu_2 v_2} d(F_{L_T}^{(2)} - F^{(2)}),
\]  
(A.11)

where \( F^{(2)} \) is the joint distribution of \( v_1 \) and \( v_2 \) while \( F_{L_T}^{(2)} \) is its (infeasible) empirical counterpart obtained by using \( (v_{1\ell}, v_{2\ell}) \), \( \ell = 1, \ldots, L_T \).

Now for the first summand of the last equality, we have

\[
\frac{1}{L_T} \sum_{\ell = 1}^{L_T} (e^{iu_2 v_{2\ell}} - e^{iu_2 v_{2\ell}}) \leq \frac{1}{L_T} \sum_{\ell = 1}^{L_T} |e^{iu_2 v_{2\ell}} - e^{iu_2 v_{2\ell}}|
\]

\[
\leq \frac{1}{L_T} \sum_{\ell = 1}^{L_T} \sum_{q = 1}^{\infty} u_2^q |v_{2\ell} - v_{2\ell}|^q
\]

\[
\leq \sum_{q = 1}^{\infty} u_2^q d_q.
\]

On the other hand, in view of A6, (6) and (9), \( |v_{i\ell}| \leq M \), where \( M > 0 \). Let \( c_{L_T} = \sup |F_{L_T}^{(2)} - F^{(2)}| \). We use the inequality

\[
\left| \int \mathcal{A} f(y_1, y_2) dg(y_1, y_2) \right|
\]

\[
\leq \left| \int \mathcal{A} g(y_1, y_2) \frac{\partial^2 f(y_1, y_2)}{\partial y_1 \partial y_2} \, dy_1 \, dy_2 \right| + 8M \sup_{(y_1, y_2) \in \mathcal{A}} |g(y_1, y_2)|
\]

\times \left( \sup_{(y_1, y_2) \in \mathcal{A}} |f(y_1, y_2)| + \sup_{(y_1, y_2) \in \mathcal{A}} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right| + \sup_{(y_1, y_2) \in \mathcal{A}} \left| \frac{\partial f(y_1, y_2)}{\partial y_2} \right| \right),
\]

where \( \mathcal{A} = [-M, M] \times [-M, M] \) (see Csörgö, 1980). Applying this inequality to the second summand of (A.11) gives

\[
|\bar{\psi}(0, u_2) - \psi(0, u_2)| = \left| \int e^{iu_2 v} d(F_{L_T}^{(2)} - F^{(2)}) \right| \leq 8M(1 + u_2)c_{L_T},
\]  
(A.12)
Hence, we obtain
\[ |B(u_2)| \leq \sum_{q=1}^{\infty} \frac{u_q^2 d_q^q}{q!} + 8M(1 + u_2)C_{L,r} = b_L(u_2). \]  
(A.13)

For \( A(u_2) \), we have
\[
\frac{\partial \hat{\psi}(0, u_2)}{\partial u_1} - \frac{\partial \psi(0, u_2)}{\partial u_1} = \frac{\partial \hat{\psi}(0, u_2)}{\partial u_1} - \frac{\partial \psi(0, u_2)}{\partial u_1} + \frac{\partial \hat{\psi}(0, u_2)}{\partial u_1} - \frac{\partial \psi(0, u_2)}{\partial u_1}
\]
\[
= \frac{1}{L_T} \sum_{l=1}^{L_T} \left( i \hat{v}_{1l} e^{i u_2 \hat{v}_{2l}} - iv_{1l} e^{i u_2 v_{2l}} \right) + \int iv_1 e^{i u_2 v_2} d(F_L^{(2)} - F^{(2)}). \]  
(A.14)

Similarly to \( B(u_2) \), we can show that the first summand of the last equality satisfies
\[
\left| \frac{1}{L_T} \sum_{l=1}^{L_T} \left( i \hat{v}_{1l} e^{i u_2 \hat{v}_{2l}} - iv_{1l} e^{i u_2 v_{2l}} \right) \right|
\]
\[
= \left| \frac{1}{L_T} \sum_{l=1}^{L_T} \left( i \hat{v}_{1l} - iv_{1l} \right) e^{i u_2 \hat{v}_{2l} - e^{i u_2 v_{2l}} \right) \right|
\]
\[
\leq \frac{1}{L_T} \sum_{l=1}^{L_T} \left( d_L + M \sum_{q=1}^{\infty} \frac{u_q^2 d_q^q}{q!} \right)
\]
\[
= d_L + M \sum_{q=1}^{\infty} \frac{u_q^2 d_q^q}{q!}. \]

Following (A.8) the second summand of (A.11) satisfies
\[
\left| \int iv_1 e^{i u_2 v_2} d(F_L^{(2)} - F^{(2)}) \right| \leq 4M^2 u_2 c_{L,r} + 8M(M + 1 + Mu_2)C_{L,r}. \]

Therefore, from the definition of \( A(u_2) \), (A.13), and \( |\partial \psi(0, u_2)/\partial u_1| < M \), we obtain
\[
|A(u_2)| \leq d_L + 2M \sum_{q=1}^{\infty} \frac{u_q^2 d_q^q}{q!} + 20M^2(1 + u_2)C_{L,r}
\]
\[
+ 8M c_{L,r} \equiv a_L(u_2). \]  
(A.15)

Hence, from (A.7) and (A.9) we obtain
\[
|A_L(t)| \leq \sum_{k=1}^{\infty} \int_0^t a_L(u_2) b_L(u_2)^{k-1} \frac{d u_2}{|\psi(0, u_2)|^{k+1}}. \]  
(A.16)
(i) Under the assumption that $|\psi(0, t)| \geq d_1 |t|^{-\beta}$ as $t \to \infty$, then there exists $a > 0$ such that $|\psi(0, t)| \geq d_1 |t|^{-\beta}$ for $|t| \geq a$. Let $b = \min_{|t| \leq a} |\psi(0, t)|$. Choosing $T$ large enough such that $T > a$, we have $|\psi(0, t)| \geq d_1 |t|^{-\beta} \geq d_1 |T|^{-\beta}$ for $t \in [-T, T] \setminus [-a, a]$. Also for $T$ large enough, then for $t \in [-a, a]$ we have $|\psi(0, t)| \geq b \geq d_1 |T|^{-\beta}$. Therefore, for $t \in [-T, T]$ where $T$ is large enough, we obtain

$$|\psi(0, t)| \geq d_1 |T|^{-\beta}.$$  

On the other hand, by the log–log law (see Chung, 1949; Serfling, 1980),

$$c_{Lt} \equiv \sup |F_L^{(2)} - F^{(2)}| = O\left(\frac{L_T}{\log \log L_T}\right)^{-1/2}. \tag{A.17}$$

Let $e_L = \max\{c_{Lt}, d_L\}$. Then we may choose $T$ appropriately such that $T^{1+\beta}e_L < 1$ as $L \to \infty$. Hence for $t \in [-T, T]$ where $T$ is large enough, (A.16) gives after some algebra

$$|A_L(t)| \leq \sum_{k=1}^{\infty} \int_0^T \frac{B^k(T^k + T^{-k-1})e_L^k}{T^{-\beta(k+1)}} \, du_2$$

$$= A T^{2(1+\beta)} e_L - \frac{A T^{1+2\beta} e_L}{1 - A T^{1+\beta} e_L},$$

for some constant $A$. Let $T = e_L^{-2/(1+\beta)}$ where $0 < \alpha < 1$, then for $t \in [-T, T]$,

$$|A_L(t)| = O(e_L^{1-\beta}).$$

It follows that for $L$ large enough and for $t \in [-T, T]$, $|A_L(t)|$ is smaller than one. Hence, from (A.6)

$$\sup_\{t \in [-T, T]\} |A_L(t)| = O\left(\sup_\{t \in [-T, T]\} |A_L(t)|\right).$$

The desired result follows.

(ii) Now assume that $|\psi(0, t)| \geq d_0 |t|^\rho \exp(-|t|^\gamma)$. By using the same argument as in (i), we can show that for $|t| \leq T$, where $T$ is large enough,

$$|\psi(0, t)| \geq d_0 T^\rho \exp(-T^\gamma).$$

Therefore, (A.16) gives after some algebra for $|t| \leq T$,

$$|A_L(t)| \leq \sum_{k=1}^{\infty} \int_0^T \frac{B^k(T^k + T^{-k-1})e_L^k}{T^{\beta(k+1)} \exp(-(k+1)T^\gamma)} \, du_2$$

$$= B e_L T^{2(1-\rho)} \exp(2T^\gamma) \frac{1}{1 - B e_L T^{1-\rho} \exp(T^\gamma)} + B e_L T^{1-2\rho} \exp(2T^\gamma) \frac{1}{1 - B e_L T^{1-\rho} \exp(T^\gamma)}.$$
where $B$ is an appropriate constant, provided $B e_L T^{1 - \beta_0} \exp(T^{\beta}/\gamma) < 1$. Let

$$T = \left[ -\frac{\gamma}{2} \log e_L \right]^{1/\beta},$$

where $0 < \alpha < 1$. Then

$$\sup_{t \in [-T,T]} |A(t)| = O\left(e_L^{1-\alpha} \left[ \frac{\gamma}{2} \log \frac{1}{e_L} \right]^{2(1-\beta_0)/\beta} \right) + O\left(e_L^{1-\alpha} \left[ \frac{\gamma}{2} \log \frac{1}{e_L} \right]^{(1-2\beta_0)/\beta} \right)$$

$$= O\left(e_L^{1-\alpha} \left( \log \frac{1}{e_L} \right)^{2(1-\beta_0)/\beta} \right).$$

Hence, using (A.6) and the same argument as in (i) yield the desired result.

References


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