A Bayesian approach to dynamic macroeconomics

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Abstract

We propose and implement a coherent statistical framework for combining theoretical and empirical models of macroeconomic activity. The framework is Bayesian, and enables the formal yet probabilistic incorporation of uncertainty regarding the parameterization of theoretical models. The approach is illustrated using a neoclassical business-cycle model that builds on the Greenwood et al. (1988, American Economic Review 78, 402–417) variable-utilization framework to study out-of-sample forecasting of output and investment. The forecasts so produced are comparable with those from a Bayesian vector autoregression. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

In conducting empirical research, guidance from theory is invaluable. Yet in macroeconomics, as theoretical models have become increasingly complex and stylized, such guidance has become increasingly difficult to incorporate in
a formal statistical sense. Here, we describe and implement a coherent statistical framework for combining the use of theoretical and empirical models. The framework is Bayesian, and enables the formal incorporation of uncertainty regarding the parameterization of theoretical models.

In principle, the incorporation of parameter uncertainty is straightforward in a Bayesian analysis: combine prior and likelihood functions to obtain posterior distributions of functions of interest. But there are two difficulties that arise in working with fully specified dynamic macroeconomic models: the exact form of the associated likelihood function is generally unknown, and even given an approximation of the likelihood function, posterior analysis cannot be carried out analytically. To resolve these difficulties, we use a Kalman filter to compute the likelihood function for the data given a log-linear approximation to the solution of the theoretical model, and conduct posterior analysis using importance-sampling techniques recently developed in the Bayesian literature.

Having obtained posterior distributions for parameters, the theoretical model may be put to one of its many uses: explaining the behavior of observed data, performing policy analysis, or forecasting future movements in macroeconomic variables. In a companion paper (DeJong et al., 2000), we studied whether a particular theoretical model suggested by Greenwood et al. (GHH, 1988) could be used to explain cyclical variation in output and investment. Here, we study whether this model can usefully be employed to perform out-of-sample forecasting. Perhaps surprisingly, the restrictions embodied in the model are quite helpful predictively, and forecasts produced from the model are comparable with those produced from a Bayesian vector autoregression (BVAR).

Our approach stands in contrast to other estimation procedures used in the macroeconomics literature. Calibration, as it is usually practiced, lacks a formal statistical foundation (see Kim and Pagan, 1994), and like other limited-information methods (e.g., Hansen’s (1982) Generalized Method of Moments; and Lee and Ingram’s (1991) Method of Simulated Moments), focuses on a limited number of moments. Classical statistical approaches, including Maximum Likelihood (McGrattan et al., 1993) and GMM (Christiano and Eichenbaum, 1992), cannot easily accommodate even the most non-controversial prior information about parameters (e.g., that the rate of time discount be less than unity). Often, particular parameters are fixed before other parameters are estimated, either to accommodate prior beliefs or to deal with identification issues (e.g., Altug, 1989; Ireland, 1999). However, the final assessment of the model does not usually incorporate this aspect of the estimation.

Our procedure is best characterized as a Bayesian approach to the fitting of dynamic stochastic general equilibrium models to data. However, all of the implications of the model for the likelihood are taken into account in estimation; in addition, it is possible to accommodate the views of the researcher – the sort that lead to ‘point mass’ priors of some of the calibration literature, or that lead
researchers using GMM or MSM to fit using a selected set of moments. Our method thus encompasses both limited-information methods such as calibration, and full-information MLE. The application that we pursue, however, lies between calibration and maximum likelihood, and illustrates the use of a specific informative, non-degenerate prior.

Although the point of this paper is to outline a method for the estimation of parameters in dynamic stochastic general equilibrium models, this method also lends itself to the evaluation of the fit of a model. Assessing fit in the Bayesian framework is always done with respect to an explicit alternative, via production of posterior odds ratios or ‘Bayes factors’. These have classical analogues in likelihood ratios, but go beyond comparison of two parameter vectors of greatest likelihood to involve information about other possible values by comparing weighted-average values of likelihoods, with weights given by the priors. What is important in this approach to fit assessment, just as was the case in estimation itself, is that it embraces all of the implications the equilibrium model carries for the data. By doing so, our method differs from many of the techniques suggested elsewhere (e.g., Gregory and Smith, 1991; Watson, 1993; Canova, 1995; DeJong et al., 1996; Geweke, 1999). In these papers, the measure of fit tends to focus on a particular feature of the likelihood function. Gregory and Smith focus on a subset of moments, and ask whether the model has sources of uncertainty rich enough to produce moments in simulated data that are statistically close to those moments calculated using historical data. Watson focuses on the spectral properties of particular series, and asks how much error must be added to the model to make it consistent with the observed data. DeJong et al. compare posterior distributions of parameters or moments from models to those produced by otherwise unrestricted VARs; Canova carries out a similar but classical exercise. Geweke generalizes and formalizes the DeJong et al. procedure for joint, multidimensional measures.

Beyond the methodological contribution, we believe that our application is of interest in its own right. Although there is considerable interest among macroeconomists in developing models that can provide reasonable forecasts of macroeconomic variables, little work has been done in trying to use fully specified, dynamic stochastic general equilibrium models for that purpose. Ingram and Whiteman (1994) investigated whether a simpler real business cycle model (that of King et al., 1988) could be used as a source of a prior distribution on comovements in variables in the data, which would be used in place of the ‘Minnesota’ prior typically employed in Bayesian vector autoregression forecasting exercises. The focus there was on supplying prior information on reduced-form VAR coefficients. Here, we operate exclusively within the context of the theoretical model; prior information is brought to bear on parameters of the economic model, but the (linearized) reduced form of the model carries with it all of the theoretical restrictions implied by the real business cycle model.
The paper is laid out in a fashion that parallels how we use our approach to analyze macroeconomic models. First, in Section 2 we introduce a specific model and discuss its properties. We next address (Section 3) the solution of the model. This results in a linear observer system. In Section 4 we discuss building the posterior distribution of the parameters of the model by combining the likelihood function for the observer system with prior distributions for the parameters. The priors are of course subjective; in practice, we try to ground them in economic theory. The observer system likelihood function can be evaluated using a Kalman filter; together with prespecified prior distributions, numerical sampling techniques can be used to build up the posterior. Finally, in Section 5 we turn to a question of interest; here, this involves the empirical predictions of the model.

2. A real business cycle model with investment shocks and variable capital utilization

While the original King et al. (1988) model is a benchmark for the Real Business Cycle literature, that model has a single shock and a single sector. Thus, the last decade has seen the evolution of RBC models along many dimensions, including among others the inclusion of a home production sector (Benhabib et al., 1991), multiple production sectors (Huffman and Wynne, 1999), human capital accumulation (Perli and Sakellaris, 1998), and multiple sources of uncertainty (Ingram et al., 1994). Hence, there are many models we could use to illustrate our approach, which would apply equally in any of them.

The model we use to illustrate our approach, analyzed first by Greenwood et al. (1988), is simple but general enough to accommodate the analysis of time series on both output and investment. In this model, a representative agent gains utility from consumption, $c_t$, and disutility from labor, $l_t$, and seeks to maximize expected discounted lifetime utility,

$$\max_{E_0} \sum_{t=1}^{\infty} \beta^t \frac{1}{1 - \gamma} \left[ \left( c_t - \frac{l_t^{1+\theta}}{1 + \theta} \right)^{1-\gamma} - 1 \right],$$

where $\beta$ controls the rate of time discount, $0 < \beta < 1$, the parameter $\gamma$ controls the degree of risk aversion, $\gamma > 0$, and $\theta$ controls the elasticity of labor supply, $\theta > 0$. The production technology available to the agent differs from the

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1 In this model, preferences are designed to remove the intertemporal aspect of the labor–leisure choice. Although this feature simplifies the interpretation of the model, it may pose problems if one is interested in analyzing historically observed labor hours.
neoclassical constant-returns-to-scale specification in two respects: there is a variable rate of capital utilization (specified by GHH) and a technology shock (our addition to the GHH specification):

\[ y_t = A_t(k_t h_t)^{\rho_t} e_t^{1-z}, \]

where \( y_t \) is output in period \( t \), \( A_t \) is the technology shock, \( k_t \) is the capital stock at the beginning of \( t \) (initial capital \( k_0 \) is given), and \( h_t \) is an index of period-\( t \) utilization. Higher capital utilization speeds depreciation, hence the evolution of capital is given by

\[ k_{t+1} = k_t (1 - \delta_0 \omega^{-\frac{1}{2}} h_t^\omega) + i_t \epsilon_t, \]

where \( \omega > 1 \), \( i_t \) is investment, and \( \epsilon_t \) is a shock to the marginal efficiency of investment. Finally, there is a resource constraint

\[ y_t = c_t + i_t. \]

In addition, we specify \( \ln(A_t) \) and \( \ln(\epsilon_t) \) to be independent univariate first-order autoregressions. (We depart from GHH in the specification of a continuous-state Markov process for \( \epsilon_t \); they assumed \( \epsilon_t \) was described by a two-state Markov process.) The autoregressions are given by

\[ \ln A_t = \rho_A \ln A_{t-1} + v_{A_t}, \]
\[ \ln \epsilon_t = \rho_\epsilon \ln \epsilon_{t-1} + v_{\epsilon_t}. \]

The choice of AR(1) specifications for the shock processes follows standard practice in the RBC literature.

In DeJong et al. (2000), we show that the model yields a simple expression for capital utilization and the supply of labor hours as a function of the beginning-of-period capital stock and \( A_t \) and \( \epsilon_t \). Hence neither variable responds to expected future shocks, which implies that the level of output is also unresponsive to expected future shocks. In the current period, positive productivity shocks directly raise labor supply, utilization and output; the increase in output is divided between consumption and investment, as agents attempt to smooth their lifetime consumption profile. Positive investment shocks increase current-period utilization, thus increasing labor supply and output. In addition, investment shocks reduce the relative price of capital, and hence induce agents to augment the future capital stock through additional investment. In periods following the realization of a positive shock of either type, responses of output and consumption remain positive, while the responses of utilization and investment become negative (i.e., utilization and investment fall below their steady-state values) due to the buildup in the capital stock undertaken in the period in which the shock was realized.
3. Solution procedure

The model on which we have focused cannot be solved analytically. While there are many methods for the numerical solution of this sort of model, the linearization method described by Christiano (1990) and Sims (1997) is the most appealing for our purposes. The method produces solutions quickly, enabling us to solve the model for many different values of the underlying parameters in a reasonable amount of time. In addition, the approximate solution produces a tractable form for the likelihood function.

The agent’s first-order conditions for choosing labor, utilization, consumption and capital are

\[ \ell_t^{\theta+1} = (1 - \alpha)y_t, \] (1)
\[ \delta_0 h_t^\alpha = \alpha e_t y_t / k_t, \] (2)
\[ \frac{U_{ct}}{e_t} = \beta E_t \left\{ U_{ct+1} \left( \frac{1}{e_{t+1}} + \alpha \frac{y_{t+1}}{k_{t+1}} \left( 1 - \frac{1}{\omega} \right) \right) \right\}, \] (3)

to which are appended the constraints and exogenous laws of motion

\[ k_{t+1} = k_t (1 - \delta_0 \omega^{-1} h_t^\alpha) + i_t, \] (4)
\[ y_t = c_t + i_t, \] (5)
\[ y_t = A_t (k_t h_t)^{\gamma \ell_t^{1-\gamma}}, \] (6)
\[ \ln A_t = \rho_A \ln A_{t-1} + \nu_A, \] (7)
\[ \ln e_t = \rho_e \ln e_{t-1} + \nu_e. \] (8)

A final necessary condition for optimality is the transversality condition, which requires that the capital stock not grow too quickly.

The set of equations (1)–(8) can be log-linearized around the non-stochastic steady state of the model’s eight variables to yield a first-order linear difference equation

\[ G_0 x_t = G_1 x_{t-1} + G_v e_t, \]
where

\[ e_t = \left[ \eta_t \bar{v}_t \right]', \]

\[ x_t = \left[ \ln \left( \frac{y_t}{y_t} \right) \ln \left( \frac{c_t}{\bar{c}} \right) \ln \left( \frac{k_{t+1}}{k_t} \right) \ln \left( \frac{h_{t+1}}{h_t} \right) \ln \left( \frac{\bar{c}_t}{\bar{c}} \right) \right]', \]

upper bars denote steady-state values, and \( E_t \eta_{t+1} = 0 \). The random variable \( \eta_{t+1} \) is introduced by dropping the conditional expectation in (3) and adding an expectation error to the right-hand side of (3). The matrices \( G_0 \) and \( G_1 \) are square with, in our case, dimension 8 \( \times \) 8; the matrix \( G_e \) has dimension 8 \( \times \) 3. If \( G_0 \) has full rank, we can write \( x_t = G_0^{-1}G_1x_{t-1} + G_0^{-1}G_e e_t \). The behavior of the system is governed by the eigenvalues of \( G_0^{-1}G_1 \). In this model, a unique solution exists when exactly one eigenvalue exceeds \( 1/\sqrt{\beta} \) in absolute value. To find the stable subspace in which the solution lies, an eigenvalue–eigenvector decomposition is calculated: \( G_0^{-1}G_1 = CAC^{-1} \), where \( C \) is the matrix of eigenvectors and the matrix \( A \) is diagonal with entries equal to the eigenvalues of \( G_0^{-1}G_1 \). We find the row of \( C^{-1} \), \( c_i \), associated with the explosive eigenvalue, and ‘zero out’ this value by imposing the condition \( c_i x_t = 0 \) on the system by using this latter equation in place of the stochastic Euler equation (3). Hence, we have a first-order system in the vector \( x_t \)

\[ x_t = Fx_{t-1} + Gv_t, \tag{9} \]

where \( v_t = \left[ v_{At} \bar{v}_{et} \right]' \) and \( G \) has dimension 8 \( \times \) 2. Though the system is of dimension eight, it is stochastically singular since there are only two random shocks (the variance matrix of \( x_t \) has rank two). Therefore, in general the model carries nontrivial predictions for any two of the variables. We focus on its implications for output and investment, and append to (9) the observation equation

\[ X_t = H'x_t, \tag{10} \]

where the 2 \( \times \) 8 matrix \( H' \) selects the output and investment series (the first and third elements of \( x_t \)).

4. Estimation

In any empirical modeling exercise, there are three potential sources of unknowns: the model itself (including the characterization of the underlying

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\(^4\) Sims (1997) demonstrates how to handle the case in which \( G_0 \) is not full rank. The method involves using the QZ algorithm to solve the generalized eigenvalue problem, \( G_0 x = \lambda G_1 x \). King and Watson (1998) describe a similar algorithm based on a Jordan decomposition. For certain problems, there may be numerical differences between the two algorithms (see Golub and Van Loan, 1989).

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probability distribution), the parameterization conditional on the model, and the data. In our structural approach, we take the specification of the model as given, though in principle even uncertainty about the nature of the model specification can be accommodated as well. Given the model, there are two sets of random variables: the parameters and the data. In the classical approach to inference, the parameters are treated as fixed and the data are treated as unknown in the sense that their probability distribution (the likelihood) is the center of focus. The question is whether the observed data could plausibly have come from the model under a particular parameterization. In the Bayesian approach the observed data are treated as fixed, and the unknown parameters are treated as random variables. The ultimate goal is in learning about the unknown parameters (and functions — like forecasts — of the parameters); this is accomplished through the (posterior) probability distribution of the parameters given the data.

Denoting the collection of parameters by \( \mu = [\alpha \beta \gamma \omega \rho \sigma^2 A \sigma^2_B] \) and the sample of observations on the observed variables by \( X \), write the posterior distribution of interest as \( P(\mu | X) \). This posterior is proportional to the product of the likelihood for the data given the parameters, and the prior for the parameters.

\[
P(\mu | X) \propto L(X | \mu)P(\mu).
\]

(The factor of proportionality is the marginal distribution of the data, which is a constant from the point of view of the distribution of \( \mu \).) The building of the posterior distribution thus requires constructing the likelihood function and specifying the prior.

The likelihood function for an observer system like (9) and (10) is straightforward to calculate using the Kalman filter. Then the mapping from the underlying economic parameters \( \mu \) to the observer system parameters (the mapping implicit in the linearization procedure discussed above) can be used to complete the specification of the likelihood function \( L(X | \mu) \).

For the observer system, the Kalman filter equations are

\[
x_{t|t} = x_{t|t-1} + P_{t|t-1} H(H'P_{t|t-1}H)^{-1}(X_t - H'x_{t|t-1}),
\]

\[
x_{t+1|t} = Fx_{t|t},
\]

\[
P_{t|t} = P_{t|t-1} - P_{t|t-1} H(H'P_{t|t-1}H)^{-1}H'P_{t|t-1},
\]

\[
P_{t+1|t} = FP_{t|t}F' + Q,
\]

where \( x_{t|t} \) is the optimal estimate of the (unobserved) state vector (the solution to the ‘signal extraction’ problem) at time \( t \) given data through time \( t \) (\( X_s, s = 1, \ldots, t \)), \( x_{t+1|t} \) is the corresponding one-step-ahead predictor, \( P_{t|t} \) is
the covariance matrix of the error in estimating the state, $E(x_t - x_{t|t})(x_t - x_{t|t})'$, $P_{t+1|t}$ is the corresponding covariance matrix of the one-step-ahead predictor, $Q = GE_vv'G'$, and the recursions are initialized by $x_{1|0} = 0$ and vec($P_{1|0}$) = $(I - F \otimes F)^{-1}$vec($Q$), the values from the unconditional distribution. (For details, see e.g., Harvey, 1989, or Hamilton, 1994.)

Conditional on an initial observation $X_0$, the likelihood can be built up using the so-called ‘prediction error decomposition’ (see Harvey, 1989, p. 126). In particular, define the prediction error as the error in predicting the observables one-step-ahead: $u_t = X_t - H'x_{t|t-1}$. Then the likelihood is given by

$$
\log L \propto -\frac{1}{2} \sum_{t=1}^T \log |P_{t|t-1}| - \frac{1}{2} \sum_{t=1}^T u_t P_{t|t-1}^{-1} u_t.
$$

Thus given data and a candidate parameter value $\mu$, the model can be transformed into the observer system (9) and (10), the Kalman filter can be applied, and the value of the likelihood function calculated. One potential source of candidate values of $\mu$ is the prior distribution.

We regard it not as a shortcoming but as a virtue of our approach that prior views regarding parameter values must be brought to bear on the analysis. These prior views may take many forms, but to incorporate them into the analysis in a formal manner, it is necessary to specify them in the form of prior probability density functions for the parameters.

Our practice has been to assume as little as possible about the volatility of the innovations of the shock processes (i.e., to use ‘diffuse’ priors), and to take guidance from the macroeconomics literature regarding the other class of parameters. Denoting the former parameters by $\mu_s$ and the remaining elements of $\mu$ by $\mu_e$, the prior is written in product form, $P(\mu) = P(\mu_s)P(\mu_e)$, making priors independent across the two classes, and diffuseness is implemented by taking $P(\mu_s)$ proportional to a constant. Since the posterior is only specified up to multiplication by a constant in the first place, we simply take $P(\mu | X) \propto L(X | \mu)P(\mu_e)$.

Our prior information for the parameters $\mu_e$ comes in the form of dispersion about specified means, and while other distributional assumptions present no additional computational assumptions, we have generally used normal prior distributions, truncating if necessary. Some parameters in the model are very important in determining the steady state, and thus it is easier to specify prior distributions for them. For example, it is customary in the real business cycle literature to take $\beta = 0.96$ for annual data, or 0.99 (as here) for quarterly data. The standard deviation in the prior was fixed at 0.00125, making the 95% highest prior density coverage interval for the annual interest rate the range 3–5%. (The distribution for $\beta$ was truncated above at 1.0.) Likewise, the parameter $\alpha$ determines the steady-state value of capital's share of income, a magnitude about which much is known. GHH calculated the average value of
As noted above, to accommodate unit-root shocks, we would need to modify the solution algorithm we employ in order to allow for drift in the model's variables. Thus, we made the upper bound of the distribution 1.0. The probability of a drawing at this value is zero.

Since there is much persistence in actual time series, and since in the model this persistence is inherited directly from the parameters \((\rho_A, \rho_e)\), there is some agreement in the literature regarding values of these parameters. We chose mean values of 0.95, and let the prior coverage intervals be \((0.9, 1.0)\). As with the discount factor, we truncated the prior distributions at 1.0.\(^5\)

Other parameters are more problematic. For example, there is little agreement in the literature regarding the coefficient of relative risk aversion \(\gamma\) other than that it is not too large. GHH used two different values, 1.0 and 2.0, in their analysis. Similarly, GHH argued that values of \(\theta\) (the inverse of the intertemporal elasticity of substitution in labor supply) between 0.45 and 3.33 are reasonable. They then argued that within this range higher substitution is more likely, and they settled on the single value 0.6. In our specification of the prior, we took the mean value of \(\gamma\) to be 1.5, and utilized a 95% coverage interval of \((1.0, 2.0)\); for \(\theta\) we specified a mean of 0.6, and took the 95% prior coverage interval to be \((0.4, 0.8)\). Finally, there is little in the literature regarding the depreciation curvature parameter \(\omega\). GHH chose a value for this parameter that implied the steady-state depreciation rate was 10% per year; we adopted a similar procedure. The prior means and standard deviations are listed in Table 1.

Having specified the likelihood and the prior, we proceed to analyze the posterior. But (11) is analytically intractable for two reasons. First, the likelihood itself is intractable. Second, even if it was not, the posterior is the product of the likelihood and the prior, and such product forms of density functions are quite difficult to analyze.

Thus we analyze the posterior distribution by numerical methods. Our intention is to generate an artificial sample \(\{\mu_k\} \) for \(k = 1, \ldots, N\) from the posterior density. Unfortunately, even this is not generally possible: if the posterior is analytically intractable, it is unlikely that random numbers can be generated from it easily. What we do instead is to generate an artificial sample from a different distribution from which it is straightforward to sample, and assign weights to the elements of the sample so that they can be thought of as originating from the posterior distribution of interest. This procedure, known as ‘importance sampling’ (see Geweke, 1989), can be illustrated

\(^5\)As noted above, to accommodate unit-root shocks, we would need to modify the solution algorithm we employ in order to allow for drift in the model’s variables. Thus, we made the upper bound of the distribution 1.0. The probability of a drawing at this value is zero.
Importance sampling is not the only procedure available for generating artificial samples of interest. For example, in some special cases (not this one), a full set of conditional distributions for subsets of the parameters given the others may be available, in which case the function of interest is the (average) value of the posterior on that interval. Given a sample \( \{\mu_k\} \) for \( k = 1, \ldots, N \) from a different density, \( I(\mu) \) (termed the ‘importance density’), the expectation is calculated as

\[
\bar{g}_n = \frac{\sum_{i=1}^{n} g(\mu_i)w(\mu_i)}{\sum_{i=1}^{n} w(\mu_i)},
\]

where the ‘weight’ function \( w(\cdot) \) is given by

\[
w(\mu_i) = \frac{P(\mu_i|X)}{I(\mu_i)}.
\]

What the weighting does is to downweight those \( \mu_i \) that are overrepresented in the importance distribution relative to the posterior, and upweight those that are underrepresented. Notice that if it is possible to sample directly from the posterior, the weights are all the same, and arithmetic averages replace weighted averages. Provided that the support of \( I(\mu) \) includes that of \( P(\mu|X) \), Geweke shows that \( \bar{g}_n \) converges almost surely to \( E[g(\mu)] \), so long as \( E[g(\mu)] \) exists and is finite.

While in principle this procedure can be applied in a broad variety of contexts, in practice convergence can be slow. The problem is that if the importance density does not closely mimic the posterior, enormous weights will be assigned occasionally to particular drawings. In order that a few drawings not dominate the results, huge numbers of drawings may be necessary. For example, Richard and Zhang (1996) report an example in which an inefficient

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6 Importance sampling is not the only procedure available for generating artificial samples of interest. For example, in some special cases (not this one), a full set of conditional distributions for subsets of the parameters given the others may be available, in which case the Gibbs sampler could be used. In general, however, it will be necessary to sample from some convenient distribution other than the posterior, and then make adjustments to the probabilities. An alternative to importance sampling is the Metropolis–Hastings algorithm, in which a ‘candidate generating density’ produces a sequence of values of the parameter vector wherein the jth entry is either the same as the previous entry or a new candidate, with probability given by the ratio of the posterior density at the candidate to its value at the previous parameter. This produces a sample in which the ‘weights’ are all the same, but highly likely values are represented by multiple occurrences. Otrok (1999) adopts an approach similar to ours, but uses the Metropolis–Hastings algorithm. Study of the relative merits of the two approaches is ongoing (see Richard, 1998; Geweke, 1997).
### Table 1
Prior, Likelihood, and Posterior Moments

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\omega$</th>
<th>$\theta$</th>
<th>$\rho_A$</th>
<th>$\rho_e$</th>
<th>$\sigma_A^2$</th>
<th>$\sigma_e^2$</th>
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</thead>
<tbody>
<tr>
<td>Prior Mean</td>
<td>0.290</td>
<td>0.990</td>
<td>1.500</td>
<td>1.600</td>
<td>0.600</td>
<td>0.95</td>
<td>0.95</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Posterior Mean</td>
<td>0.284</td>
<td>0.989</td>
<td>1.402</td>
<td>1.565</td>
<td>0.519</td>
<td>0.97</td>
<td>0.94</td>
<td>1.4e-5</td>
<td>5.1e-6</td>
</tr>
<tr>
<td>Likelihood Mean</td>
<td>0.441</td>
<td>0.952</td>
<td>7.279</td>
<td>2.019</td>
<td>1.629</td>
<td>0.97</td>
<td>0.86</td>
<td>2.9e-5</td>
<td>1.3e-5</td>
</tr>
<tr>
<td>Prior S.D.</td>
<td>0.025</td>
<td>0.001</td>
<td>0.250</td>
<td>0.150</td>
<td>0.100</td>
<td>0.025</td>
<td>0.025</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Posterior S. D.</td>
<td>0.022</td>
<td>0.001</td>
<td>0.224</td>
<td>0.061</td>
<td>0.103</td>
<td>0.014</td>
<td>0.018</td>
<td>5.8e-6</td>
<td>1.1e-6</td>
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<tr>
<td>Likelihood S. D.</td>
<td>0.072</td>
<td>0.013</td>
<td>4.085</td>
<td>0.210</td>
<td>0.628</td>
<td>0.016</td>
<td>0.059</td>
<td>2.2e-5</td>
<td>4.2e-6</td>
</tr>
</tbody>
</table>

(a) Prior, posterior and likelihood moments of the parameter vector

(b) Posterior and likelihood correlations among parameters (prior correlations are zero; likelihood correlations are in parentheses)

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\omega$</th>
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<th>$\sigma_e^2$</th>
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<tbody>
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<td>Prior</td>
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importance sampler would have required years of CPU time to produce results as reliable as those produced in seconds from an efficient sampler.

In designing an efficient sampling scheme, two characteristics of the importance density are key: it must have tails that are fat relative to the posterior, and its center and rotation must not be too unlike the posterior. Tail behavior is handled by working with multivariate-$t$ densities since, under weak conditions, the posterior is asymptotically normal (Heyde and Johnstone, 1979; Sweeting, 1992). The multivariate-$t$ can be written as the product of a normal conditional on a covariance matrix scaling factor times an inverted gamma distribution for the scale parameter. Small degrees of freedom in the inverted gamma ensure that the multivariate-$t$ has fatter tails than the normal.

We determine the moments of the importance sampler adaptively, beginning with either prior moments or by using values obtained through numerical maximization of the posterior. Given the initial importance density, a sample of say 1000 drawings is taken. These drawings are used to compute posterior moments, and these are used in the importance density for the next step. At each stage, the quality of the importance sample is monitored by

$$\sigma = \max_i w(\mu_i)/\Sigma[w(\mu_i)],$$

i.e., the largest weight relative to the sum of all weights. This function penalizes disparity in the weights; if the importance sampler mimics the posterior perfectly so that the weights are all unity, the value of $\sigma$ is the inverse of the sample size. The results that follow are based on a sample of size 80,000 in which $\sigma$ is roughly 2%.

Beyond the calculation of posterior moments and the distributions themselves (in the form of weighted histograms), our procedure can be used in a variety of contexts. In DeJong et al. (2000), we focused on the calculation of posterior distributions for underlying shocks. Here, we address the construction of predictive distributions of future values of the observable variables.

Given an importance sample and the observer system (9) and (10), predictive distributions are straightforward to calculate. For each element of the importance sample, what is involved is a dynamic simulation of the state transition equation, beginning with a drawing of $x_{T+1}$ from a normal distribution with covariance matrix $P_{T+1}$ (where $T$ denotes the last date at which observations of the data are available). Combining this drawing with drawings from $G_{T+1}, G_{T+2}, \ldots, G_{T+j}$ yields simulated values $x_{T+1}, \ldots, x_{T+j}$. These simulated values are assigned the importance weight of the underlying element of the sample, and a distribution is built up. Specifically, the function $g(\mu_i)$ introduced above is an indicator function in this case; its value is one if $x_{T+j}$ falls in a particular histogram bin, and zero otherwise. Predictive densities are approximated by calculating $\bar{g}_n$ for each of a sequence of finely specified bins. The next
section discusses such distributions, along with posterior distributions of the deep parameters.

5. Results

We measure investment using nonresidential fixed investment, and output using the sum of investment and consumption of nondurables and services. The series are measured in 1987 dollars, and were converted to *per capita* terms by dividing by the noninstitutionalized population over 16 years of age. Linear trends were extracted from the logarithms of each series. The sample is quarterly, runs from 1950:I to 1996:IV, and was obtained from CITIBASE. The model is estimated using observations through 1986:IV; the remaining observations were withheld to enable us to evaluate out-of-sample forecasts.

Table 1 provides the posterior moments and correlation matrix of the parameters. We assessed the numerical accuracy of the entries in the table in two ways (in addition to our monitoring of the performance of the importance density). First, we computed numerical standard errors using the procedure outlined in Geweke (1989). Second, we employed a resampling procedure in which we generated 500 independent sub-samples of (appropriately weighted) drawings from the importance density and calculated posterior means in each of the sub-samples. We then computed the standard deviation of the means. Each of these procedures provides a measure of the accuracy with which posterior means are estimated. The largest numerical standard error we encountered was 11% of the mean, for $\sigma^2_A$. All the others were $< 3\%$.

In order to assess the influence of our prior on the results, we also conducted the entire analysis using a prior that was uninformative over the parameters. With such a prior, the posterior is simply the likelihood renormalized to be a density. The results of this exercise are also given in Table 1.

The pattern that emerges from comparing the likelihood, prior, and posterior means is that the location of the informative prior plays an important role in determining the location of the posterior distribution. The means of the posterior distributions are virtually the same as means of the prior distributions, and reflect movement toward more economically plausible values of the parameters than are produced by the likelihood alone. In particular, the likelihood prefers

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7 There is some evidence of a trend break in the mid-1970s. We have analyzed the sensitivity of our results to the assumption of a single trend by calculating the value of the likelihood under the alternative hypothesis that a trend break occurred in the first quarter of 1974. This had little effect on our results: with one exception, posterior means of each parameter differ by no more than one posterior standard deviation across trend-removal procedures. The exception is for the persistence parameter of the investment shock, which drops to 0.91 from 0.94.

8 We thank Chris Sims for suggesting this procedure.
specifications characterized by very low labor shares (large $\alpha$), impatience (small $\beta$), and excessive risk aversion (large $\gamma$). The mean of $\theta$ under the likelihood function is in the range considered by GHH, but represents a much lower intertemporal elasticity of substitution than they assumed (and than we adopted in our prior).

Clearly, the prior plays an important role in determining the location of the posterior means for the parameters of the model. The beginnings of an explanation for this can be seen in the table entries for ‘likelihood standard deviations’. The large standard deviations suggest that at least in some dimensions the likelihood is nearly flat relative to the prior distribution, despite the relatively generous dispersions we accommodated in the prior. Of course the full explanation requires knowledge of correlations in the prior and likelihood, a point to which we return below.

As is usually the case in dynamic models, the data are reasonably informative about persistence, and thus about the shock parameters. Indeed, the prior and posterior distributions of the shock parameters are relatively distinct. The posterior mean of $\rho_A$ is more than one posterior standard deviation above its prior mean, while the posterior mean of $\rho_e$ is about half a posterior standard deviation smaller than its prior mean. Moreover, the posterior standard deviations of these parameters are considerably smaller than their prior standard deviations. Not only do the data indicate that productivity shocks are more persistent than investment shocks, they also indicate that productivity shocks are more volatile than investment shocks: the posterior mean of $\sigma_e^2$ is just over a third as large as the posterior mean of $\sigma_A^2$; this difference is large relative to the dispersion in either posterior distribution. Taken together, these results indicate that, in the context of the model, productivity shocks are the primary culprits behind the business cycle fluctuations exhibited by output and investment.

While the priors and marginal posteriors of the behavioral parameters exhibit relatively close correspondence, prior and posterior correlations across parameters do not: despite our assignment of zero prior correlation, many of the parameters exhibit distinctly nonzero posterior correlations. The most interesting entry in the correlation table is that for the correlation between $\beta$ and $\omega$, which is $-0.54$. We believe this strong negative correlation arises because the data have a strong predilection for certain values of the rate of depreciation. In this model, the steady-state rate of depreciation is given by

$$\delta = \frac{1/\beta - 1}{\omega - 1}.$$  

Steady-state depreciation, then, is a decreasing function of both $\beta$ and $\omega$: to keep steady-state depreciation constant, a rise in $\beta$ must be accompanied by a decline in $\omega$. (For example, the movement from the likelihood mean to the posterior mean reflects just such an increase in $\beta$ and decline in $\omega$.) In the posterior, the data prefer a steady-state rate of depreciation of 1.97% per quarter (slightly
<8% annually). This rate is somewhat lower than that typically used in the business cycle literature, though it is similar to the magnitude reported by Ingram et al. (1994).

The posterior and likelihood correlations reported in Table 1 stem from the interaction of the likelihood and prior. Because prior correlations are zero, if likelihood contours are roughly elliptical (as they would be for VAR parameters), posterior correlations would be lower (in absolute value) than likelihood correlations. In this case, however, the likelihood function is a highly nonlinear function of the parameters of interest, and the likelihood is distinctly non-elliptical. This means that conditional correlations often vary substantially over the parameter space; this variation generates differences that are in some cases surprising. Two notable examples are given by correlations between $\beta$ and $\gamma$ ($-0.27$ in the posterior, $0.28$ in the likelihood), and $\gamma$ and $\omega$ ($0.50$ in the posterior and $-0.56$ in the likelihood). These differences largely reflect variation in the shape of the likelihood contours across the parameter space. For example, conditional on the subset of the parameter space in which $\beta$, $\gamma$, and $\omega$ are within two prior standard deviations of their prior means, the likelihood correlation between $\beta$ and $\gamma$ is $-0.21$, and is $0.51$ between $\gamma$ and $\omega$.

An especially interesting example of the interaction of prior and likelihood means and correlations involves labor’s share ($x$) and the discount factor ($b$). The means associated with the likelihood are $0.44$ for $x$ and $0.95$ for $b$, with a strong negative correlation of $-0.71$. Serendipitously, the prior mean ($0.29$ for $x$ and $0.99$ for $b$) lies along the population regression line dictated by the shape of the likelihood contours. Thus although the data are reasonably informative about these two parameters in most directions (i.e., the gradients of the likelihood function are steep off the regression line), the data are uninformative in one direction, and thus the prior, which happens to lie in that direction, determines the posterior mean.

We now turn to the results of our forecasting exercise. As noted above, we chose 1986:IV as the endpoint of our ‘observed’ sample, and compared one-through 40-step-ahead forecasts obtained from the observer equation induced by the estimated model with subsequent realizations of the data observed through 1996:IV. Since the model only carries implications for the stationary components of the data, we only used it to forecast deviations of the data from their linear trends. In order to provide a basis for judging the model’s forecasting performance, we also obtained forecasts of these deviations over this same period from a four-lag BVAR (estimated using default values of the ‘Minnesota’ prior – see Doan et al., 1984, for details).\(^9\)

\(^9\)Typically, BVARs with the Minnesota prior are implemented using an equation-by-equation mixed-estimation scheme which is Bayesian in spirit, but not in fact. Our implementation of the Minnesota prior is in fact fully Bayesian and is carried out by importance sampling using as the importance density the flat-prior Normal–Wishart posterior for the unrestricted VAR.
Fig. 1 depicts 50% quantiles and 95% coverage intervals of the predictive densities generated by the theoretical model (under both the posterior and likelihood functions), and the BVAR, as well as the data observed over the period 1985:I–1996:IV. The three sets of predictive densities are remarkably similar, both in terms of location and dispersion, over both short- and long-range horizons. They track output and investment quite closely over roughly the first two years of the forecast horizon, but then largely fail to capture the impact of the subsequent recession. While recognizing that the similarity of these forecasts may not obtain in general, these results indicate that the forecasting performance gained from the adoption of the ‘Minnesota’ prior can also be enjoyed given the adoption of a forecasting model founded upon a coherent general-equilibrium model. The simplicity of the particular model examined here makes this result particularly encouraging.

We conclude with a note concerning the sensitivity of our forecasting results to our prior. Given the close correspondence of our prior and posterior estimates of the parameters of the theoretical model, the potential sensitivity of our predictive densities to the adoption of alternative priors is nontrivial. But as the close correspondence between the posterior and likelihood quantiles depicted in Fig. 1 illustrates, the prior does not exert undue influence on the predictive densities in this case. In particular, the relatively tight prior we specified over the parameters of the theoretical model induced very little shrinkage beyond that induced by the adoption of the theoretical model.

In Fig. 2, we illustrate predictive densities obtained from all three forecasting models at the 2- and 30-step forecast horizons. This figure provides a clearer picture of the close correspondence between the posterior and likelihood densities, and amplifies small differences between these densities and those obtained from the BVAR. Specifically, the BVAR densities are relatively tight at short forecasting horizons, and diffuse at long horizons, while the model predictive densities are relatively diffuse at short horizons, but reflect greater certainty about long-run forecasts than do the BVAR predictives. But these differences are relatively minor; there is considerable overlap between all three sets of distributions at all horizons.

6. Conclusion

We have proposed the use of a coherent statistical framework for formally bringing to bear theoretical models of dynamic macroeconomic activity in addressing empirical issues of interest to policy makers, forecasters, and practitioners interested in explaining the behavior of observed data. The framework is Bayesian: for a given theoretical model, it involves combining the likelihood function induced by the model with a prior distribution specified over the
Fig. 1. Predictive densities of 50% quantiles and 95% coverage intervals generated by the theoretical model and the BVAR (period 1985:I–1996:IV).
Fig. 2. Predictive densities obtained from all three forecasting models at the 2- and 30-step forecast horizons.
model’s parameters, and using the resulting posterior distribution to address empirical questions of interest.

Of interest to us in this application was the ability of a neoclassical business cycle model to generate accurate forecasts of the cyclical behavior of output and investment. Ability has been demonstrated in this case: the performance of the model is comparable to that of a Bayesian VAR, a result we find to be impressive. Measurement with theory appears to have its merits.

Acknowledgements

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