Multi-task agency: a combinatorial model

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Received 20 October 1998; received in revised form 21 March 2000; accepted 29 March 2000

Abstract

An agent allocates unobservable effort over a portfolio of projects; the principal observes only which projects succeed and which fail. An explicit solution to the multi-task agency problem is found using Möbius inversion in the lattice of project portfolios.

An application is made to the structure and management of scientific research organisations. Optimal structures are compared with those derived by standard cost-benefit techniques. Systematic biases are identified in the selection of project type, in the level of investment in scientific infrastructure such as libraries, and in the diversification of tasks within the organisation. © 2001 Elsevier Science B.V. All rights reserved.

JEL classification: D82; H43

Keywords: Multi-task agency; Project evaluation

1. Introduction

The multiple-task principal-agent problem has been used in several studies of organisational design and job design (for example, Holmstrom and Milgrom, 1991, 1992; Meyer et al., 1996; Slade, 1996). Because of the intractability of the multi-agency problem, virtually all studies use the simplifying assumption of normally distributed risk and exponential utility. Under these assumptions, the problem has a closed form solution that is linear in incentives. Notable exceptions are Chambers and Quiggin (1996), and Bergemann (1992).

In this paper, we study a different specification of the multi-agency problem. We allow many tasks, but we simplify the type of task. Each task is linked to a project with a binary “succeed” or “fail” outcome. This is unlike other models that have been studied for which the outcome is typically a continuously distributed random variable. By exploiting combinatorial identities, we are able to find a closed form solution.
The application that we have in mind is the management of large public scientific organisations such as the international agricultural research institutions of the consultative group on international research (CGIAR), or the laboratories and research centres of the national agricultural research institutions throughout the world. The performance of these institutions is a subject of current policy concern (Alston et al., 1995; Just and Huffman, 1992; Huffman and Just, 1994; Norton et al., 1992; World Bank, 1996).

Bardsley (1999) considers the research management problem from an agency perspective. The agents are scientists, or autonomous laboratory managers who allocate effort and resources among a portfolio of projects. The principal is the central research manager who allocates funds between programs and attempts to direct those programs to achieve policy objectives. The central manager observes only whether projects succeed or fail. It is shown that there are important links between the riskiness of selected projects, the difficulty of monitoring projects, and the kind of incentives that can be imposed on agents. These arguments are different from, and independent of the Arrow and Lind (1970) analysis of risk bearing by public institutions. Bardsley (1999) focuses mainly on the single-activity case with a brief discussion of a two-project example.

Here, we consider the general problem where there are \( n \) activities in which the agent might engage. It is important to consider this case which is technically much more difficult as the essence of the problem is the efficient level of risk exposure of the agent. Diversification of effort across many projects may have a major effect on the agent’s exposure to risk with flow-on effects to the optimal incentive structure. Given the difficulties of the general case, we focus on the case where projects are very risky (the probability of success is close to zero), and we assume that the agent’s effort function is sufficiently convex to justify a first-order approach. We examine in particular, the effect of project-by-project evaluation, using conventional cost-benefit techniques (see for example Alston et al. (1995)). We find that project-by-project evaluation, ignoring effects on the agent’s risk exposure, may lead to a bias in the research project portfolio or in the infrastructure that supports research.

The structure of the paper is as follows. The formal agency model is set-up in Section 2, and its properties are derived. This section also contains an example, which shows how the optimal contract can be calculated in practice. In Section 3, the model is used to examine two questions: will standard project evaluation techniques lead to optimal investment in research infrastructure such as libraries? will standard project evaluation techniques lead to the selection of projects that are too risky or too safe? Systematic biases are identified in both the cases. Finally, some general conclusions are drawn in Section 4.

2. The model

After establishing the combinatorial notation and the structure of the game, the optimal decision rules for both principal and agent are derived. The key technical result (Theorem 1) is that the Lagrange multipliers that feature in the principal’s decision rule are all non-negative. This will allow us to draw qualitative conclusions about the structure of the contract, and it will allow us to make definite statements below about the direction of certain biases in the portfolio choice.
2.1. Notation

We consider a set $N = \{1, \ldots, n\}$ of projects, indexed by integers. A portfolio is a subset $J \subset N$. To reduce notation when the context is clear the set $\{j\}$ will be written $j$ and the set $J \setminus \{j\}$ with $j$ deleted will be written $J - j$. The size of $J$, that is the number of projects in the portfolio $J$ will be denoted $|J|$. The set of portfolios, partially ordered by inclusion is a lattice (Aigner, 1997; Wehrhahn, 1990). If $J \subset K$ then the set of portfolios $\{Z : J \subset Z \subset K\}$ between $J$ and $K$ is called a lattice interval; it is denoted $[J,K]$. Where the context is clear, the empty portfolio $\emptyset$ will be denoted $0$ (it is the smallest element of the lattice), and the portfolio of all projects $N$ will be denoted $1$ (it is the largest element of the lattice). It will frequently be necessary to sum some quantity $x_Z$ over a lattice interval. This sum will typically be denoted $\sum_{Z \in [J,K]} x_Z$. The sum $\sum_{Z \in \{0,1\}} x_Z$ over all portfolios will be abbreviated $\sum_Z x_Z$.

For each project $i$, $p_i$ is the probability that the project succeeds. The projects are assumed to be statistically independent. The probability that $K$ is included in the portfolio of successful projects is

$$P_K = \prod_{k \in K} p_k,$$

while the probability that $K$ is exactly the portfolio of successful projects is

$$P^K_N = \prod_{k \in K} p_k (1 - p_k).$$

We note the fundamental identity that, for any lattice quantity $x_K$,

$$\sum_{J} p_N^J x_J = \sum_{J, K, J \subset K} p_K \mu^K_J x_J = \sum_{K} p_K \Delta x_K,$$

where $\mu^K_J$ is the Möbius function on the Boolean lattice of subsets of $N$, and $\Delta$ is the Möbius lattice derivative (see Appendix A for details and examples of these concepts). Note that

$$\mu^K_J = \begin{cases} 0, & \text{if } J \not\subset K, \\ 1, & \text{if } J = K, \\ (-1)^{|K|-|J|}, & \text{if } J \subset K. \end{cases}$$

and

$$\Delta x_K = \sum_{J \in \{0,K\}} \mu^K_J x_J.$$

Thus, the lattice derivative is an alternating sum which is best illustrated by an example. Let $N = \{1, 2, 3\}$; then

$$\Delta x_{1,2,3} = x_{1,2,3} - x_{1,2} - x_{2,3} - x_{3,1} + x_1 + x_2 + x_3 - x_0.$$
2.2. Structure of the game

The risk neutral principal receives a benefit $b_j$ if project $j$ is successful. It is assumed that benefits are additive, so $b_J = \sum_{j \in J} b_j$ is the total benefit if $J$ is the portfolio of successful projects. The principal rewards the agent with a payment $x_j$ if $J$ is the portfolio of successful projects. In contrast to the benefits, $x_J$ need not be additive. The expected pay-off to the principal is thus

$$V = \sum_{j \in N} p_j b_j - \sum_{K} p_K^J x_K = \sum_{K} p_K^J (b_K - x_K) = \sum_{J} P_J (\Delta b_J - \Delta x_J).$$  \hfill (1)

The agent chooses the vector $\mathbf{p} = (p_1, \ldots, p_n)$ of success probabilities which requires effort $e(\mathbf{p})$. If $\mathbf{p} = \mathbf{0} = (0, \ldots, 0)$ then all the projects will certainly fail; if $\mathbf{p}$ is close to $\mathbf{0}$ then we will say that the outcome is risky. If $\mathbf{p} = \mathbf{1} = (1, \ldots, 1)$ then all the projects will certainly succeed; if $\mathbf{p}$ is close to $\mathbf{1}$ then we will say that the outcome is safe. It will be assumed that, for all $i$, $e_i(\mathbf{p}) > 0$, $e_i(\mathbf{p}) > 0$, and $e(\mathbf{p}) \to \infty$ as $p_i \to 1$. It will also be assumed that, at the margin, effort is separable between projects so $e_{i,j}(\mathbf{p}) = 0$.

The agent’s utility is separable in income and effort, so expected utility is

$$U = \sum_{K} p_K^J u_K - e(\mathbf{p}) = \sum_{J} P_J \Delta u_J - e(\mathbf{p}),$$  \hfill (2)

where $u_J = u(x_J)$ is the utility associated with the transfer $x_J$. For convenience the inverse utility function will be denoted $x$, so $x_J = x(u_J)$. It will be assumed that the agent is risk averse which implies that $x'(u) > 0$, $x''(u) > 0$.

Of the existing literature, Bergemann (1992) is most similar in structure to the problem considered here. Bergemann considers, however, continuously differentiable distribution functions rather than the discrete two-point distributions considered here, and our model does not fall under his assumptions. Where he uses calculus-based monotone likelihood methods, we use a combinatorial approach.

2.3. The agent’s problem

The agent chooses $\mathbf{p}$ to maximise $U$ subject to a reservation utility of zero; unless this level of expected utility can be ensured, the agent will not participate. We study only the case where $U$ is a convex function of $\mathbf{p}$ near $\mathbf{0}$; this will be so if the effort function $e(\mathbf{p})$ is sufficiently convex \footnote{1 We can give reason as follows. Let $\mathbf{p}^*$ be the agent’s optimal effort as calculated from the first-order conditions (3) and (4) below, given the contract $x_K$. If the effort function $e(\mathbf{p})$ is made more convex while holding fixed $e(\mathbf{p}^*)$ and the first-order quantities $e_i(\mathbf{p}^*)$, then the optimal contract $x_K$ will not change and $\mathbf{p}^*$ will still be the optimal effort.}. This assumption will allow us to apply the first-order approach to the agency problem.

The participation constraint is

$$e(\mathbf{p}) = \sum_{J} P_J \Delta u_J = \sum_{K} p_K^J u_K.$$  \hfill (3)
The Kuhn Tucker incentive compatibility conditions are

\[ \sum_{j \in J} P_{j \rightarrow j} \Delta u_j = e_j(p), \quad \text{if } p_j > 0, \quad \sum_{j \in J} P_{j \rightarrow j} \Delta u_j \leq e_j(p), \quad \text{if } p_j = 0; \]  

which can be rewritten

\[ \sum_{j \in K} P_j^{K-j} u_K - \sum_{j \notin K} P_j^{K-j} u_K = e_j(p), \quad \text{if } p_j > 0, \]

\[ \sum_{j \in K} P_j^{K-j} u_K - \sum_{j \notin K} P_j^{K-j} u_K \leq e_j(p), \quad \text{if } p_j = 0. \]  

2.4. The principal’s problem

By studying the principal’s optimisation problem we get good information on the marginal reward function \( x'_K \). Under appropriate assumptions about risk aversion, this leads to information on the optimal reward function \( x_K \). We also derive useful information about the sign of Lagrange multipliers.

Consider first, the least cost contract that the principal would choose in order to implement \( p \), leaving aside for the moment the question of which probability vector \( p \), the principal would wish to implement. Assuming that the first-order approach is valid, the principal chooses the \( u_K \) to minimise the expected cost \( P_J x_J \) subject to the constraints (4) and (5). Let \( \gamma \) be the Lagrange multiplier attached to the participation constraint, and let \( \lambda_j \) be the Lagrange multipliers attached to the incentive compatibility constraints. Then the Lagrangean is

\[ L(p) = \sum_j P_j^J x_J + \gamma \left( e(p) - \sum_j P_j^J u_j \right) + \sum_j \lambda_j \left( e_j(p) \sum_{j \in K} P_j^{K-j} u_j + \sum_{j \notin K} P_j^{K-j} u_j \right). \]  

The first-order conditions are

\[ (x'_K - \gamma) P_j^K - \sum_{j \in K} \lambda_j K_{j \rightarrow j} P_j^{K-j} + \sum_{j \notin K} \lambda_j P_j^{K-j} = 0 \]

where \( x'_K \) is an abbreviation for \( x'(u_K) \).

If the probabilities are all non-zero, then the first-order conditions may be written in the following convenient form:

\[ x'_K = \gamma + \sum_{j \in K} \frac{\lambda_j}{p_j} - \sum_{j \notin K} \frac{\lambda_j}{1 - p_j}. \]
For example, let \( n = 2 \) and \( N = \{1, 2\} \). Then

\[
x'_{1,2} = \gamma + \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad x'_1 = \gamma + \frac{\lambda_1}{p_1} - \frac{\lambda_2}{1 - p_2}, \\
x'_2 = \gamma - \frac{\lambda_1}{1 - p_1} + \frac{\lambda_2}{p_2}, \quad x'_0 = \gamma - \frac{\lambda_1}{1 - p_1} - \frac{\lambda_2}{1 - p_2}.
\]

Now consider the choice of \( p \). The principal maximises

\[
L = \sum_{j \in N} p_j b_j - L(p),
\]

where the payments \( x_j \) take on the cost minimising values determined by (7).

The objective in the principal’s cost minimisation problem (6) is convex in the \( u \) (because of the assumptions made above on the degree of risk aversion), and the participation constraint is linear. By a standard Kuhn Tucker argument, it is straightforward to show that \( \gamma \geq 0 \) (relaxing the agent’s participation constraint or reducing the agent’s cost will benefit the principal). To determine the sign of the other Lagrange multipliers is a little more delicate.

**Theorem 1.** If \( p \) is a risky portfolio (\( p \) is close to \( 0 \)) then \( \lambda_j \geq 0 \). The inequality is strict unless \( p_j = 0 \).

**Proof.** See Appendix A.

This result says that if a project \( j \) is active then reducing the agent’s marginal cost \( e_j(p) \) will reduce the principal’s cost.

**Theorem 2.** Assume that \( p_j > 0 \) for all \( j \). Then

1. \( x' \) is an additive set function over the lattice of portfolios: \( x'_{K \cup L} + x'_{K \cap L} = x'_{K} + x'_{L} \);
2. \( \Delta x'_j = x'_j - x'_0 = \frac{\lambda_j}{p_j(1 - p_j)} \);
3. \( \Delta x'_K = 0 \), if \( |K| > 1 \);
4. \( \Delta x'_K = x'_0 + \sum_{\ell \in K} \frac{\lambda_{\ell}}{p_{\ell}(1 - p_{\ell})} \).

**Proof.** See Appendix A.

It is interesting to inquire whether the reward structure is submodular, additive, or supermodular. The preceding result shows that it is additive at the margin.

**Theorem 3.** Let \( \rho(x) = -u''(x)/u'(x) \) be the Arrow Pratt coefficient of risk aversion; then

1. if \( \rho'(x) < -\rho^2(x) \) then \( u_K \) is a supermodular lattice function; if \( \rho'(x) > -\rho^2(x) \) then it is submodular;
2. if \( \rho'(x) < -2\rho^2(x) \) then \( u_K \) is a supermodular lattice function; if \( \rho'(x) > -2\rho^2(x) \) then it is submodular.
Proof. To prove part (1), consider the function $\phi$ defined implicitly by $\phi(x'(u)) = x(u)$. The marginal reward function $x'_K$ is additive, and hence, it is both submodular and supermodular. Supermodularity is preserved under a convex transformation (Topkis, 1978), so it is sufficient to show that $\phi$ is a convex function whenever $\rho'(x) < -\rho^2(x)$. But

$$\phi''(x'(u)) = -\frac{\rho'(x) + \rho(x)^2}{\rho(x)^2 x'(u)^2},$$

which is opposite in sign to $\rho'(x) + \rho^2(x)$.

To prove part (2), consider the function $\psi$ defined implicitly by $\psi(x'(u)) = u$. It is sufficient to show that $\psi$ is convex whenever $\rho'(x) < -2\rho^2(x)$. But

$$\psi''(x'(u)) = -\frac{\rho'(x) + 2\rho(x)^2}{\rho(x)^3 x'(u)},$$

which is opposite in sign to $\rho'(x) + 2\rho^2(x)$.

Thus, unless risk aversion declines very rapidly with wealth, the contract will have a submodular structure. The interpretation is that the agent is rewarded to a decreasing degree for multiple successes. The borderline case that separates submodular from supermodular payments is when utility is logarithmic, since in this case $\rho'(x) = -\rho^2(x)$. The borderline case that separates submodular from supermodular payoffs in utility units is the square root utility function, since in this case $\rho'(x) = -2\rho^2(x)$.

The structure of the optimal contract is particularly clear if the utility function is logarithmic, since in that case the marginal reward $x'_K$ equals the actual payment $x_K$. In that case, the payment made to the agent if $K$ is the set of projects that succeeds is

$$x_K = x_0 + \sum_{i \in K} \frac{\lambda_i}{p_i(1 - p_i)}.$$

Thus projects are rewarded quite independently. In the general case, the reward function takes the form

$$x_K = \phi\left(x_0 + \sum_{i \in K} \frac{\lambda_i}{p_i(1 - p_i)}\right),$$

where $\phi$ is a transformation (concave in the payment submodular case, convex in the supermodular case) that depends only on the agent’s utility function, and the $\lambda_i \geq 0$.

2.5. An example

With a small number of projects, the optimal portfolio is easy to calculate. We illustrate with a two-project example. Consider first, the minimum cost portfolio. The incentive compatibility and individual rationality constraints are

$$e(p_1, p_2) = (1 - p_1)(1 - p_2)u_0 + p_1(1 - p_2)u_1 + p_2(1 - p_1)u_2 + p_1 p_2 u_{12},$$
$$e_1(p_1, p_2) = (u_1 - u_0) + p_2(u_0 - u_1 - u_2 + u_{12}),$$
$$e_2(p_1, p_2) = (u_2 - u_0) + p_1(u_0 - u_1 - u_2 + u_{12}),$$

with $p_1 + p_2 = 1$. If the utility function is logarithmic, the optimal portfolio is

$$x_K = x_0 + \frac{\lambda_1}{p_1(1 - p_1)} + \frac{\lambda_2}{p_2(1 - p_2)},$$

where $\lambda_i$ are weights determined by the utility function.

In the general case, the optimal portfolio is

$$x_K = \phi\left(x_0 + \sum_{i \in K} \frac{\lambda_i}{p_i(1 - p_i)}\right),$$

where $\phi$ is a transformation (concave in the payment submodular case, convex in the supermodular case) that depends only on the agent’s utility function, and the $\lambda_i \geq 0$.
where $u_0$, $u_1$, $u_2$ and $u_{12}$ are the conditional rewards in utility terms, $p_1$ and $p_2$ the probabilities of success, and $e$ the effort function. As shown above, the minimum cost conditions imply that the contract (in financial terms) is additive at the margin

$$x'(u_0) - x'(u_1) - x'(u_2) + x'(u_{12}) = 0,$$

where $x$ is the payment or inverse utility function. Given the desired success rates $(p_1, p_2)$, then these four equations can be solved for the $u_i$ and the for least cost contract $(x_0, x_1, x_2, x_{12}) = (x(u_0), x(u_1), x(u_2), x(u_{12}))$ that implements this outcome. Depending upon the cost function and the utility function, this system can be expected to be non-linear, but easy to solve numerically.

Fig. 1 shows a typical example of how the optimal contract varies with $p_1$ and $p_2$. We recall that $x_0$ is the payment if both projects fail, $x_1$ the payment if only project 1 succeeds, $x_2$ the payment if only project 2 succeeds, and $x_{12}$ the payment if both projects succeed. In this example, we assume a logarithmic utility function and a cost function of the form

$$e(p_1, p_2) = \frac{\alpha_1}{1 - p_1} - \alpha_1 + \frac{\alpha_2}{1 - p_2} - \alpha_2$$

with $\alpha_1 = 0.1$, $\alpha_2 = 0.1$. 
Table 1
Optimal contracts, symmetric costs \((\alpha_1 = 0.1, \alpha_2 = 0.1)\)

<table>
<thead>
<tr>
<th>Project values</th>
<th>Probabilities</th>
<th>Contract</th>
<th>Expected payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b_1)</td>
<td>(b_2)</td>
<td>(p_1)</td>
<td>(p_2)</td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
<td>0.55</td>
<td>0.55</td>
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<tr>
<td>1.00</td>
<td>2.00</td>
<td>0.54</td>
<td>0.65</td>
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<tr>
<td>1.00</td>
<td>3.00</td>
<td>0.54</td>
<td>0.69</td>
</tr>
<tr>
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<td>4.00</td>
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<td>0.72</td>
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<td>6.00</td>
<td>0.52</td>
<td>0.75</td>
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</table>

Once we know the contract, then one can calculate the Lagrange multipliers \(\lambda_1\) and \(\lambda_2\), and the shadow prices \(\mu_1 = \partial \mathcal{L}(p_1, p_2)/\partial p_1\) and \(\mu_2 = \partial \mathcal{L}(p_1, p_2)/\partial p_2\). A simple calculation shows that

\[
\lambda_1 = p_1(1 - p_1)(x_1 - x_0),
\]

\[
\lambda_2 = p_2(1 - p_2)(x_2 - x_0),
\]

\[
\gamma = x_1' + p_1(x_1' - x_0') + p_2(x_2' - x_0'),
\]

\[
\mu_1 = (x_1 - x_0) - \lambda_2(u_0 - u_1 - u_2 + u_{12}) \\
+ p_2(x_0 - x_1 - x_2 + x_{12}) + \lambda_1 e_{11} + \lambda_2 e_{21},
\]

\[
\mu_2 = (x_2 - x_0) - \lambda_1(u_0 - u_2 - u_1 + u_{21}) \\
+ p_1(x_0 - x_2 - x_1 + x_{21}) + \lambda_2 e_{22} + \lambda_1 e_{12}.
\]

The shadow prices \(\mu_1\) and \(\mu_2\) measure the marginal cost of increasing \(p_1\) and \(p_2\). If the value of the projects is \(b_1\) and \(b_2\), then \(b_1\) and \(b_2\) represent the marginal benefits of increasing \(p_1\) and \(p_2\). The optimal contract, given the project values \(b_1\) and \(b_2\) and the cost parameters \(\alpha_1\) and \(\alpha_2\), is found by equating marginal costs and benefits, and solving the equations

\[
b_1 = \mu_1, \quad b_2 = \mu_2.
\]

Table 1 shows a range of optimal contracts under an assumption of symmetric costs \((\alpha_1 = 0.1, \alpha_2 = 0.1)\), while Table 2 shows optimal contracts under an assumption of asymmetric costs (project 2 is more costly than project 1, \(\alpha_1 = 0.05, \alpha_2 = 0.15\)).

3. Implications

We now consider some of the implications for the management of research\(^2\). It will be assumed throughout this section that the research portfolio is risky in the sense of

\(^2\) These results generalise the results derived in Bardsley (1999) for a single-project problem, and confirm the general validity of the conclusions drawn there.
Table 2
Optimal contracts, asymmetric costs ($\alpha_1 = 0.05$, $\alpha_2 = 0.15$)

<table>
<thead>
<tr>
<th>Project values</th>
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<td>0.69</td>
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</table>

the preceding section. As a consequence, all of the Lagrange multipliers $\gamma$ and $\lambda_i$ are non-negative. This is the main theoretical conclusion that we carry over to a discussion of some management implications.

3.1. Investment in research infrastructure

Consider the non-project-specific resources that are provided to scientists as part of the institutional infrastructure of any scientific organisation. Examples would be the provision of libraries, computing facilities, general scientific equipment, and the opportunity to attend conferences and to visit other scientific institutions. Let $x$ be expenditure by the principal on a complementary input that increases the productivity of scientists. It will be assumed that the required effort $e = e(p, x)$ depends both on the probability of success $p$ and on $x$, and that $x$ reduces the effort required both in total and at the margin

$$e_x < 0, \quad e_{i,x} < 0.$$ (10) (11)

What are the implications of the incentive effect for the investment decision? Assume that the principal determines the optimal value of $x$ by undertaking a standard cost-benefit evaluation that takes into account the direct cost savings but that ignores the indirect effects due to changes in the agent’s behaviour. There are two reasons why this calculation might be of interest. The first is that this is in fact the standard practice for making such decisions. The second is that it is relatively easy to make this calculation, as it does not require any analysis of the agent’s behaviour. Will this lead to under or over investment?

**Theorem 4.** Under the maintained assumptions, a standard cost-benefit analysis will lead to under investment in complementary inputs such as libraries and scientific equipment.

**Proof.** The marginal benefit of increasing $x$ can be found, by the envelope theorem, by differentiating the Lagrangian (9); the marginal cost is clearly 1. The optimal choice of $x$ is then given by the relation

$$\gamma e_x + \sum_i \lambda_i e_{i,x} = 1.$$
This equation can be interpreted as follows. The first term is the direct cost saving (the smaller payment that is required by the agent) due to the investment \( x \), assuming \(^3\) that there is no change in \( p \). The Lagrange multiplier \( \gamma \) just converts units of marginal utility into units of the transfer payment made from the principal to the agent. The second term is the indirect benefit due to the fact that the agent will choose to work harder, and will implement a more favourable \( p \).

A standard project analysis looks only at the direct cost savings, assuming that there is no change in the agent’s action. It ignores the indirect or incentive benefit. Since the \( \lambda_i \) are positive, investment in \( x \) will be suboptimal.

Since the investment reduces the agent’s marginal effort while leaving unchanged the agent’s expected reward from success, one would expect a general increase in the agent’s level of effort. However, there may be some re-allocation of effort between projects, and one cannot guarantee that the success rate will increase on all projects. One can show by a similar argument that “on average” the probability of success will rise.

**Theorem 5.** Under the maintained assumptions, \( \sum_i (-e_i, x) \frac{dp_i}{dx} \geq 0 \).

Note that, by assumption, the weights \( (-e_i, x) \) are positive.

### 3.2. Choice of projects

Now consider the type of projects that the principal will wish to include in the portfolio. Assume that the type of the project is indexed by the gross benefits \( b \) that are achieved if the project succeeds, and that more valuable projects are more difficult to carry out successfully. Let \( \mathbf{p} \) be the vector of success probabilities and let \( \mathbf{b} \) be the vector of gross project benefits, and let \( e(\mathbf{p}, \mathbf{b}) \) be the effort function. It costs more, both in total and at the margin, to achieve the same success probability with a more valuable project:

\[
e_{b_i} \geq 0, \quad e_{p_i, b_j} \geq 0.
\]

Under these assumptions, projects with a higher \( b_i \) can be characterised as more risky in the sense that the effort curve is shifted up.

The expected benefit to the principal from project \( i \) is \( p_i b_i \). The marginal benefit from increasing \( b_i \), that is from choosing a slightly more valuable project is \( p_i \). The marginal cost may be calculated using the envelope theorem from Eq. (6). Thus, the condition for choosing \( b_i \) optimally is

\[
p_i = \gamma e_{b_i} + \sum_j \lambda_j e_{p_j, b_i}.
\]

\(^3\) In this analysis, we take the parameter \( p \) as the specification of the agent’s action. One can derive a similar result if we let the agent’s action be described by any parameter \( z \) which, together with \( x \), influences \( p \). The benefits of investment can be split into the benefits holding \( z \) fixed and the incentive benefits that occur because the agent chooses a better value of \( z \). Once again, we show that investment is sub-optimal.
It is clear that if the incentive effect is ignored the marginal cost will be underestimated. There will be a bias towards projects that are too difficult and too risky.

**Theorem 6.** Under the maintained assumptions, a standard cost-benefit analysis that neglects incentive effects will be biased towards the selection of projects that are too risky.

4. Conclusion

The combinatorial multi-task model is a computationally tractable model of task design and contract specification. We illustrate the model with an application to scientific research. Public research is undertaken by large, complex and decentralised organisations. While the managers of these research institutions pay attention to the cost benefit calculations produced by their project analysts, this is only a small part of their task. Much of their work is concerned with the management of people and of groups of people. This paper suggests that this concern is intrinsic to the nature of the task, and that the proper economic analysis of research projects should take into account the interaction of project choice with and incentive structure of the organisations that carry out the research as well as the direct costs and benefits of projects.

A narrow reliance on traditional project evaluation techniques may lead to a systematic bias. The conclusions drawn from such a framework need to be tempered in a way that might be characterised as “friendly to scientists”. However, this conclusion is not driven by any benevolent concern for the welfare or the peace of mind of scientists. Risk aversion imposes an important constraint on the rewards and incentives that can be imposed in organisations. Policies that moderate the agent’s unnecessary exposure to risk allow stronger and sharper incentives to be imposed. These stronger incentives lead to greater effort and a closer alignment between the agent’s personal objectives and the corporate objectives of the principal.

Appendix A. Möbius inversion on lattices

Let $L$ be a lattice, and let $f : L \to \mathbb{R}$ be a real valued function. Think of $f$ as a probability density function, and let $F : L \to \mathbb{R} : F(K) = \sum_{J \leq K} f(J)$ (it will be assumed that the sum is always finite). $F$ can be thought of as the analog of a probability distribution function.

The problem of recovering $f$ from $F$ is known as Möbius inversion (Aigner, 1979; Wehrhahn, 1990). If $L$ is the real line, and the sum is interpreted as an appropriate integral, then by the fundamental theorem of calculus $f(K) = F'(K)$ almost everywhere, and Möbius inversion amounts to the fact that the derivative is an inverse operator to the indefinite integral.

If $L$ is a finite lattice then there exists a lattice invariant $\mu_J^L$ defined recursively by

$$
\mu_J^L = \begin{cases} 
0, & \text{if } K \leq J, \\
1, & \text{if } K = J, \\
-\sum_{K < Z \leq J} \mu_Z^L, & \text{if } K < J.
\end{cases}
$$
such that
\[ f(K) = \Delta F(K) = \sum_{Z \leq K} \mu_Z^K F(Z), \]
where the lattice derivative \( \Delta \) is defined by this equation.

If \( L \) is the lattice of integers \( Z \), then \( \Delta \) is the usual finite difference operator: \( \Delta f(x) = f(x) - f(x - 1) \). If \( L \) is the real line \( R \) then, under an appropriate interpretation, \( \Delta \) is the derivative of the Dirac delta function.

If \( L \) is the Boolean lattice if subsets of \( N = \{1, \ldots, n\} \), ordered by set inclusion, then
\[ \mu^K_J = (-1)^{|J| - |K|} \]
where \(|J|\) denotes the number of elements of the set \( J \). On a Boolean lattice \( \mu^K_J \) takes on the values 0, 1, and \(-1\), so the lattice derivative \( \Delta \) takes the typical form of an alternating sum over subsets (the “inclusion exclusion principle”). The following lemmas about Möbius inversion on a Boolean lattice will be useful.

**Lemma 7.** Let \( P_J = \prod_{j \in J} p_j \), and let \( P^K_N = \prod_{k \in K} \prod_{l \notin K} p_k (1 - p_l) \). Then \( P^K_N = \sum_{K \subset J} \mu^K_J P_J \).

**Proof.** If the projects \( i, j, \ldots \) are independent, then \( P_J \) is the probability that all the projects in the portfolio \( J \) succeed, while \( P^K_N \) is the probability that all the projects in the portfolio \( J \) succeed while those that are not in \( J \) fail. The result can be proved by a simple inclusion-exclusion argument, but it is best illustrated by an example. Let \( N = \{1, 2, 3\} \) and let \( J = \{1\} \). Then
\[ p^{1,2,3}_1 = p_1(1 - p_2)(1 - p_3) = p_1 - p_1p_2 - p_1p_3 + p_1p_2p_3 = \mu^1_1 p_1 - \mu^{1,2}_1 p_1p_2 - \mu^{1,3}_1 p_1p_3 + \mu^{1,2,3}_1 p_1p_2p_3. \]

**Corollary 8.** Let \( A = \sum_J P_J \Delta u_J = \sum_{K \subset J} P_J \mu^K_J u_K = \sum_{K \subset N} P^K_N u_K \). Then
\[ \frac{\partial^2 A}{\partial u_K \partial p_j} = \begin{cases} P^K_{K-j}, & \text{if } j \in K, \\ -P^K_{K-j}, & \text{if } j \notin K. \end{cases} \]

**Lemma 9.** \( x_J = \sum_{K \subset J} \Delta x_K \).

**Proof.** Recall that \( \Delta x_K = \sum_{Z \leq K} \mu_Z^K x_Z \). The result is immediate by Möbius inversion. It is best illustrated by an example. Let \( J = \{1, 2\} \).
\[ \Delta x_0 = x_0, \]
\[ \Delta x_1 = x_1 - x_0, \]
\[ \Delta x_2 = x_2 - x_0, \]
\[ \Delta x_{1,2} = x_{1,2} - x_1 - x_2 + x_0, \]
\[
\sum_{K \subseteq \{1,2\}} \Delta x_K = x_{1,2}.
\]

**Lemma 10.** If \( \Delta x_J \leq 0 \) for all \( J \) such that \(|J| > 1\) then \( x \) is a submodular set function.

**Proof.**

\[
\begin{align*}
&x_{A \cup B} - x_A - x_B + x_{A \cap B} = \\
&\sum_{A \subseteq K \subseteq A \cup B} \Delta x_K + \sum_{A \subseteq K \subseteq A \cup B} \Delta x_K.
\end{align*}
\]

By assumption, all terms on the right are non-negative.

**A.1. Proof of the key results**

**Proof** (Theorem 1). Let us first show that \( \lambda_j = 0 \) if \( p_j = 0 \). Consider the portfolio \( K = \{j\} \). Note that \( PK_j = p_j \prod_{i \neq j} (1 - p_i) = 0 \), and \( PK_{N-j} = \prod_{i \neq j} (1 - p_i) \neq 0 \). This latter inequality holds because of the assumptions on the effort function: no matter how large the effort, success cannot be made certain for any project. Then by Eq. (7) we see that \( \lambda_j = 0 \).

If \( p_j > 0 \) then the constraint binds. In order to show that \( \lambda_j > 0 \) in this cost minimisation problem, it is sufficient to show that the dot product between the gradient of the cost function and the gradient of the constraint function is strictly positive (these are both vectors in a \( 2^{|N|} \) dimensional space indexed by portfolios \( J \subset N \)). That is to say, if a project is active (so \( p_j > 0 \)), then an increase in the agent’s marginal cost will strictly increase the principal’s cost.

The gradient of the cost function is

\[
\text{grad} \sum_{K \subseteq J} P_j \mu_K x_K = [P^K_N x'_K] \rightarrow x'_0[\delta_0(K)] \text{ as } p \rightarrow 0.
\]

Recall that 0 is the empty portfolio, so \( u_0 \) is the agent’s income utility when all projects fail, and \( x'_0 \) is the marginal cost of providing this utility; \( \delta_0(K) \) is the delta function, defined by \( \delta_0(K) = 1 \) if \( K = 0 \), while \( \delta_0(K) = 0 \) if \( K \neq 0 \).

Since \( p_j \neq 0 \) the \( j \)th incentive compatibility constraint can be written

\[
e_j(p) - \sum_{j \in K} \frac{P^K_k u_K}{p_j} + \sum_{j \notin K} \frac{P^K_k u_K}{1 - p_j} \geq 0
\]

so the gradient of the constraint function is \( [\alpha_K] \), where

\[
\alpha_K = \begin{cases} 
- \frac{P^K_N}{p_j}, & \text{if } j \in K, \\
+ \frac{P^K_N}{1 - p_j}, & \text{if } j \notin K.
\end{cases}
\]
Notice that
\[
\alpha_K \rightarrow \begin{cases} 
1, & \text{if } K = 0, \\
-1, & \text{if } K = j, \\
0, & \text{otherwise},
\end{cases}
\]
as \( p \to 0 \). Thus, the dot product tends to a strictly positive quantity \( x_0' \) as \( p \to 0 \). By
continuity, the dot product must be positive for \( p \) close to 0.

**Proof** (Theorem 2). Assertions (i) and (ii) follow from the first-order conditions by direct
calculation. It is known (see Topkis (1978)) that an additive lattice function is a valuation,
and can be written \( x_K' = x_0' + \sum_{i \in K} \Delta x_i' \); this proves (iv).

To demonstrate (iii), note that by Möbius inversion
\[
x_K' = \sum_{J \subset K} \Delta x_J = \Delta x_K' + \sum_{J \notin K, |J| > 1} \Delta x_J' + \sum_{J \in K} \Delta x_J' + x_0'.
\]
But \( x' \) is a valuation so
\[
\Delta x_K' + \sum_{J \notin K, |J| > 1} \Delta x_J' = 0.
\]
The result follows by induction.

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