Mediation of Mathematical Meaning Through the Graphic Calculator

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ABSTRACT In a climate where the use of calculators is being criticised as one of the reasons for failure in mathematics, this paper explores how graphic calculators can be used to support students’ mathematical development through the processes of visualisation, formalisation and interpretation. It explores the current research through a vignette of one student’s experience and argues for a reconceptualisation of assessment procedures that makes use of the representational power of the graphic calculator.

Introduction

If men learn writing, it will implant forgetfulness in their souls; they will cease to exercise memory because they will rely on that which is written, calling to remembrance no longer from within themselves, but by means of external marks. You have found a charm not for remembering but for reminding, and you are providing your pupils with the semblance of wisdom, not the reality. (Plato, Phaedrus, 380BC.)

This amusing quotation takes on a rather more poignant character when we note that a recent newspaper report (Pyke, 1996) stated that A-level examinations in England and Wales would no longer permit the use of calculators in some parts of the examination. Perhaps we shall soon be ordered to do our washing by hand, sweep our carpets with a broom and abandon our cars for a horse and cart? The rationale behind this statement regarding calculators is to ‘increase the rigour of a qualification which has found itself recently under attack’. Such statements illustrate popular rhetoric; that calculators of any description are ‘a bad thing’; they decrease students’ understanding of mathematics; they make them lazy and they do not support the development of mathematical understanding. There are of course some plausible arguments as to why there might justifiably be times
when students are asked to tackle certain types of tasks mentally even within an examination, and these can be constructed so as to render the calculator redundant. There are many such situations but these two examples will serve to make the point: in pure mathematics, a question involving reasoning or proof, whilst, in applied mathematics, a question which requires the formulation of a mathematical model will each demand thinking skills, which technology can not offer.

However, the use of calculators can sometimes support and develop mental methods making them more robust. A student experienced in the use of a graphic calculator is likely to develop a sense of the characteristics of graphs of functions through many comparisons between emerging mental models and feedback from the calculator screen. Of course, an examination question which simply asks the student to sketch a graph may become trivial if such calculators are available, but a newly constructed question could instead emphasise the characteristics of a graph.

This article reviews the literature on the use of calculators as a tool for supporting mathematical thinking through graphic and other representations and makes a case in support of the graphic calculator in developing mathematical thinking.

A Vignette

Leah has started an Advanced Level course in mathematics. She bought the graphic calculator recommended by her teacher while at the airport at the start of her holiday. She started to play with it – following the instructions in the manual. She turned to her mother and showed her the quadratic function \( y = x^2 + 2x + 1 \) that she had just graphed. She then decided to change the constant term without any prompting. Her mother asked “Why’s that?” when she showed her what happened. Leah said she didn’t know, but thought she would see what happened if she changed the constant again. She continued this for a number of values, as shown in Figure 1.

Leah was asked to guess what would happen to the graph each time, before she pressed the buttons. Her conjectures became more and more assured and she was eventually convinced that the constant term determined the intercept on the y-axis. She tried it again, changing the values of \( a \) and then \( b \). She was surprised that increasing the coefficient of \( x^2 \) made the graph ‘thinner’ and not ‘fatter’ as she had conjectured. She also noticed that the coefficient of \( x \) had an effect on where the turning point on the graph fell, and she was surprised that whatever she changed, the constant term still determined where the graph cut the y-axis. Her conjectures continued and prompted by her mother she began to form and test relationships between the coefficients; she conjectured what a particular graph would look like and where it would be placed on the grid.

Figure 1. Graphic calculator screen showing a family of quadratics.
Several thoughts come to mind from this vignette of Leah. First, how easily she accepted the two representations of the function, and secondly, how she wanted to explore what it was about the algebraic representation that determined the shape of the graph. How much longer would it have taken her if she had to draw each one by hand? How convinced would she have been of the results? And even more importantly, would she have asked the same questions without the calculator to hand?

The graphic calculator can be a tool for mediating mathematical meaning. It is the students who controls and formulates the input and makes sense of the output. In this paper, we wish to review these two aspects of Leah’s story. In the next section, we will consider the process of formalising mathematical thinking into conventional representations. Later, we shall discuss the process of interpreting images as presented by the graphic calculator. Of course, these two process are deeply linked, and the relationship between them will provide the focus for the final section of this paper.

**Formalising**

Mathematics has many of the characteristics we associate with a language. Admittedly it is more precise than conventional language and this precision has a cost in that, as a language, it is rather constrained in the variety of ways that shades of meaning can be expressed. Nevertheless, thinking mathematics, speaking mathematics and writing mathematics involve different modes of expression, as is the case with conventional language.

One of the difficulties that faces mathematics teachers is that they, as the only people in the classroom already inculcated into the culture and domain of mathematics, have to act as the arbiter of whether a student’s way of expressing mathematical thinking is valid. Technology can be a vital medium for validating student’s naive attempts to express their mathematical ideas. When a student expresses an idea on a graphic calculator, the feedback from the screen will usually indicate whether the mathematical idea expressed was that intended. However the student also needs to be sure that the representation they see on the screen is what was
intended. Mediation between student, teacher and screen can now take
place, but the representation has come as a result of interaction between the
student and the screen and not between the student and the teacher. The
negotiation of meaning and interpretation becomes more equal between
student and teacher, and the novice/expert divide is reduced (Alexander,

Formalising is the process by which mathematical thinking is turned
into a conventional representation, such as an algebraic expression, or a
graph, or a table of numbers. We do not know in the vignette above what
Leah’s expectations were when she entered the equation \( y = x^2 + 2x + 1 \), but
it surprised her that the graph of \( y = 2x^2 + 2x + 1 \) was thinner than the first
graph when she had conjectured that it would be ‘fatter’. The formalisation
of this thinking – the graph, indicated that this was not so and led her into a
cycle of conjecture, test, interpret, .... Teachers can enter the world of a
student’s thinking through the screen as if it was a window on the student’s
mind. Leah was the one who had asked the question and it was her
endeavours that helped her to understand that the higher the coefficient of
\( x \) the narrower the curve became, but it is the teacher who could use that
window to help scaffold Leah’s developing understanding.

However the use of symbolic language like that above can cause
difficulties for some students. Leah felt comfortable with the formalisation
and was able to generalise, but that is not always so readily accomplished.
Ruthven (1994) suggests how generating successive terms of sequences on a
graphic calculator meant that ‘rules had to be formulated in terms of the
formal language of the calculator’:

Here, different strategies for finding an iterative rule led to different
symbolic formulations of the rule: the terms 2, 4, 8, 16, ..., for example,
producing rules such as Ans + Ans or 2 x Ans; or 2, 4, 10, 28, ...
suggesting rules such as Ans – 2 + 2 x Ans, as well as 3 x Ans – 2 and
Ans + Ans + Ans – 2. (Ruthven, 1994, p. 163)

The step to formalising Ans as ‘n’ is a trivial one and the process here
develops an understanding of equivalent expressions.

This issue of formalising can of course be an important feature of
computer software, as well as graphic calculators. An excellent example of
the formalising process is given by Ainley (1995) in describing the work of
two 11 year old boys with a spreadsheet. These children were investigating
the problem in Figure 2.

After first working with art straws to make the sheep pens, the boys
realised that it was possible to calculate the length of the pen from the
width. At this point they began to enter the data directly into the
spreadsheet. An intervention by the researcher encouraged the two boys to
‘teach’ their rule to the spreadsheet. This required them to formalise their
thinking into the algebraic language of the spreadsheet. They needed to
express, in the spreadsheet’s language, the notion of ‘double’ and the notion of ‘take away from’. The latter proved extremely difficult for them.

Figure 2. The sheep pen activity

We conjecture that such activities give children a sense of the purpose and the power of formalising. They realise that unlike their teacher, the spreadsheet simply will not be able to interpret non-formal rules, such as ‘take away from’. It is our belief that this experience of using formalising contributes to children’s success in understanding variables. In common with other computer based environments, children’s thinking is supported by feedback given by the computer on their attempts to give a formalisation. (Ainley, 1995, p. 33)

Ruthven (1995) in his research about the effects of calculator use in schools argues that the centrality of the user is vital when a calculator is used to graph symbolic expressions, particularly when they are fairly complex, say in fraction format. ‘...more is required than simply transcribing the expression; it has to be translated from fraction format to line format acceptable to the machine.’ (p. 241).

This may be seen by many teachers as an obstacle, a reason for not using graphic calculators. However, it is also a learning opportunity, since the line format emphasises the nature of a fraction as a division. Thus, the child has to re-present the mathematical concept in a way acceptable to the graphic calculator, but in so doing may make new connections, between fraction and division in this case.

Interpreting

The graphic calculator requires formal input, as discussed above, and gives formal output. Hence, the graph, or the algebraic expression, or the table of values, will be presented in a conventional format, which requires interpretation by the child. Ruthven (1995) notes that ‘... the resulting graphic display needs to be interpreted. Here, whichever default setting is used for the range of the axes, the user needs to recognise that only a
restricted portion of the graph is shown, and to find an appropriate re-scaling of the axes.’ (p. 241).

There are many issues associated with the interpretation of graphs. We discuss some of the issues raised in our minds by Leah’s vignette. Goldenberg (1987) has described the ‘illusions’ which can be caused when technology presents graphs for the child to interpret. These illusions can be caused, for example, by the technology choosing to show only a portion of the graph. If Leah had chosen a large negative constant, the calculator may have presented the graph as two apparently disconnected curves (see Figure 3).

Such an image may easily have confused Leah or led her in new possibly unproductive directions in her investigation. Similarly scaling can present illusory effects. Leah’s quadratic can appear linear if the axes are re-scaled to zoom in on smaller and smaller sections of the parabola. (Tall, 1985, has used this feature of graphing software in an imaginative way to encourage the visualisation of the derivative of a function in terms of ‘local straightness’.)

The graphic calculator is complex. Pimm’s work (1995) focuses on imagery and symbols in constructing mathematical meaning. He argues that modern calculating devices are far less transparent with regard to their functioning than their earlier counterparts, the abacus and the slide rule. He describes the calculator’s mechanisms as ‘opaque’ so therefore they ‘offer very little support ... leaving pupils free to form their own imagery with regard to using such devices to gain either numerical fluency or understanding’ (p. 81).

Certainly there is no direct link between the operation of a modern calculator and the output. In contrast, the old-style mechanical calculating machines incorporated the notion of multiplication as repeated addition which was physically represented by the turning of handles, and the notion of place value manifested itself explicitly in terms of shifting mechanisms.

There is therefore a need for calculator-based pedagogy to help children make links. There are many examples in the CAN project (see for example Duffin, 1996) of how calculator-based activities can support the
child’s mathematical thinking. The graphic calculator is a relatively recent tool and more work is needed in this direction. Some examples can be taken from the literature for computer-based activities. Pratt (1995) has described a pedagogical approach, referred to as ‘active graphing’, in which children collate data from an experiment into a spreadsheet as they carry out the experiment, and generate scatter graphs as they proceed, using these graphs to try to interpret the experiment. The links between the graph, their memories of the experiment and the table of values help the child to develop their understanding of the graph itself as an analytical tool. This active graphing approach could be used with graphic calculators as a way of helping children to interpret graphical representations.

*Formalising and Interpreting*

In fact the two processes highlighted above are linked, with the interpretation of the graphic calculator’s feedback leading to new conjectures and new mathematical ideas. These can then be formalised once more and tested on the calculator. Leah’s attempts to understand the role of the constant in the quadratic equation offers a clear example of this iterative cycle.

Pimm recognises that the graphic calculator can permit the user to develop an aspect of imagery that is often inadequately developed. Smart (1995) provides evidence to this effect. She found that while using graphic calculators, students displayed ‘unusually high levels of visual strategies to solve algebraic problems’ after working with graphic calculators for just one term. Her research focused particularly on girls’ mathematical development and she observed that visual skills increased significantly alongside a developing confidence in mathematics. Previous research on mathematical development (Gipps & Murphy, 1994) has suggested that from adolescence, girls visual-spatial skills are less well developed then boys. This indicates the benefits of the representational function of the graphic calculator in promoting this development in girls. Although graphic calculators help students develop a link between the symbolic/algebraic form and the visual/graphical form, Julie (1993) is concerned that the latter is used to support the former and as a result the visual/graphical form is somewhat devalued. Smart found that the visual/graphic mode of thought when working on equations became a legitimate mode for developing girls’ understanding and they would explain the difference between two equations, \( y=(x+2)^2 \) and \( y=(x-2)^2 \), say, by tracing diagrams on their desks or on scraps of paper.

This class used the graphic calculators to build a more fundamental visual image of equations given in symbolic form. Then, in preference to the manipulation of symbols, they employed their visual knowledge to help
make generalisations and solve any new problems. (Smart, 1995, pp. 202-203)

Ruthven’s earlier work supports this view (Ruthven, 1991). He evaluated a project sponsored by the National Council for Educational Technology (NCET) in which small groups of classroom teachers participated in using graphic calculators with post-16 students studying an Advanced Level examination course in mathematics. Control groups were set up who were taught the same course without the use of graphic calculators. Test items were devised in such a way as to give no direct advantage to groups of
students who had access to graphic calculators. There were two sets of items broadly defined as symbolisation tasks (an algebraic description of a graph) and interpretation tasks (of verbally-contextualised graphs). An example of each is given in Figures 4 and 5.

Figure 4. A symbolisation task.

Figure 5. An interpretation task.

He found that students’ performance was superior in the project group compared to the control group on the symbolisation items but not on the interpretation items. He argued that the difference in the symbolisation results was due to the fact that regular use of a graphic calculator was likely to:

- Rehearse specific relationships between particular symbolic and graphic forms [...] Moreover, reliable access to graphic calculators is likely to
encourage both students and teachers to make more use of graphic approaches in solving problems and developing new mathematical ideas, not only strengthening these specific relationships, but rehearsing more general relationships between graphic and symbolic forms. (Ruthven, 1991, p. 447)

Another set of factors attributable to the success of the project students was that access to a graphic calculator for checking results and conjectures might cause reduced uncertainty and diminished anxiety. In the interpretation tasks none of the graphic calculator facilities was able to support students in their execution of the tasks. However Ruthven suggests that in the long term this might be facilitated by asking students to explore relationships between graphs and functions and those of their differential and integral functions.

Other uses of graphic calculators have been to explore particular cases and then to make generalisations. The calculator allows a large number of cases to be explored and then graphic calculators take this a step further by allowing generalisation for graphic representations. Teacher support is crucial however. For example, two girls were working on the equation $y=2x+1$ using a graphic calculator (Open University, 1994). The task they had been set was to draw other lines that were parallel to the first. They soon realised, through trial and improvement, that as long as they kept the equation in the form $y=2x+c$, the lines would all be parallel. However, they were not able to draw a parallel line below the line $y=2x+1$, when asked by their teacher, indicating that the generality was only partially understood. The mediation of a teacher was necessary to show them that $y=2x$ and $y=2x-1$ were part of the same ‘family’ and that the value of $c$ could range from plus infinity to minus infinity. The emphasis on the graphic calculator as only a tool to support and explore mathematical understanding is apparent yet again, as is the need for mediation between student, teacher and screen.

Another teacher, Jo, to whom the authors spoke, works intensively with graphic calculators throughout the secondary school and was working on a similar task to that just described. The class had drawn graphs of the form $y=x+c$ and she then asked them to explore $y=2x+c$. She used the phrase ‘$y = two \times plus c$’. Many students entered $y=xx$ into their machines as their formalisation of ‘two $x$’ and were surprised when a parabola appeared. The difficulties students have with mathematical notation are well known (see Kieran, 1992, for an overview of these issues) and were discussed earlier; it is often described as learning another language. Here was a situation where students were given access to a very graphic display of the difference between $xx$ and $x+x$ (or $2x$), and it also illustrates the difficulty pupils have with using the correct mathematical notation. The calculator has an inbuilt interpretation for $xx$ which is not necessarily that of the pupil, and it is again the teacher’s role to intervene and point out how important is knowledge
and correct usage of acceptable mathematical notation especially where calculators or computers are involved. In the next stage, Jo might have asked the students to graph \( y=2x \), \( y=x+x \) and \( y=x^2 \), and/or to numerically calculate \( y \) for different values of \( x \). (One of the advantages of the graphic calculator is that it will do both). The experience of the students is therefore much richer and more visual through offering them alternative ways of ‘seeing’ and, given that students have preferred ways of learning (Gardner, 1994), allowing them to make choices about how they might convince themselves of a correct answer when encountering similar situations in the future.

**Conclusion**

We have set out the role that the graphic calculator can play in supporting the fundamentally important mathematical processes of formalising and interpreting. It is essential that teachers, both in continuing and in initial teacher education, can identify how a technology-based pedagogy can support rather than hinder good practice.

It is well understood that pedagogy can be driven by assessment, and this is why it is so important that our systems of assessment are seen to encourage such ways of teaching and learning. In fact, some examination questions would become trivial with the introduction of the graphic calculator (particularly as the symbolic manipulators built within them increase in power). For example:

Find the equations of the asymptotes of the curve
It is perhaps such concerns that feed the popular demand for the restriction or banning of calculators in examinations. We would advocate that the examination should be instead reconceptualised in terms of what can now be explored that could not be so readily or easily have been explored before, and in ways which demonstrate conceptual understanding rather than rehearsal of traditional techniques that were essential without the aid of tools that did not exist 20 years ago, but are now less important.
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References


