The density of the extinction probability of a time homogeneous linear birth and death process under the influence of randomly occurring disasters

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Abstract

Under the influence of randomly occurring disasters, the eventual extinction probability, \( q \), of a birth and death process, \( Z \), is a random variable. In this paper, we obtain an integral expression for the probability density function \( g(x) \) of \( q \) under the assumption that the population process \( Z \) is a time homogeneous linear birth and death process and the disasters occur according to an arbitrary renewal process so that its interarrival times have a density. An example is provided to demonstrate how to evaluate the integral numerically. © 2000 Elsevier Science Inc. All rights reserved.

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Keywords: Birth and death process; Extinction probability; Random disasters; Galton–Watson process

1. Introduction

Consider a birth and death process \( Z = \{Z(t); t \geq 0\} \), with \( Z(0) = 1 \). If the evolution of \( Z \) is under the influence of a sequence of randomly occurring disasters, then the probability of extinction of \( Z \) is a random variable and not a constant as is the case in the absence of outside influence.

In this paper we consider a time homogeneous linear birth and death process \( Z \), under the influence of randomly occurring disasters whose timing is governed by an arbitrary renewal process. Many natural phenomena can be modeled by this process. For example, a population of boars in the Central Range of Taiwan are living under the risk of forest fires. We may assume that the forest fires occur according to a Poisson process, with each forest fire killing a binomial
proportion of the boar population. Between fires, the size of boar population is modeled by, e.g., a
time-homogeneous Markov branching process. A second example is a population of insects on a
farm. To protect the crops, farmers spray pesticide repeatedly over time. If we assume that the
waiting time between sprays are i.i.d. random variables with a common probability density
function \( f(t) \), then the randomly occurring disasters (sprays) form a renewal process. Each spray
causes a binomial proportion of death of the insects. Between two sprays, the population of in-
sects follows a linear birth and death process.

Still another example is in pharmacology. Suppose a certain anticancer drug is administered to
patients. The drug stays in the body for a period of time whose duration (the residence time)
depends on the rate of metabolism of each individual patient. Here, the drug may be viewed as a
continuous stream of ‘disasters’, and each disaster kills a binomial proportion of cancer cells.
Between two disasters, cancer cells grow and die following a linear birth and death process.

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authors like Athreya and Karlin [1], Kaplan, Sudbury and Nilsen [2]. Some excellent summaries
can be found in [3,4]. The topic was further studied by Brockwell et al. [5]; Pakes [6]; Zeiman [7];
Grey and Lu [8].

The main results we obtained are the integral form of the density of the asymptotic distribution
of the probability of population extinction and its numerical solution if the random disasters are
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Grey and Lu [8].
By a standard renewal approach (see, e.g., [9]) it is easy to see that the probability generating function \(E(s^{Z(t)})\) of the population size right after the first disaster is \(M((1 - \varepsilon)s + \varepsilon, t_1)\).

Now consider the embedded process \(Z_n = Z((\sum_{i=1}^{n} t_i)^+)\) which is the number of individuals that survive the \(n\)th disaster. Define \(T_n = (t_1, t_2, \ldots, t_n) \forall n \geq 1\) and \(S(T_n) = (t_2, t_3, \ldots, t_n) \forall n \geq 2\), and let \(F_n(s, T_n) = E(s^{Z_n})\) be the probability generating function of \(Z_n\). Then by the i.i.d. property of \(t_1, t_2, \ldots\), we obtain

\[
F_1(s, T_1) = F_1(s, t_1) = M((1 - \varepsilon)s + \varepsilon, t_1),
\]

and for \(n \geq 2,\)

\[
F_n(s, T_n) = E(s^{Z_n}) = E(E(s^{Z_n} \mid Z_1)) = E(F_{n-1}(s, S(T_n)))
\]

\[
= F_1(F_{n-1}(s, S(T_n)), t_1)
\]

\[
= M((1 - \varepsilon)F_{n-1}(s, S(T_n)) + \varepsilon, t_1).
\]

Define \(v = v(t) = (\mu - \lambda)t, \beta = \beta(t) = (\mu/(\mu - \lambda))(\exp(\mu - \lambda)t - 1).\) Then (see e.g., [10] or [11])

\[
M(s, t) = E(s^{X(t)}) = \frac{\beta - (\beta - \varepsilon)s}{1 + \beta - (1 + \beta - \varepsilon)s}.
\]

Let \(\beta_i = \beta(t_i), v_i = v(t_i), S_n = \sum_{i=1}^{n} v(t_i)\) and define

\[
A(t_i) = -(1 - \varepsilon)(\beta_i - e^{v_i}),
\]

\[
B(t_i) = \beta_i - \varepsilon(\beta_i - e^{v_i}),
\]

\[
C(t_i) = -(1 - \varepsilon)(1 + \beta_i - e^{v_i}),
\]

\[
D(t_i) = 1 + \beta_i - \varepsilon(1 + \beta_i - e^{v_i}).
\]

Then

\[
F_1(s, t_1) = M((1 - \varepsilon)s + \varepsilon, t_1) = \frac{\beta_1 - (\beta_1 - e^{v_1})(1 - \varepsilon)s + \varepsilon}{1 + \beta_1 - (1 + \beta_1 - e^{v_1})(1 - \varepsilon)s + \varepsilon}
\]

\[
= \frac{A_1s + B_1}{C_1s + D_1},
\]

where

\[
A_1 = A(t_1), \quad B_1 = B(t_1), \quad C_1 = C(t_1), \quad D_1 = D(t_1).
\]

Similarly, use (2.1),

\[
F_2(s, t_2) = F_1(F_1(s, t_2), t_1) = \frac{(A_2s + B_2)}{(C_2s + D_2)},
\]

where

\[
A_2 = A_1A(t_2) + B_1C(t_2) = (1 - \varepsilon)e^{v_2} - (1 - \varepsilon)(\beta_2 + \varepsilon)e^{v_1} - (1 - \varepsilon)^2\beta_1,
\]

\[
B_2 = A_1B(t_2) + B_1D(t_2) = \varepsilon e^{v_2} + (1 - \varepsilon)(\beta_2 + \varepsilon)e^{v_1} + (1 - \varepsilon)^2\beta_1,
\]

\[
C_2 = C_1A(t_2) + D_1C(t_2) = (1 - \varepsilon)e^{v_2} - (1 - \varepsilon)(\beta_2 + \varepsilon)e^{v_1} - (1 - \varepsilon)^2\beta_1 - (1 - \varepsilon)^2,
\]

\[
D_2 = C_1B(t_2) + D_1D(t_2) = \varepsilon e^{v_2} + (1 - \varepsilon)(\beta_2 + \varepsilon)e^{v_1} + (1 - \varepsilon)^2\beta_1 + (1 - \varepsilon)^2.
\]
By induction, it can be shown that

$$A_n = (1 - \varepsilon)e^{Sn} - \sum_{i=1}^{n-1} (1 - \varepsilon)^i (\beta_{n-i+1} + \varepsilon)e^{S_{n-i}} - (1 - \varepsilon)^n \beta_1,$$

$$B_n = \varepsilon e^{Sn} + \sum_{i=1}^{n-1} (1 - \varepsilon)^i (\beta_{n-i+1} + \varepsilon)e^{S_{n-i}} + (1 - \varepsilon)^n \beta_1,$$

$$C_n = (1 - \varepsilon)e^{Sn} - \sum_{i=1}^{n-1} (1 - \varepsilon)^i (\beta_{n-i+1} + \varepsilon)e^{S_{n-i}} - (1 - \varepsilon)^n \beta_1 - (1 - \varepsilon)^n,$$

$$D_n = \varepsilon e^{Sn} + \sum_{i=1}^{n-1} (1 - \varepsilon)^i (\beta_{n-i+1} + \varepsilon)e^{S_{n-i}} + (1 - \varepsilon)^n \beta_1 + (1 - \varepsilon)^n.$$  \hfill (2.6)

And the extinction probability right after the \(n\)th disaster is

$$q_n = F_n(s, T_n) \big|_{s=0} = \frac{B_n}{D_n}.$$ \hfill (2.7)

When \(\varepsilon = 0\) (i.e., disasters do not cause death) and \(\lambda > \mu\), then \(B_n\) and \(D_n\) become telescoping series that \(B_n = (\mu/(\mu - \lambda)) \exp(\mu - \lambda) \sum_{i=1}^{n} t_i + \mu/(\lambda - \mu) \rightarrow \mu/(\lambda - \mu)\) and \(D_n = B_n + 1\). Therefore \(q_n \rightarrow \mu/\lambda\) a.e. If \(\varepsilon = 0\) and \(\lambda < \mu\), then \(q_n \rightarrow 1\) a.e. The special case \(\lambda = \mu\) also implies \(q_n \rightarrow 1\).

3. Probability of extinction under random disasters

To study the eventual extinction probability of the process \(Z\) we utilize the embedded process \(\{Z_n\}_{n=1}^{\infty}\) in which \(Z_n\) is the population size immediately after the \(n\)th disaster. This is a Galton–Watson process in a random environment. We shall consider the supercritical case, \(\lambda > \mu\), in which the extinction is not certain. By (2.7), the extinction probability of \(Z_n, q_n,\) right after the \(n\)th disaster is a random variable which depends on the timing of all previous disasters, i.e.,

$$q_n = q_n(t_1, t_2, \ldots, t_n).$$

Let \(T_n\) and \(S(T_n)\) be defined as in Section 2, then \(q_n\) satisfies the following iterative property:

$$q_{n+1}(T_{n+1}) = q_{n+1}(t_1, t_2, \ldots, t_{n+1})$$

$$= P\left\{\sum_{i=1}^{n+1} t_i = 0\right\}$$

$$= \sum_{m=0}^{\infty} P\{Z(t_1^+ = m)(q_n(t_2, \ldots, t_{n+1}))^m$$

$$= F_1(q_n(t_2, \ldots, t_{n+1}), t_1)$$

$$= F_1(q_n(S(T_{n+1})), t_1).$$
Passing to the limit, the probability of extinction, \( q = \lim_{n \to \infty} q_n \), of the process \( \{Z(t); t \geq 0\} \) is easily seen to satisfy the equation

\[
q(T) = \lim_{n \to \infty} q_n(T_n) = F_1(q(S(T)), t_1),
\]

(3.2)

where \( T = (t_1, t_2, \ldots) \) and \( S(T) = (t_2, t_3, \ldots) \).

It is worth noting that the almost sure convergence limit, \( q \), of \( q_n \), is a random variable which depends only on the interarrival times and the killing rate.

Based on this equation we shall derive the distribution function of \( q \), \( G(x) = P\{q \leq x\} \), and show that the density \( g(x) \) of \( q \) exists under very mild conditions. The proofs are long and will be presented in Sections 3.1 and 3.2. They are followed by an example in Sections 3.3 and 3.4.

### 3.1. Differentiability of \( G \)

**Theorem 1.** Under the assumptions stated in Section 2. Let \( F_1(s, t) = M((1 - \varepsilon)s + \varepsilon, t) \), where

\( M(s, t) \) is as defined in (2.2). Let \( F_1^* \) be the inverse of \( F_1 \) with respect to its first argument, and \( F_1^{**} \) be the inverse of \( F_1^* \) with respect to its second argument (see (3.8) and (3.10) below). Assume that the density \( f(t) \) of the interarrival times of disasters is such that \( \partial / \partial x[f(F_1^{**}(x, y))(\partial / \partial y)F_1^{**}(x, y)] \) exists and is integrable on \([a, 1)\), \( a = \max(0, (r - \varepsilon)/(1 - \varepsilon)) \), \( r \) is the natural extinction probability, i.e. \( r = \mu / \lambda \). Then \( G(x) \) is differentiable on (0,1) with the only exception \( x = r \).

**Proof.** By (3.2), conditioning on \( t_1 \), we obtain

\[
G(x) = P[q(T) \leq x] = E\{P[q(T) \leq x | t_1]\}
\]

\[
= \int_0^\infty f(t)P[q(T) \leq x | t_1 = t] \, dt
\]

\[
= \int_0^\infty f(t)P\{F_1(q(S(T)), t) \leq x\} \, dt
\]

(3.3)

\[
= \int_0^\infty f(t)P\{q(S(T)) \leq F_1^*(x, t)\} \, dt
\]

\[
= \int_0^\infty f(t)G(F_1^*(x, t)) \, dt
\]

since \( q(T) \) and \( q(S(T)) \) have the same probability distribution. To simplify (3.3), we notice that when \( x \) is fixed, \( F_1^*(x, t) \) is a monotone function of \( t \), therefore if we change variable (fix \( x \)) by \( y = F_1^*(x, t) \), then \( t = F_1^{**}(x, y) \) for some function \( F_1^{**} \). Also note that the integration limit \( t = 0 \) in (3.3) corresponds to \( y = (x - \varepsilon)/(1 - \varepsilon) \) (for \( M((1 - \varepsilon)s + \varepsilon, 0) = (1 - \varepsilon)s + \varepsilon = x \) implies \( s = (x - \varepsilon)/(1 - \varepsilon) \)) and \( t = \infty \) corresponds to \( y = 1 \), therefore by change of variable the last integral can be written as below and we have
\[ G(x) = \int_{\frac{x-\epsilon}{1-\epsilon}}^{1} f(F_1^{**}(x,y)) \frac{\partial F_1^{**}(x,y)}{\partial y} \, dy. \]  
\hspace{1cm} (3.4)

It is obvious that adding disasters only increases extinction probability; thus \( 0 < r \leq q \leq 1 \). Therefore \( G(x) = 0 \) for all \( 0 \leq x < r \), and \( G(x) \) is differentiable (in fact, \( G'(x) = 0 \)) at least for \( 0 \leq x < r \). Let us write \( G(x) = \int_{\frac{x-\epsilon}{1-\epsilon}}^{1} H(x,y)G(y) \, dy \), where \( H(x,y) = f(F_1^{**}(x,y))\frac{\partial F_1^{**}(x,y)}{\partial y} \). First we show that \( G(x) \) is continuous on \([0,1] \). We already know that \( G(x) = 0 \) for all \( x \) in \([0,r] \). It suffices to consider \( r < x_0 < x < 1 \). We have

\[
G(x) - G(x_0) = \int_{\frac{x-\epsilon}{1-\epsilon}}^{1} H(x,y)G(y) \, dy - \int_{\frac{x_0-\epsilon}{1-\epsilon}}^{1} H(x_0,y)G(y) \, dy
\]

\[
= \int_{\frac{x-\epsilon}{1-\epsilon}}^{1} [H(x,y) - H(x_0,y)]G(y) \, dy - \int_{\frac{x_0-\epsilon}{1-\epsilon}}^{1} H(x_0,y)G(y) \, dy.
\]

The continuity of \( H(x,y) \) in \( x \) and boundedness of \( G(y) \) imply that the first integral of the last equality tends to zero when \( x \) tends to \( x_0 \) from the right. The integrability of \( H(x_0,y) \) with respect to \( G(y) \) ensures that the last integral also tends to zero as \( x \) tends to \( x_0 \) (see e.g. [12]). Hence \( G(x) \) is right continuous. The proof for the left continuity is similar. Next we show that \( G(x) \) is differentiable on \((0,1)\). Consider for \( r < x_0 < x < 1 \),

\[
\lim_{x \to x_0} \frac{G(x) - G(x_0)}{x - x_0} = \lim_{x \to x_0} \int_{\frac{x_0-\epsilon}{1-\epsilon}}^{1} \left[ \frac{H(x,y) - H(x_0,y)}{x - x_0} \right] G(y) \, dy
\]

\[
- \lim_{x \to x_0} \int_{\frac{x_0-\epsilon}{1-\epsilon}}^{1} \left[ \frac{H(x,y) - H(x_0,y)}{x - x_0} \right] G(y) \, dy.
\]

The first integral tends to

\[
\int_{\frac{x_0-\epsilon}{1-\epsilon}}^{1} \left. \left( \frac{\partial H(x,y)}{\partial x} \right) \right|_{x = x_0} \, dy
\]

by the Lebesgue Dominate Convergence Theorem. This integral is finite by the assumption of \( H(x,y) \). The second one tends to zero and the third one tends to a finite value both by the Mean–Value Theorem for integrals, similarly for the case when \( x \) tends to \( x_0 \) from the left. This proves Theorem 1. \( \square \)

**Remark.** Given an arbitrary \( f(t) \), it is not difficult to check the integrability of the function

\[
\frac{\partial}{\partial x} \left[ f(F_1^{**}(x,y)) \frac{\partial}{\partial y} F_1^{**}(x,y) \right]
\]

since \( (\partial/\partial y)F_1^{**}(x,y) \) is a function of \( y \) alone (see 3.11).

### 3.2. Density of \( q \)

Differentiating \( G(x) \) in (3.3) to obtain the density of \( q \), \( g(x) = G'(x) \), then
\[ g(x) = \int_0^\infty f(t)g(F_1^*(x,t)) \frac{\partial F_1^*(x,t)}{\partial x} \, dt, \]

or equivalently (replace \( t \) by \( F_1^*(x,y) \))

\[ g(x) = \int_{-\infty}^t f(F_1^*(x,y))g(y) \left( \frac{\partial F_1^*(x,t)}{\partial x} \bigg|_{t=F_1^*(x,y)} \right) \frac{\partial F_1^*(x,y)}{\partial y} \, dy. \] (3.5)

Unfortunately there is no close form solution of \( g(x) \) for the general case in (3.5); however for the special case that the occurrence of disasters follows a Poisson process, we develop a simple method that can evaluate the numerical solution easily.

### 3.3. An example

To illustrate how \( g(x) \) can be evaluated, we assume that the occurrence of disasters follows a Poisson process with intensity \( \alpha > 0 \), then the interarrival density is \( f(t) = \alpha e^{-\alpha t}, t > 0 \).

By (2.2) and (2.3),

\[ M(s,t) = \frac{(\mu - \mu e^{(\lambda-\mu)t}) - (\lambda - \mu e^{(\lambda-\mu)t})s}{(\mu - \lambda e^{(\lambda-\mu)t}) - (\lambda - e^{(\lambda-\mu)t})s}, \] (3.6)

\[ F_1(s,t) = M((1-\varepsilon)s + \varepsilon, t) = \frac{(\mu - \mu e^{(\lambda-\mu)t}) - (\lambda - \mu e^{(\lambda-\mu)t})((1-\varepsilon)s + \varepsilon)}{(\mu - \lambda e^{(\lambda-\mu)t}) - (\lambda - e^{(\lambda-\mu)t})((1-\varepsilon)s + \varepsilon)}. \] (3.7)

To invert \( F_1(s,t) \) in (3.7) for fixed \( t \), set \( x = F_1(s,t) \), we obtain

\[ s = F_1^*(x,t) = \frac{1}{1 - \varepsilon} \left\{ \frac{(\mu - \lambda e^{(\lambda-\mu)t})x - (\mu - \mu e^{(\lambda-\mu)t})}{(\lambda - \lambda e^{(\lambda-\mu)t})x - (\lambda - e^{(\lambda-\mu)t})} - \varepsilon \right\}, \] (3.8)

so that

\[ \frac{\partial}{\partial x} F_1^*(x,t) = \frac{1}{1 - \varepsilon} \frac{(\lambda - \mu)^2 e^{(\lambda-\mu)t}}{[(\lambda - \lambda e^{(\lambda-\mu)t})x - (\lambda - e^{(\lambda-\mu)t})]^2}. \] (3.9)

Next we invert \( F_1^*(x,t) \) with \( x \) fixed. Put \( y = F_1^*(x,t) \), then

\[ t = F_1^{**}(x,y) = \frac{1}{\lambda - \mu} \log \left\{ \frac{(\lambda((1-\varepsilon)y + \varepsilon) - \mu)(1-x)}{(\lambda x - \mu)(1 - ((1-\varepsilon)y + \varepsilon))} \right\}, \] (3.10)

hence

\[ \frac{\partial}{\partial y} F_1^{**}(x,y) = \frac{1 - \varepsilon}{[\lambda((1-\varepsilon)y + \varepsilon) - \mu][1 - ((1-\varepsilon)y + \varepsilon)]}. \] (3.11)

If we write \( \gamma = \alpha/\lambda - \mu \), then (3.5) becomes
\[ g(x) = \frac{x(\lambda x - \mu)^{\gamma - 1}}{(1 - x)^{\gamma + 1}} \int_{\frac{1}{\lambda}}^{1} \left[ \frac{1 - ((1 - \varepsilon)y + \varepsilon)}{\lambda((1 - \varepsilon)y + \varepsilon) - \mu} \right]^7 g(y) \, dy, \] (3.12)

for \( \mu/\lambda < x < 1 \) and \( g(x) = 0 \) for all \( x \) in \([0, \mu/\lambda] \).

Put

\[ c = \int_{\frac{1}{\lambda}}^{1} \left[ \frac{1 - ((1 - \varepsilon)y + \varepsilon)}{\lambda((1 - \varepsilon)y + \varepsilon) - \mu} \right]^7 g(y) \, dy \] (3.13)

then obviously

\[ g(x) = \frac{x(\lambda x - \mu)^{\gamma - 1}}{(1 - x)^{\gamma + 1}} \cdot c \]

for all \( x \) in the interval \([\mu/\lambda, (\mu/\lambda) + (1 - \mu/\lambda)\varepsilon] \). To determine the value of \( g(x) \) for \( x \) in the interval \((\mu/\lambda) + (1 - \mu/\lambda)\varepsilon, 1\), we observe that

\[ \int_{\frac{1}{\lambda}}^{1} \left[ \frac{1 - ((1 - \varepsilon)y + \varepsilon)}{\lambda((1 - \varepsilon)y + \varepsilon) - \mu} \right]^7 g(y) \, dy = c - \int_{\frac{1}{\lambda}}^{\varepsilon/\lambda} \left[ \frac{1 - ((1 - \varepsilon)y + \varepsilon)}{\lambda((1 - \varepsilon)y + \varepsilon) - \mu} \right]^7 g(y) \, dy. \]

The integral can be estimated subinterval by subinterval that \( g(x) \) is a certain multiple of \( c \) by Simpson’s rule. This will be demonstrated in Section 3.4.

### 3.4. Numerical results

Suppose \( \lambda = 6, \mu = 2, \varepsilon = 0.4, \alpha = 4 \). Then (3.12) becomes

\[ g(x) = (4/(1 - x)^2) \int_{(x - 0.4)/0.6}^{1} h(y)g(y) \, dy, \quad 1/3 \leq x < 1, \]

where \( h(y) = [1 - (0.6y + 0.4)]/[6(0.6y + 0.4) - 2] \). Let

\[ c = \int_{1/3}^{1} h(y)g(y) \, dy, \]

then for all \( x \) in \([1/3, 0.6] \), \( g(x) \) is a multiple of \( c \). For example,

\[ g(1/3) = 9 \int_{1/3}^{1} h(y)g(y) \, dy = 9c, \quad g(0.5) = 16c, \quad g(0.6) = 25c, \ldots \]

The values of \( g(x), x \in (0.6, 0.76] \), can be estimated from the values of \( g(x) \) for \( x \) in \([1/3, 0.6] \). For example,

\[ g(0.7) = (4/0.09) \int_{0.5}^{1} h(y)g(y) \, dy = (4/0.09) \left( c - \int_{1/3}^{0.5} h(y)g(y) \, dy \right). \]
Table 1
The density of the extinction probability, \( g(x) \), given by (3.12) \((\lambda = 6, \mu = 2, \varepsilon = 0.4, \alpha = 4)\)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) ) as multiple of ( c )</th>
<th>Numerical values of ( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.333</td>
<td>9.0000c</td>
<td>0.72809</td>
</tr>
<tr>
<td>0.367</td>
<td>9.9723c</td>
<td>0.80675</td>
</tr>
<tr>
<td>0.400</td>
<td>11.1111c</td>
<td>0.89888</td>
</tr>
<tr>
<td>0.433</td>
<td>12.4567c</td>
<td>1.00774</td>
</tr>
<tr>
<td>0.467</td>
<td>14.0625c</td>
<td>1.13764</td>
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<td>0.500</td>
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<td>1.29438</td>
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<tr>
<td>0.533</td>
<td>18.3673c</td>
<td>1.48590</td>
</tr>
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<td>0.567</td>
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The integration limit $[1/3, 0.5]$ is contained in the interval $[1/3, 0.6]$ therefore the integral
\[ \int_{1/3}^{0.5} h(y)g(y) \, dy \]
can be estimated by Simpson's rule. (For example, divide $[1/3, 0.5]$ into
\[ \{8/24, 9/24, 10/24, 11/24, 12/24\} \]
then \[ \int_{1/3}^{0.5} h(y)g(y) \, dy \approx 0.3636843c \]
and therefore $g(0.7) \approx 28.2807c$. Continue in this way and using the fact that $\int_{1/3}^{1} g(x) \, dx = 1$, the constant $c$ can be determined as $c = 0.080899$. Table 1 lists 20 values of $g(x)$ and the graph of $g(x)$ is sketched in Fig. 1.

References