Values of games with probabilistic graphs

Emilio Calvo\textsuperscript{a,}\textsuperscript{*}, Javier Lasaga\textsuperscript{b}, Anne van den Nouweland\textsuperscript{c}

\textsuperscript{a}Departament d’Anàlisi Econòmica, Universitat de Valencia, Av. dels Tarongers, s/n, Edificio Departamental Oriental, 46022 Valencia, Spain

\textsuperscript{b}Departamento de Economía Aplicada I, Avda. Lehendakari Agirre, 83, 48015 Bilbao, Spain

\textsuperscript{c}Department of Economics, 435 PLC, 1285 University of Oregon, Eugene, OR 97403-1285, USA

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Abstract

In this paper we consider games with probabilistic graphs. The model we develop is an extension of the model of games with communication restrictions by Myerson (1977). In the Myerson model each pair of players is joined by a link in the graph if and only if these two players can communicate directly. The current paper considers a more general setting in which each pair of players has some probability of direct communication. The value is defined and characterized in this context. It is a natural extension of the Myerson value and it turns out to be the Shapley value of a modified game. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The way cooperative games with side payments in characteristic function form are defined is to assign a real number to each coalition of agents. This number may be interpreted as the economic possibilities presented to the particular coalition at hand, regardless of the actions of the other agents in the group. It is generally assumed that there are no restrictions on communication between players and, hence, every subgroup of players can effectively cooperate. But there are many economic and political contexts in which restrictions in communication arise, and then full cooperation fails.

One of these possible settings appears when direct communication is not necessarily transitive. A simple example of this kind is found in the administrative machinery of a university, where one has to know the proper channels in order to get things done. To

\textsuperscript{*}Corresponding author. Tel.: +34-63-82-82-46; fax: +34-63-82-82-49; e-mail: emilio.calvo@uv.es
capture nontransitive communications structures, Myerson (1977) introduced graphs to model communication channels between players. In such a communication graph the players are the nodes and the presence of a link between two players indicates that these two players can communicate directly. The graph induces a partition on the set of players into connected components which are the coalitions of players that can negotiate effectively in the game. For an overview of the line of research on games with communications restrictions readers are referred to Borm et al. (1994) and van den Nouweland (1993).

In this paper we will extend the model of Myerson (1977). In general it will not be the case that two players either are able to communicate and hence cooperate or are not able to do so, but rather that they will be able to cooperate up to a certain degree. For example, in many markets we observe the existence of intermediaries who try to bring buyers and sellers in contact with one another. These intermediaries play an important role in clearing the market, and they are paid for their intermediation. We could model this situation assuming that there are communication links only between buyers and intermediaries, and between intermediaries and sellers. But a more accurate description of this situation should allow us to incorporate into the model the possibility that sellers and buyers reach agreements directly, without intermediation. The probability that direct transactions will take place depends on several factors, one of the most significant being the cost of gaining information about different offers and demands in the market. Introducing these probabilities into the communication graph enables us to find the value of the players in the market as a function of the uncertainties involved in the bilateral relationships.

Another typical example appears in voting games, where parties in a parliament are considered. In order to get a measure of the power of different agents in majority voting situations, these situations can be modeled as cooperative (voting) games and then some solution concept can be used to assign an index of power to each party in the voting situation. Well known power indices are the Shapley–Shubik index (cf. Shapley and Shubik, 1954) and the Banzhaf index (cf. Banzhaf, 1965). Both aforementioned papers show that simply counting the number of votes of each party does not provide a reliable indication of the power of the parties in general. In order to get a good indication of the power of each party it is important to consider so called winning coalitions, i.e., coalitions of parties that hold a majority of the votes. Then, instead of merely counting votes, the power of a party is the probability that it has of being pivotal to the success of a winning coalition. This approach, however, fails to take into account sociological, political, and ideological aspects that influence the degree of compatibility of the objectives of different parties. These aspects may obviously influence the power of a party. A party that has sociological, political, or ideological objectives that are opposite to the objectives of most other parties may often vote differently from most parties. This will diminish the chances this party has of being critical to the success of a winning coalition. We can associate with each pair of parties a probability that reflects the a priori degree of compatibility of both parties. An extension of the Shapley–Shubik index to this type of model is then defined, and this generalized index provides an a priori indication of the power of political parties when these kinds of sociological, political, and ideological aspects are taken into consideration.

The structure of the paper is as follows. Following this introduction, Section 2 is
devoted to the introduction of Myerson’s (1977) model of games with deterministic graphs. We will discuss this model by means of two typical examples in which we show how to introduce bilateral probabilities of communication into the model. In Section 3 we formally define probabilistic graphs, we show how characteristic functions are modified by such graphs, and we define a generalization of the value for this model. In Section 4 we offer an axiomatic characterization of this value and Section 5 contains concluding remarks.

2. Deterministic graphs

Let \( N = \{1, 2, \ldots, n\} \) be a set of players and let \( L(N) = \{\{i, j\}|i \in N, j \in N, i \neq j\} \) be the set of unordered pairs of distinct players of \( N \). We will refer to these unordered pairs as communication links. A communication graph on \( N \) is any subset of links \( L \) of \( L(N) \). Let \( L_N \) be the set of all communication graphs on \( N \). For any such graph \( L \), any coalition \( S \subseteq N \), and any players \( i \) and \( j \) in \( S \), we say that \( i \) and \( j \) are connected by \( L \) within \( S \) if and only if there exists some chain of communication links within \( S \) such that \( i \) and \( j \) are connected through the chain, i.e. if \( i = j \) or there exist \( \{h_1, h_2, \ldots, h_k\} \subseteq S \) such that \( h_1 = i, h_k = j \) and \( \{h_t, h_{t+1}\} \subseteq L \) for every \( t = 1, \ldots, k - 1 \). Let \( S/L \) denote the unique partition of \( S \) into the groups of players which are connected by \( L \) within \( S \). \( S/L \) can be interpreted as the collection of smaller coalitions into which \( S \) would break up, if players could only coordinate along the links in \( L \). We say that \( S \) is internally connected by \( L \) if and only if \( S/L = \{S\} \). Given any communication graph \( L \), we can think of the internally connected coalitions as the set of coalitions that can negotiate effectively in a game.

A coalitional game is a pair \((N, v)\), where \( N \) is the set of players and \( v \) is a characteristic function \( v:2^N \rightarrow \mathbb{R} \), with \( v(\emptyset) = 0 \), where each number \( v(S) \) is interpreted as the wealth of transferable utility which the members of \( S \) would have to divide among themselves if they were to cooperate together and with no one outside \( S \). Let \( G_N \) be the set of all coalitional games with player set \( N \).

The introduction of communication restrictions (modeled by means of a graph) changes the payoffs attainable by coalitions. Given any communication graph \( L \), and a coalitional game \((N, v)\), define \( v_L \) to be the characteristic function that would result if we altered the situation represented by \( v \), requiring that players can only communicate along links in \( L \), so that (cf. Myerson, 1977)

\[
\forall S \subseteq N, v_L(S) = \sum_{T \in S/L} v(T).
\]

A game with a communication graph is a triple \((N, v, L)\), where \((N, v) \in G_N \) and \( L \in L_N \). An allocation rule for games with communication graphs is a function \( \psi:G_N \times L_N \rightarrow \mathbb{R}^N \), where \( \psi_i(N, v, L) \) should be interpreted as the utility payoff which player \( i \) would expect in game \((N, v)\) if \( L \) represents the restrictions on cooperative agreements between the players. Allocation rules can be found by applying solution

\footnote{We will slightly abuse notation and write \((N, v, L) \in G_N \times L_N \).}
concepts such as the Shapley value (cf. Shapley, 1953), the nucleolus (cf. Schmeidler, 1969), and others to the game \((N, v_L)\) associated with a game with a communication graph \((N, v, L)\). In this paper we will restrict ourselves to the Shapley value\(^2\) \((\mathcal{SH})\) and, following Aumann and Myerson (1988), we will refer to the allocation rule that we obtain in this way as the Myerson value.

**Definition 1.** Let \((N, v, L)\) be a game with a communication graph. Then the **Myerson value** of \((N, v, L)\), \(\mathcal{M}(N, v, L) \in \mathbb{R}^N\), is defined by

\[
\mathcal{M}(N,v,L) = \mathcal{SH}(N,v_L).
\]

Next, we show by means of two examples a drawback of this deterministic approach and a way to overcome it.

2.1. **Example 1**

Consider an agent who wants to sell his house and another agent who would be interested in buying the house. It is quite usual for agents who want to buy or sell a house to go to a broker, because a broker is someone who has a lot of information concerning possible buyers and sellers of houses. So, the broker can act as an intermediary between the buyer and the seller. However, if the broker acts as an intermediary, then he or she has to be paid for this intermediation.

We consider a small example. There are three agents, a seller \((s)\), a buyer \((b)\), and a broker \((i)\), so \(N = \{s, b, i\}\). If the house of the seller can be sold to the buyer, then a surplus of say 1 unit is created. The characteristic function is \(v(S) = 1\) if and only if \(S\) is equal to \(\{s, b\}\) or \(\{s, b, i\}\), and \(v(S) = 0\) otherwise. Suppose now that both the seller and the buyer know the broker, but they cannot interact directly. Hence the communication graph in this case is \(L = \{\{s, i\}, \{b, i\}\}\) and the modified characteristic function is \(v_L(S) = 0\), for all \(S \subseteq \{s, b, i\}\), and \(v_L(\{s, b, i\}) = 1\). Computing the Myerson value we find that \(\mathcal{M}(N, v, L)\) assigns 1/3 to each of the three players.

If we assume that the seller and the buyer can also communicate directly, without the intermediation of the broker, then the graph coincides with the complete graph \(L(\{s, b, i\})\) and the modified characteristic function \(v_{L(\{s, b, i\})}\) coincides with the original characteristic function \(v\). Hence \(\mathcal{M}(N, v, L(\{s, b, i\})) = \mathcal{SH}(N, v)\), which in our case gives \(\mathcal{SH}(N, v) = \mathcal{SH}_k(N, v) = 1/2\) and \(\mathcal{SH}_k(N, v) = 0\), where \(\mathcal{SH}_k(N, v)\) denotes the payoff to player \(k \in N\).

Both ways of modelling seem not to capture the situation realistically. To be more realistic the model must take into account the fact that the buyer and the seller may not know of each other but may try to find each other without the intermediation of the broker to avoid the brokerage fee. Then there is some probability, possibly very small, that they can make the transaction without the intermediation of the broker. In order to introduce this fact into the model, suppose that the probability that the seller and the

\(^2\)We refer the reader to Shapley (1953) for a definition of the Shapley value.
The majority game is completely determined by the set of minimal winning coalitions of the parties. This fact influences the power of each party, for those who are ideologically closer to each other. Denote by $v_p$ the characteristic function that takes into account these probabilities of communication. Then $v_p(\{s, b\}) = p$ and $v_p(S) = v(S)$ otherwise. If we compute the Shapley value for $v_p$, then we find that the broker gets a payoff of $v_p(N, v_p) = (1 - p)/3$ and that the seller and the buyer each get $v_p(N, v_p) = (2 + p)/6$. Hence, we see that the payoff of the broker decreases when the probability of direct cooperation between the seller and the buyer increases.

Notice that in this example it holds that $v_p$ is a convex combination of the original characteristic function $v$, which is the characteristic function we find with the complete graph, and the characteristic function $v_L$ that corresponds to the graph with links between the buyer and the broker and between the seller and the broker only. In formula: $v_p = pv + (1 - p)v_L$, which corresponds to the fact that $p$ is the probability that the complete graph $L(\{s, b, i\})$ is actually formed and $(1 - p)$ is the probability that $L = \{(s, i), (b, i)\}$ is formed. Therefore, using linearity of the Shapley value, we have

$$v_p(N, v_p) = pv(N, v) + (1 - p)v(N, v_L).$$

2.2. Example 2

Voting situations in a parliament can be modeled by a weighted majority game, that is, assume there are $n$ parties, where party $i$ has $s_i$ seats, and a coalition of parties needs more than half the number of votes to win a ballot. This is a cooperative game $(N, w)$ with player set $N = \{1, 2, \ldots, n\}$ and characteristic function $w$ defined by $w(S) = 1$ if $\sum_{i \in S} s_i > \frac{1}{2} \sum_{i \in N} s_i$, and $w(S) = 0$ otherwise, for all $S \subseteq N$. A coalition of parties $S$ is called winning if $w(S) = 1$ and losing if $w(S) = 0$. Coalition $S$ is said to be minimal winning if $w(S) = 1$ and $w(T) = 0$ for all strict subcoalitions $T$ of $S$. We denote by $MW$ the set of all minimal winning coalitions of $(N, w)$. Note that the characteristic function of a weighted majority game is completely determined by the set $MW$.

Measures of power in such games count how often a member of a winning coalition is pivotal, i.e. how often this member’s defection from such a coalition would cause it to be losing. We consider here the Shapley–Shubik index. Suppose that a bill is to be decided upon by an assembly, and that the bill under consideration aligns the voting parties in order of their enthusiasm for the proposal. Given any such alignment, there will be a single pivotal voting party, i.e. one who, by joining the more enthusiastic parties, brings the coalition up to winning strength. Assuming a priori that all $n!$ orderings of the voting parties are equally likely, the Shapley–Shubik index for each party is precisely the probability that it is pivotal. Mathematically, the Shapley–Shubik index is the Shapley value of the weighted majority game.

This approximation takes into account only the number of seats that each party has, but fails to recognize that certain parties are ideologically closer together than others. This fact influences the power of each party, for those who are ideologically closer to
each other are more likely to cooperate than parties which are extreme and isolated in
their aims.

A way to include compatibilities of parties in the model is to define a graph for the set
of parties in which each pair of parties \(i\) and \(j\) is joined by a link if and only if these two
parties are ideologically compatible. Parties can only cooperate if they are connected on
the graph directly by a link, or indirectly through other parties. Now we have a triple \((N, w, L)\),
where \((N, w)\) is a weighted majority game and \(L\) is a communication graph on \(N\).
The introduction of \(L\) modifies the voting game, and we denote by \(MW_L\) the set of all
minimal winning coalitions for the game \(w_L\). The modified power index is then the
Shapley value of \((N, w_L)\).

As an illustration, consider a fictitious parliament, in which there are four parties;
\(N = \{e, l, c, r\}\), where \(e\) denotes the “extreme left” party, \(l\) the “left” party, \(c\) the
“center” party, and \(r\) the “right” party. Suppose that the set of minimal winning
coalitions, which is determined by the number of seats each party holds, is
\(MW = \{\{e, l\}, \{e, r\}, \{l, r\}\}\). In this case \(c\) is a null party and the remaining three parties are symmetric
in the voting game. Hence, the power index corresponding to this situation is
\[
\mathcal{F}(N, w) = \frac{1}{3}, \text{ for } i = e, l, r, \text{ and } \mathcal{F}(N, w) = 0.
\]

Suppose now that we place the parties in the socioeconomic left–right dimension and
consider the communication graph \(L_1 = \{\{e, l\}, \{l, c\}, \{c, r\}\}\). Here, in \(L_1\), parties can
communicate directly only if they are placed face to face in the left–right line.
Therefore, as an example, \(l\) and \(r\) can only reach some kind of agreement if \(c\) is included
in this agreement: this can be interpreted by the fact that both \(l\) and \(r\), given their
ideological differences, could only justify an agreement before their respective voters, if
it were also accepted by an intermediate party, \(c\), in the ideological spectrum.

In the game modified according to \(L_1\), \((N, w_{L_1})\), the set of minimal winning coalitions
is \(MW_{L_1} = \{\{e, l\}, \{l, c, r\}\}\). Note that here \(c\) becomes pivotal for some orders. The
modified power index is
\[
M_i(N, w, L_1) = M_i(N, w, L_1) = \frac{1}{12},
\]
\[
M_i(N, w, L_1) = \frac{3}{12}, \text{ and } M_i(N, w, L_1) = \frac{7}{12}.
\]

Alternatively, we can consider the graph \(L_2 = \{\{e, l\}, \{l, c\}, \{l, r\}, \{c, r\}\}\), which is less
extreme than \(L_1\), because it only excludes direct communication between extreme left
party \(e\) and both parties \(c\) and \(r\). Now the set of minimal winning coalitions is
\(MW_{L_2} = \{\{e, l\}, \{l, r\}\}\) and the modified power index is
\[
M_i(N, w, L_2) = M_i(N, w, L_2) = \frac{1}{6},
\]
\[
M_i(N, w, L_1) = \frac{4}{6}, \text{ and } M_i(N, w, L_1) = 0.
\]
Graphs $L_1$ and $L_2$ differ only in link $\{l, r\}$. Its absence yields party $c$ a positive power given its intermediation role between parties $l$ and $r$.

However, in a parliament two parties are not generally wholly compatible or incompatible, but they are usually compatible to a certain degree. Therefore, it is more appealing to associate with each pair of parties $i$ and $j$ a number $p_{ij}$ between 0 and 1 that reflects their degree of compatibility. This number $p_{ij}$ should be interpreted as the probability that the two parties will agree on a particular issue. For example, suppose that link $\{h, l, r, j\}$ has a probability of $p \in [0, 1]$. Then, graphs $L_1$ and $L_2$ happen with probabilities $(1 - p^2)$ and $p^2$ respectively. Defining $w_i = (1 - p^2)$ and $w_j = p^2$, we obtain as power indices for the new situation:

$$\mathcal{P}(N, w_i^p) = \frac{3 - p}{12}, \quad \mathcal{P}(N, w_j^p) = \frac{7 + p}{12},$$

Here, the power of party $c$ is obtained now as a function of the degree of incompatibility between parties $l$ and $r$. As these parties become more compatible, the power of party $c$ declines.

### 3. Probabilistic graphs

In this section we will extend the model of Myerson (1977) in order to be able to work with probabilistic graphs. A game with a probabilistic graph is a triple $(N, v, p)$, where $(N, v)$ is a coalitional game and $p: \{(i, j) | i, j \in N, i \neq j\} \to [0, 1]$ is a function that assigns to each pair of agents $i$ and $j$ the probability that these two agents can communicate directly. The probabilities are assumed to be independent. Sometimes we will refer to the function $p$ as a system of probabilities. Further, we will often denote $p_{ij}$ instead of $p(\{i, j\})$.

Let $(N, v, p)$ be a game with a probabilistic graph. With this game we will associate a new coalitional game $(N, v_p)$, called the communication game, that incorporates both the economic possibilities of the agents described by the coalitional game $(N, v)$ and the probabilities of bilateral communication described by the system of probabilities $p$. Since we are dealing with probabilities of communication, we will consider expected profits in the new game.

Let $i, j \in N, i \neq j$. Then, with probability $p_{ij}$ agents $i$ and $j$ are able to communicate. If this is so, then they can cooperate and obtain $v(\{i, j\})$. But with probability $1 - p_{ij}$ the agents cannot communicate and in this case they cannot obtain more than $v(\{i\}) + v(\{j\})$.

Therefore, the expected profit of agents $i$ and $j$ is:

$$v_p(\{i, j\}) = p_{ij}v(\{i, j\}) + (1 - p_{ij})(v(\{i\}) + v(\{j\})).$$

Generalizing the idea that is at the basis of this definition, we can define the expected profit of arbitrary coalitions of agents. Let $S \subseteq N$ be a fixed coalition of agents and define $L(S) = \{(i, j) | i, j \in S, i \neq j\}$, the set of all possible communication links between agents in...
S. We will often denote typical links in \( L(S) \) by \( l \). For each set of links \( L \subseteq L(S) \), the probability that \( L \) is the communication graph that is realized among the agents in \( S \) is:

\[
p^*(L) = \prod_{i \in L} p_i \prod_{j \notin L} (1 - p_j).
\]

Now, suppose \( L \subseteq L(S) \) is the set of communication links that is realized. Note that the graph \( L \) induces a partition of \( S \) into communication components \( S \). Correspondingly, the worth obtainable by coalition \( S \) if \( L \subseteq L(S) \) is realized is \( v_L(S) \). Now, we can define the expected profit of coalition \( S \), namely

\[
v_p(S) = \sum_{L \subseteq L(S)} p^*(L) v_L(S).
\]

The procedure described above is a generalization of the procedure followed by Myerson (1977). To see this, note that a deterministic communication graph \( L \) on \( N \) can be identified with a function \( p : \{i, j\} \mapsto \{0, 1\} \), defined by \( p(\{i, j\}) = 1 \) if \( \{i, j\} \subseteq L \) and \( p(\{i, j\}) = 0 \) if \( \{i, j\} \notin L \). It is easily seen that for this \( p \) it holds that \( v_p = v_L \).

We want to define allocation rules for games with probabilistic graphs, i.e., rules that associate a vector of payoffs with each game with a probabilistic graph. Formally, denoting by \( G_P \) the space of coalitional games with player set \( N \) and by \( P_N \) the set of systems of probabilities for player set \( N \), an allocation rule is a function \( \psi : G_P \times P_N \rightarrow \mathbb{R}^N \). In this paper we will extend the Shapley value to games with probabilistic graphs and we will refer to the allocation rule that we obtain in this way as the Myerson value.

**Definition 2.** Let \((N, v, p)\) be a game with a probabilistic graph. Then the **Myerson value** of \((N, v, p)\), denoted \( \mathcal{M}(N, v, p) \), is defined by

\[
\mathcal{M}(N, v, p) = \mathcal{SH}(N, v_p).
\]

### 4. An axiomatic characterization

We show in this section that the Myerson value can be axiomatically characterized using two of its properties, component efficiency and fairness.

Consider a coalition that has probability 0 of communicating with any player outside the coalition and that is minimal with respect to this property. Component efficiency is an axiom that states that the total payoff to the players in such a coalition should be equal to the expected profit of the coalition. In order to introduce component efficiency formally, we need some notations. Let \((N, v, p)\) be a game with a probabilistic graph. With this game we associate a deterministic graph \((N, L_p)\) defined as follows: \( l = \{i, j\} \in L_p \) if and only if \( p_{ij} > 0 \). The graph \((N, L_p)\) induces a partition of \( N \) into

\[
\text{Although the formal expression of } v_p \text{ bears some similarity with the formula of Owen's (1972) multilinear extension, we point out that the game } v_p \text{ is not the linear extension of the game } v.
\]
communication components. We will refer to this partition as \( N/p \). Now, we are ready to introduce the property component efficiency formally.

**Definition 3.** An allocation rule \( \psi: G_N \times P_N \rightarrow \mathbb{R}^N \) is component efficient if for all games with a probabilistic graph \((N, v, p)\) and all communication components \( C \subseteq N/p \) it holds that

\[
\sum_{i \in C} \psi(N, v, p) = v_p(C).
\]

In the context of weighted majority games, where it can never be the case that there are two disjoint components of parties that are both winning, component efficiency simply states that the total power is divided among the parties that are in the component consisting of a winning coalition of parties.

The fairness axiom states that when the possibility for direct communication between two players is destroyed, other things being equal, then the payoffs of both these players change by the same amount, so either they both lose the same amount or they both gain the same amount.

**Definition 4.** An allocation rule \( \psi: G_N \times P_N \rightarrow \mathbb{R}^N \) is fair if for all games with a probabilistic graph \((N, v, p)\) and all \( i, j \in N \) it holds that

\[
\psi_i(N, v, p) - \psi_i(N, v, p_{-ij}) = \psi_j(N, v, p) - \psi_j(N, v, p_{-ij}),
\]

where \( p_{-ij}(k, l) = p(k, l) \) if \( \{k, l\} \neq \{i, j\} \), and \( p_{-ij}(\{i, j\}) = 0 \).

In the context of weighted majority games the fairness axiom states that when a party decides to try and oppose another party to diminish its power, then this will cause an equal loss in power for this party and its opponent.

**Theorem 1.** The Myerson value is the only allocation rule \( \psi: G_N \times P_N \rightarrow \mathbb{R}^N \) satisfying component efficiency and fairness.

**Proof.** To prove that the Myerson value satisfies component efficiency, let \((N, v, p)\) be a game with a probabilistic graph and let \( C \subseteq N/p \). We split up \((N, v_p)\) into two games, \((N, v^C)\) and \((N, v^{N\setminus C})\), where for all \( S \subseteq N \):

\[
v^C(S) = v_p(S \cap C) \quad \text{and} \quad v^{N\setminus C}(S) = v_p(S \setminus C).
\]

Since \( C \) is a component of \((N, L_p)\) we know that \( v_p = v^C + v^{N\setminus C} \). It follows from the dummy property of the Shapley value\(^5\) that \( \mathcal{F}(N, v^C) = 0 \) for all \( i \in C \) and that \( \mathcal{F}(N, v^{N\setminus C}) = 0 \) for all \( i \in N \setminus C \). Hence,

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\(^5\)An equivalent definition of the fairness property is obtained when replacing \( 'p_{-ij}(\{i, j\}) = 0' \) by \( 'p_{-ij}(\{i, j\}) \in [0, 1]' \). The statement of the axiom would then be as follows. When the probability for direct communication between two players changes, then the payoffs to both players change by the same amount. We refer the reader to Shapley (1953) for the properties of the Shapley value used in this proof.
\[
\sum_{i \in C} \mathcal{M}(N, v, p) = \sum_{i \in C} \mathcal{F}(N, v^C) + \sum_{i \in C} \mathcal{F}(N, v^{N\setminus C}) = \sum_{i \in C} \mathcal{F}(N, v^C) = \sum_{i \in N} \mathcal{F}(N, v^C) = v^C(N) = v_p(C),
\]

where the first and the fourth equality follow from the additivity and efficiency of the Shapley value.

To show fairness, let \((N, v, p)\) be a game with a probabilistic graph and let \(i, j \in N\), \(i \neq j\), with \(p_{ij} > 0\). Set \(w = v_p - v_{p_{-ij}}\), where \(p_{-ij}\) is identical to \(p\) except for \(p_{-ij}(\{i, j\}) = 0\). Note that \(v_p(S) = v_{p_{-ij}}(S)\) for all \(S \subseteq N\) with \(\{i, j\} \subseteq S\). So, if \(S \subseteq N\) such that \(i \in S\) or \(j \in S\), then \(w(S) = 0\). So, the only coalitions with nonzero worth in the game \((N, w)\) are coalitions containing both \(i\) and \(j\). Hence, it follows from symmetry of the Shapley value that \(\mathcal{F}(N, w) = \mathcal{F}(N, w)\). Using linearity of the Shapley value, we obtain

\[
\mathcal{F}(N, v_p) - \mathcal{F}(N, v_{p_{-ij}}) = \mathcal{F}(N, v_p) - \mathcal{F}(N, v_{p_{-ij}}).
\]

To prove uniqueness, suppose that \(\psi^1\) and \(\psi^2\) are two allocation rules that are component efficient and fair. We will prove that \(\psi^1\) and \(\psi^2\) must be identical.

First, note that component efficiency of \(\psi^1\) and \(\psi^2\) implies that \(\psi^1\) and \(\psi^2\) are identical for games with probabilistic graphs where the probabilities of communication are zero for all pairs of players. Now, suppose that \(\psi^1\) and \(\psi^2\) are not identical. Then, let \((N, v, p)\) be a game with a probabilistic graph with a minimum number of links with positive probability such that \(\psi^1(N, v, p) \neq \psi^2(N, v, p)\). By the minimality of \((N, v, p)\), we know that for any link \(\{i, j\}\) with \(p_{ij} > 0\), it holds that \(\psi^1(N, v, p_{-ij}) = \psi^2(N, v, p_{-ij})\). Hence, fairness of both \(\psi^1\) and \(\psi^2\) implies that

\[
\psi^1(N, v, p) - \psi^1(N, v, p_{-ij}) = \psi^2(N, v, p) = \psi^2(N, v, p_{-ij})
\]

\[
= \psi^1(N, v, p_{-ij}) - \psi^1(N, v, p_{-ij}) = \psi^2(N, v, p) = \psi^2(N, v, p_{-ij})
\]

From this we see that

\[
\psi^1(N, v, p) - \psi^2(N, v, p) = \psi^1(N, v, p) - \psi^2(N, v, p),
\]

whenever \(i\) and \(j\) are in the same communication component \(C \subseteq N/p\). Thus, we can find numbers \(d_C, C \subseteq N/p\), such that \(\psi^1(N, v, p) - \psi^2(N, v, p) = d_C\) for all \(i \in C\) and all \(C \subseteq N/p\). Now, we use component efficiency of both \(\psi^1\) and \(\psi^2\), which implies that for all \(C \subseteq N/p\) it holds that

\[\text{We do indeed use additivity here. As is well-known, additivity does not usually make sense in a context of voting games. Note, however, that in the instance where we use it here there will be no problem, because the decomposition of } v_p \text{ is done in such a way that one of the two games } v^C \text{ and } v^N \text{ is the zero game, and hence the sum of the two games is a well defined voting game.}\]
\[\sum_{i \in C} \psi_i^1(N, v, p) = \sum_{i \in C} \psi_i^2(N, v, p) = v_p(C).\]

Hence, we have \(0 = \sum_{i \in C} \psi_i^1(N, v, p) - \sum_{i \in C} \psi_i^2(N, v, p) = |C|d_C\), and so \(d_C = 0\).

Therefore, we conclude that \(\psi_i^1(N, v, p) = \psi_i^2(N, v, p)\). \(\blacksquare\)

**Remark 4.1.** In this theorem it is possible to replace the axiom of fairness by the requirement of balanced contributions\(^2\). An allocation rule has balanced contributions if the loss (or gain) that the isolation of a player \(i\) inflicts on a player \(j\) is equal to the effect that the isolation of player \(j\) has on player \(i\). Formally, an allocation rule \(\psi: G_N \times P_N \rightarrow \mathbb{R}^N\) has balanced contributions if for all games with a probabilistic graph \((N, v, p)\) and all \(i,j \in N\) it holds that

\[\psi_i(N, v, p) - \psi_j(N, v, p_{-i}) = \psi_j(N, v, p) - \psi_i(N, v, p_{-j}),\]

where \(p_{-i}(\{k, l\}) = p(\{k, l\})\) if \(i \notin \{k, l\}\), and \(p_{-i}(\{k, l\}) = 0\) if \(k = i\) or \(l = i\). The Myerson value is the unique allocation rule \(\psi: G_N \times P_N \rightarrow \mathbb{R}^N\) satisfying component efficiency and balanced contributions.

**Remark 4.2.** For practical purposes it may be useful to decompose the Myerson value as shown in the following proposition.

**Proposition 2.** Let \((N, v, p)\) be a game with a probabilistic graph. Then

\[M(N, v, p) = \sum_{L \subseteq (N)} p^N(L).M(N, v, L).\]

**Proof.** We show that \(\sum_{L \subseteq (N)} p^N(L).M(N, v, L)\) verifies component efficiency and fairness. Then theorem 1 can be invoked to prove the proposition.

To prove component efficiency, let \(C\) be a component of \(N/p\). If \(i \in C\) and \(j \in N \setminus C\) then \(p_{ij} = 0\), and therefore \(p^N(L) = 0\) for all \(L\) such that \(L \subseteq (L(C) \cup L(N \setminus C))\). Consider \(L \subseteq (L(C) \cup L(N \setminus C))\). Then \(C\) is a union of components in \(N/\{i\}\). Using this and the component efficiency of the deterministic Myerson value, we obtain

\[\sum_{i \in C} M_i(N, v, L) = \sum_{k \in N \setminus C, k \subseteq C} \left[ \sum_{i \in k} M_i(N, v, L) \right] = \sum_{k \in N \setminus C, k \subseteq C} \sum_{k \in N \setminus C} v(K) = v_{H(L)}(C),\]

where \(H(L) = L \cap L(C)\). Defining \(H'(L) = L \cap L(N \setminus C)\), it follows that

\(^2\)The proof of this statement is quite similar to the proofs given in Myerson (1980) and van den Nouweland (1993) (Theorem 3.2.1) and we refer the reader to these papers.
The probability of communication between the two players increases, all else being equal.

Players always \((weakly)\) benefit when the underlying game is superadditive. In a context of games with probabilistic graphs stability states that two players always \((weakly)\) benefit from reaching a bilateral agreement whenever the game \((N, v)\) is superadditive. He defined stability to be the property that two players benefit from reaching a bilateral agreement whenever the game \((N, v)\) is superadditive.

A formal statement of the stability property is:

\[
\sum_{L \subseteq L(N)} p^N(L, \mathcal{M}(N, v, L)) = \sum_{L \subseteq (L(C) \cup L(N\setminus C))} p^N(L, \mathcal{M}(N, v, L)) = \sum_{L \subseteq (L(C) \cup L(N\setminus C))} p^N(L) \sum_{L \subseteq L(N\setminus C)} \mathcal{M}(N, v, L) = \sum_{L \subseteq L(N)} p^N(L) \sum_{L \subseteq L(N\setminus C)} \mathcal{M}(N, v, L)
\]

To show fairness, first note that

\[
\sum_{L \subseteq L(N)} p^N_{-ij}(L, \mathcal{M}(N, v, L)) = \sum_{L \subseteq L(N) \setminus \{i, j\}} p^N_{-ij}(L, \mathcal{M}(N, v, L)) = \sum_{L \subseteq L(N) \setminus \{i, j\}} (p^N(L) + p^N(L_{+ij})) \mathcal{M}(N, v, L) = \sum_{L \subseteq L(N) \setminus \{i, j\}} p^N(L) \mathcal{M}(N, v, L_{-ij}),
\]

where \(L_{-ij} = L \setminus \{i, j\}\) and \(L_{+ij} = L \cup \{i, j\}\). Using this and fairness of the deterministic Myerson value we obtain

\[
\sum_{L \subseteq L(N)} p^N(L, \mathcal{M}(N, v, L)) - \sum_{L \subseteq L(N)} p^N_{-ij}(L, \mathcal{M}(N, v, L)) = \sum_{L \subseteq L(N)} p^N(L)(\mathcal{M}(N, v, L) - \mathcal{M}(N, v, L_{-ij}))
\]

\[
= \sum_{L \subseteq L(N)} p^N(L, \mathcal{M}(N, v, L)) - \sum_{L \subseteq L(N)} p^N_{-ij}(L, \mathcal{M}(N, v, L)),
\]

which proves fairness. □

Next we show that the Myerson value for games with a probabilistic graph is stable in the sense of Myerson (1977). He defined stability to be the property that two players always \((weakly)\) benefit from reaching a bilateral agreement whenever the game \((N, v)\) is superadditive. In a context of games with probabilistic graphs stability states that two players always \((weakly)\) benefit when the underlying game is superadditive and the probability of communication between the two players increases, all else being equal.

\(^9\)Note that \(1 - p_l = 1\) for all \(l \in (L(C) \cup L(N\setminus C))\).

\(^8\)A formal statement of the stability property is: \(\psi(N, v, L \cup \{i, j\}) = \psi(N, v, L)\) for all \(L \subseteq L(N)\).

\(^9\)A game \((N, v)\) is superadditive if \(v(S \cup T) = v(S) + v(T)\) for all disjoint coalitions \(S\) and \(T\).
Definition 5. An allocation rule $\psi:G_n \times P_n \rightarrow \mathbb{R}^N$ is stable if for all games with a probabilistic graph $(N, v, p)$ where the coalitional game $(N, v)$ is superadditive and for all $i, j \in N$ it holds that
\[
\psi_i(N, v, p) \succeq \psi_i(N, v, q) \text{ and } \psi(N, v, p) \succeq \psi(N, v, q),
\]
where $q \in P_n$ is such that $q(k, l) = p(k, l)$ if $\{k, l\} \neq \{i, j\}$, and $p(\{i, j\}) \succeq q(\{i, j\})$.

Proposition 3. The Myerson value is stable.

Proof. Let $(N, v)$ be a superadditive game and let $p$ and $q$ be two systems of probabilities such that $q(k, l) = p(k, l)$ if $\{k, l\} \neq \{i, j\}$, and $p(\{i, j\}) \succeq q(\{i, j\})$. First note that it follows from proposition 2 that
\[
\mathcal{M}(N, v, p) = \sum_{L \subseteq (L(N) \setminus \{\{i, j\}\})} P^N(L) \mathcal{M}(N, v, L)
\]
\[
= \sum_{L \subseteq (L(N) \setminus \{\{i, j\}\})} [p^N(L) \mathcal{M}(N, v, L) + p^N(L \cup \{i, j\}) \mathcal{M}(N, v, L \cup \{i, j\})].
\]
(1)

For all $L \subseteq (L(N) \setminus \{\{i, j\}\})$ define
\[
P^N_{-ij}(L) = \prod_{i \in L} p_i \prod_{j \in (L(N) \setminus (L \cup \{i, j\}))} (1 - p_j).
\]

Then (1) can be rewritten as
\[
\sum_{L \subseteq (L(N) \setminus \{\{i, j\}\})} P^N_{-ij}(L) [p_{ij}(\mathcal{M}(N, v, L \cup \{i, j\}) - \mathcal{M}(N, v, L)) + \mathcal{M}(N, v, L)].
\]

Note that $P^N_{-ij}(L) = Q^N_{-ij}(L)$ for all $L \subseteq (L(N) \setminus \{\{i, j\}\})$. Therefore, for $k = i, j$ we now have
\[
\mathcal{M}_k(N, v, p) - \mathcal{M}_k(N, v, q) = \sum_{L \subseteq (L(N) \setminus \{\{i, j\}\})} P^N_{-ij}(L) ([p_{ij} - q_{ij}] \mathcal{M}_k(N, v, L \cup \{i, j\}) - \mathcal{M}_k(N, v, L)).
\]

This last expression is nonnegative, because $p_{ij} - q_{ij} \geq 0$ by hypothesis and $\mathcal{M}_k(N, v, L \cup \{i, j\}) - \mathcal{M}_k(N, v, L) \geq 0$ by the stability of the deterministic Myerson value. ■

It is possible to define a potential function for games with probabilistic graphs along the lines of Hart and Mas-Colell (1989) (Shapley value) and Winter (1992) (Myerson value). Since it is a straightforward generalization, we will not include it in this paper.
5. Concluding remarks

**Remark 5.1.** Some analysts may be unwilling to use the independence assumption in order to determine the probabilities involved in each communication graph. Recall that by using this assumption we only need the value of $p_i$'s to calculate the probability of each graph $L \subseteq L(S)$. It is possible to drop independence and still have a way to measure the expected value of the players: One need only use as the primitive a probability function configuration $(p^S)_{S \subseteq N}$ defined directly over the set of all possible communication graphs, that is, for every $S \subseteq N$, $p^S: \{L | L \subseteq L(S)\} \rightarrow [0, 1]$, such that each $p^S$ is a probability distribution $(p^S(L) \geq 0 \text{ for all } L \subseteq L(S) \text{ and } \Sigma_{L \subseteq L(S)} p^S(L) = 1)$. Given $(p^S)_{S \subseteq N}$, the formula of $v_p$ applies for the calculation of the expected characteristic function.

Note that with the independence assumption, if we have $n$ parties we only need know $\binom{n}{2}$ probabilities, while without this assumption we need to know $\Sigma_{S \subseteq N} \binom{|L|}{2}$ probabilities.

**Remark 5.2.** The model in this paper can be applied to voting situations in order to take into account the (in)compatibilities of the parties in such situations. Our model offers an alternative to the traditional spatial voting models that are used to model ideological considerations of political parties. When applying the model in this paper to voting situations, the critical point is clearly the correct calculation of degrees of compatibility. Many factors come together to determine these. The main one is the location of the parties in political space. This space is made up of various dimensions: the socio-economic left–right dimension, the religious vs. lay dimension, the center vs. periphery dimension, etc. The traditional approach to modelling ideological factors is a geometric model where voters are situated in some ideological space, where distance between voters represents ideological difference. Owen (1971) and Shapley (1977) proposed adapting the Shapley–Shubik index to such spatially placed voters. For a detailed explanation of this approach as well as for further references, see the survey by Straffin (1994). Once the parties have been placed in such an ideological space, a system of probabilities can be derived from the distances between parties. There have been serious attempts to measure the positions of parties in some policy spaces. A useful source of references regarding this approach can be found in Laver and Schofield (1990).

Nevertheless, we wish to warn against any naive direct translation of distances in political space into degrees of compatibility. It is often the case that parties which are close in this space have more difficulties to reach agreements with each other than with parties that are further away. The explanation is that parties compete for voters in the neighborhood of their ideological territory. In this competition the closest parties are the greatest enemies. This is similar to a pattern sometimes observed in interactions between countries. If neighboring countries fight to determine their borders, they often form alliances with the neighbors of their neighbors (the enemy of my enemy is my friend). In politics similar behavior prevails when a party splits into two. At first the two new parties stress their differences more than their similarities and are thus less likely to reach agreements. This is also typical behavior between radical factions in the extreme left-wing spectrum. However, when parties reach an agreement about how to share their
voter territory and they are very close, it is also usual for them to form a stable coalition for elections in order to increase their number of seats, thus acting as a single party.

Though parties’ spatial location is an important determinant of the bilateral probabilities of communication, there are also other factors which play a role. One of the most important is the leader’s psychological profile. Character compatibilities do not necessarily correspond to ideological affinities, and good or bad “chemistry” between negotiators affects the course of negotiations. Finally, we would add to the list the influence of power distribution in collateral institutions, such as regional parliaments and town halls, in which parties are also involved. The predisposition to reach agreements with another party is also a strategic consideration based on the overall repercussions that such an agreement might have for other institutions. So the values of the $p_{ij}$’s also form part of the interaction strategies of parties. The model presented in this paper allows us to study the power impact of such changes.

In spatial voting models suitable dimensions have to be chosen and parties have to be identified with a particular point in the ideological space. In models of games with probabilistic graphs degrees of compatibility have to be chosen. It seems that the degrees of compatibility are more general because they encompass more than ideological considerations. On the other hand, degrees of compatibility might be more sensitive to changes in the political environment (in a broad sense). All this might imply that spatial voting models are more appropriate when one wants to investigate long-term relations and that games with probabilistic graphs are more appropriate to study short-term relations.

**Remark 5.3.** For practical applications the critical point in the process is the correct calculation of the probabilities $p_{ij}$. This is a delicate issue and each context has its own difficulties. As an example, Calvo and Lasaga (1997) calculated a system of probabilities of bilateral communication between the political parties in the Spanish Parliament of June 1993. To obtain these estimates, before the elections a survey was carried out among political commentators from press, radio and television. The survey consisted of a questionnaire in which the journalists were asked to give a number between 0 and 100 for each pair of parties $i$ and $j$. This was to reflect the probability that parties $i$ and $j$ would come to an agreement in parliament. The system of probabilities was worked out by simply taking the average of all of the reactions to the survey.

**Remark 5.4.** There are several ways in which the model of Section 3 can be generalized. One direction is to consider a priori unions of the parties. Doing so generates the possibility to consider government coalitions as part of the description of the voting situation. Since the government will generally consist of more than one political party, and since these parties are more or less committed to some governmental policy, one can imagine that by making the a priori union of the governmental coalitions part of the model one will get even more accurate indices of the power of the parties in the parliament. Generalizing the model in this direction means extending the models of games with a priori unions as defined by Owen (1977), and of games with a priori unions and deterministic communication graphs as studied by Vázquez-Brage et al.
One can also extend the model of Hart and Kurz (1983, 1984), in order to study endogenous formation of government coalitions. Some of the possible extensions described in this paragraph are considered in Calvo and Lasaga (1997). An interesting result in the paper by Calvo and Lasaga is that they are able to explain the fact that sometimes government coalitions are not minimal winning coalitions (as might be expected according to Riker’s (1962) size principle or the minimal winning coalition principle), but rather include more parties than would be necessary to have a majority of the votes. This result stems from the incorporation of probabilities of cooperation into the model: it often happens that if the parties which form the alliance have low probability communication values one with another, the introduction of a newcomer into the alliance who has high probabilities of cooperation with these parties has a positive effect and increases the expected power of all of them. In the stability analysis of the Spanish parliament performed in Calvo and Lasaga (1997) an example of this fact will be found. This communication gain effect can be seen as an additional theoretical support for the empirical evidence of the occurrence of surplus coalition (i.e. nonminimal winning) governments.

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