Variable intervals model

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Abstract

The variable intervals model is a generalization of Fishburn’s Intervals model. It fully characterizes the complete acyclic relation when the alternatives set is countable. In the uncountable case, a perfect separability condition has to be added. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A number of models are now in use for the modeling of preferences. The most famous are the semiorder model and the interval order model (a review of the variants of the semiorder model can be found in Fishburn, 1997). However, a common point in all of these models is that their asymmetric parts are transitive.

The purpose of this paper is to study the relationship between the variable intervals model (first formulated by Abbas and Vincke, 1993), which is a generalization of the intervals model (Fishburn, 1970), and the complete acyclic binary relations. The paper is organized as follows. Section 2 reviews some basic definitions in ordered sets theory. Section 3 is devoted to the variable intervals model. This model characterizes the complete acyclic relations when the set of alternatives is countable. In the uncountable case, a particular separability condition has to be added. According to the variable intervals model, an agent having a complete acyclic preference has an underlying true preorder (complete and transitive) preference. Section 4 deals with the issue of the best approximation of this underlying preference. Section 5 presents some conclusions.

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2. Preliminaries

Let $X$ be a set of alternatives. A binary relation $Q$ on $X$ is a subset of $X \times X$ denoted $X^2$. From $Q$, we define the following three binary relations:

$$Q^c = \{(x, y) \in X^2: (x, y) \not\in Q\}, \text{ the complement of } Q;$$

$$Q^- = \{(x, y) \in X^2: (y, x) \notin Q\}, \text{ the converse of } Q;$$

$$Q^d = \{(x, y) \in X^2: (y, x) \in Q\}, \text{ the dual relation of } Q.$$

$(x, y) \in Q$ is usually denoted by $xQy$.

**Partition of a binary relation:** Any binary relation $Q$ on $X$ can be divided into an asymmetric component, denoted $P_Q$, defined by $\forall x, y \in X, xP_Q y \iff \neg(yQx)$ and a symmetric component, denoted $I_Q$, defined by $\forall x, y \in X, xI_Q y \iff xQy$ and $yQx$. We shall write $Q = P_Q + I_Q$.

We can write $X^2 = Q + Q^c$ and $Q^- = I_Q + P_Q$, where $J_Q$ is the ‘incomparability’ relation defined by $\forall x, y \in X, xJ_Q y \iff \neg(xQy)$ and $\neg(yQx)$.

Let us define the following properties of a binary relation $Q$ on the set $X$.

$Q$ reflexive: $\forall x \in X, xQx$

$Q$ irreflexive: $\forall x \in X, \neg(xQx)$

$Q$ is complete: $Q \cup Q^- = X^2$, that is $Q^- \supseteq Q^c$

$Q$ is connected: $x \neq y$ and $xQy$ imply $yQx$

$Q$ asymmetric: $Q \subseteq Q^d$

$Q$ antisymmetric: $x \neq y$ and $xQy$ imply $yQ^d x$

$Q$ symmetric: $Q \subseteq Q^-$

$Q$ transitive: $xQy$ and $yQz$ imply $xQz$

$Q$ negatively transitive: $\neg(xQy)$ and $\neg(yQz)$ imply $\neg(xQz)$

$Q$ quasi-transitive: the asymmetric component of $Q$ (denoted $P_Q$) is transitive

$Q$ acyclic: $\neg(x, P_Q x, P_Q x, \ldots, P_Q x, P_Q x)$, for any $n$.

**Transitive closure:** The transitive closure of $Q$ denoted $TQ$ is a binary relation on $X$ defined by $\forall x, y \in X, xTQ y \iff \exists x_1, x_2, \ldots, x_n \in X, xQx_1 Qx_2 \ldots Qx_n Qy$.

**Equivalence:** This is represented by the relation $E_Q$, defined by $\forall x, y \in X, xE_Q y \iff Q(x) = Q(y) \text{ and } Q^-(x) = Q^-(y)$, where $Q(x) = \{b \in X: xQb\}$ is the lower section associated with $x$ and $Q^-(x) = \{a \in X: aQx\}$ is the upper section associated with $x$.

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1 Most definitions are borrowed from Monjardet (1978).
**Lower section partial preorder:** This is the partial preorder (reflexive and transitive relation) denoted $T_I$ and defined by: $\forall x, y \in X, x T_I y \iff Q(x) \supseteq Q(y)$.

**Upper section partial preorder:** This is the partial preorder (reflexive and transitive relation) denoted $T_e$ and defined by: $\forall x, y \in X, x T_e y \iff Q^-(x) \subseteq Q^-(y)$.

**Section partial preorder:** This is the partial preorder defined by: $\forall x, y \in X, x T y \iff Q^{-}(x) \subseteq Q^{-}(y)$ and $Q^{-}(x) \subseteq Q^{-}(y)$.

**Definitions:** A *partial preorder* is a reflexive and transitive binary relation. A *preorder* is a complete and transitive binary relation. A *strict partial order* is an asymmetric and transitive binary relation. An *ordering* is a reflexive, antisymmetric and transitive binary relation. A *Ferrers relation* is a binary relation whose lower or upper section partial preorder is complete. An *interval duorder* is a reflexive Ferrers relation; it is a complete quasi-transitive relation fulfilling the following Halphen condition in Halphen (1955): $\forall x, y, z, z' \in X, x I_P z \iff not(Q^{-}(y))$ and $Q^{-}(x) \supseteq Q^{-}(y)$ imply not($Q^{-}(x)$ or not($Q^{-}(y)$).

**Remark 1:** The terms interval duorder and semiduorder are borrowed from Doignon et al. (1986). An interval order is defined as the dual relation of an interval duorder (and vice versa); equivalently, the asymmetric component of an interval duorder is an interval order. Thus an interval order is an irreflexive Ferrers relation. A *semiorder* is defined as the dual relation of a semiduorder (and vice versa); equivalently, the asymmetric component of a semiduorder is a semiorder.

**Remark 2:** From a theoretical point of view, it is irrelevant whether we work with a binary relation $Q$ or with its dual $Q^d$. As the complete acyclic relations and the asymmetric acyclic relations are dual, we will work with the complete acyclic relations.

**Remark 3:** The symmetric component of a binary relation is called indifference and is generally not transitive. Equivalence $E_Q$ is always transitive because it is an equivalence relation (reflexive, symmetric and transitive). When a binary relation $Q$ is a partial preorder, indifference $I_Q$ coincides with equivalence $E_Q$.

**Definition (Numerical representation of a binary relation):** Let $Q$ be a binary relation on $X$. Say that $(X, Q)$ or $Q$ has a *numerical representation* if there exists a function $u$: $X \rightarrow \mathbb{R}$ with $\forall x, y \in X, x Q y \iff u(x) \geq u(y)$. One also says that $Q$ is *representable by a numerical function* $u$ or that a *numerical function* $u$ represents $Q$. The order homomorphism $u$ is usually called a *utility function*.
3. Representation of complete acyclic relations

Let \( R \) be a complete acyclic relation on a set \( X \). Of course, \( R \) cannot always be represented by a utility function. Moreover, because \( R \) is not necessarily a Ferrers relation, nor can we always use the result by Doignon et al. (1984) regarding the representation of Ferrers relations.

Let us now set out the result in Subiza (1994) concerning the representation of complete acyclic relations.

**Definition** (Utility Correspondence): If \( R \) is a complete relation defined on a set \( X \), then \( \mu \) defined from \( X \) onto \( R \), is a utility correspondence for the relation \( R \) if: (a) for all \( x \in X \), \( \mu(x) \) is non-empty and bounded; and (b) for all \( x, y \in X \), \( xP_R y \Rightarrow [\mu(x) \cap \mu(y) = \emptyset \) and \( \sup \mu(x) \geq \sup \mu(y)] \).

Subiza shows (Proposition 1) that the representation by utility correspondence characterizes the complete acyclic relations in the case of countable sets of alternatives. In the general case, she states the following condition: \( R \) satisfies condition [P] for the set \( D \) if when \( x, y, z, z' \) exists such that:

1. \( x \leq_R z, \ y \leq_R z' \)
2. \( xTP_R z, z'TP_R y \) where \( TP_R \) is the transitive closure of \( P_R \)
3. \( \mathcal{D}(x,z) = \mathcal{D}(z',y) \) where \( \mathcal{D}(x,z) = \{d \in D: xTP_R dTP_R z \} \)
4. \( \mathcal{J}(x,z) = \mathcal{J}(z',y) \) where \( \mathcal{J}(x,z) = \{d \in D: xI_R d \leq_R z \} \).

Then \( x \leq_R y \).

Her main result (Theorem 2) states that: if \( R \) is a complete relation separable on a set \( X \) (i.e. there exists a countable set \( D \) included in \( X \) such that for any \( x, y \) belonging to \( X \), \( xP_R y \) implies there exists \( d \) belonging to \( D \) such that \( xP_R dP_R y \)) such that for the countable set \( D \), property [P] is fulfilled then there exists a utility correspondence for the relation \( R \) if and only if \( R \) is acyclic.

We want to show here that the variable intervals model, which is a generalization of Fishburn’s intervals model (Fishburn, 1970), fully characterizes the complete acyclic relations when the set of alternatives \( X \) is countable. In the uncountable case, a necessary and sufficient perfect separability condition is given. From the point of view of graph-theory, if \( \Gamma \) is a graph representing a complete acyclic relation, both the utility correspondence and variable intervals models refer to a real-valued sets valuation of the vertices of \( \Gamma \). The interest of the variable intervals model is, however, to show that these sets are built by a valuation of the vertices and a valuation of the edges of a hypergraph \( \Gamma^* \). Let us finally note that the notion of a variable intervals’ representation of a complete acyclic relation was first formulated by Abbas and Vincke (1993).²

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² Subiza uses an asymmetric primitive preference relation. However, (see Remark 2) it is equivalent, by duality, to analyze a complete relation or an asymmetric relation.

³ I thank the referee for pointing this paper out to me.
Definition: Let $J$, $J'$ be two real-valued intervals. $J > J'$ if $\forall t \in J$, $\forall z \in J'$, $t > z$.

Definition (Variable Intervals Model): Let $R$ be a binary relation on a set $X$. $(X, R)$ satisfies the variable intervals model if there exist two functions $(f, s)$ with $f : X \to \mathbb{R}$ and $s : X \times X \to \mathbb{R}_+$ such that:

1. $xP_y \iff J(x, y) > J(y, x)$
2. $xI_y \iff J(x, y) \cap J(y, x) \neq \emptyset$

where $J(x, y) = [f(x), f(x) + s(x, y)]$ is a real-valued interval.

Remark 4: (3) $J(x, y) > J(y, x) \iff f(x) > f(y) + s(y, x)$. (4) $J(x, y) \cap J(y, x) \neq \emptyset \iff f(x) \leq f(y) + s(y, x)$ and $f(y) \leq f(x) + s(x, y)$.

Remark 5: In the representation by utility correspondence, the sets $\mu(x)$ are not necessarily connected. For instance, if $R$ is a complete acyclic binary relation on $X = \{x, y, z\}$ with $xP_yP_z$ and $xI_z$ then it is impossible for $\mu(x), \mu(y), \mu(z)$ to all be connected since they are subsets of $\mathbb{R}$. The following is a utility correspondence representation of $R$: $\mu(x) = [a, b] \cup [c, d]$; $\mu(y) = [b', c']$; $\mu(z) = [a, a']$ with $a < a' < b < b' < c' < c < d$.

Lemma 1: Let $R$ be a complete acyclic binary relation on a set $X$ and let $\mathcal{H}$ denote the set of preorders $H$ such that $P_R \subseteq P_H \subseteq H \subseteq R$. $\mathcal{H}$ is not empty.

Proof: Let $R$ be a complete acyclic binary relation, we will prove that $\mathcal{H}$ includes a complete ordering, $R = P_R + I_R$. Let $TP_R$ be the transitive closure of $P_R$. $TP_R$ is a strict partial order, then according to the Szpilrajn theorem (Szpilrajn, 1930), it has a linear (order) extension. Let $H$ be the dual relation of this linear order, $H$ is a complete ordering which is obviously included in $\mathcal{H}$. $\square$

Proposition 1: Let $X$ be a countable set and let $R$ be a binary relation on $X$, the following two conditions are equivalent.

1. $R$ is complete acyclic.
2. $(X, R)$ satisfies the variable intervals model.

Proof:

$(2) \Rightarrow (1): R$ is complete. Let $x, y \in X$. There are three cases to consider:
1st case: $f(x) > f(y)$;
2nd case: $f(x) < f(y)$;
3rd case: $f(x) = f(y)$. If $f(x) > f(y)$, then either $f(x) > f(y) + s(y, x)$ and then $xP_y$, or $f(x) \leq f(y) + s(y, x)$ and then $xI_y$ if $f(y) \leq f(x) + s(x, y)$ and $yP_x$ otherwise. If $f(x) = f(y)$ then $f(x) \leq f(y) + s(y, x)$ and $f(y) \leq f(x) + s(x, y)$, so $xI_y$. Thus $R$ is complete.
$R$ is acyclic: Let us assume that $xP_y$ and $yP_z$. Then $f(x) > f(y) + s(y, x)$ and $f(y) > f(z) + s(z, y)$. If $zP_x$, that is $f(z) > f(x) + s(x, z)$, then $f(z) > f(x) > f(y) > f(z)$. This is a contradiction. The same reasoning can be applied for the case $xP_y$ and $yP_z$.

(1) $\Rightarrow$ (2): Let $R$ be a complete acyclic relation. We know by Lemma 1 that there exists a preorder $S$ such that $P \subseteq S \subseteq R$. Let us take such a preorder $S$. Since $X$ is countable, then, according to Cantor (1895), $S$ is representable by a numerical function: $\exists u: X \rightarrow R$ such that $\forall x, y \in X$, $xHy \Leftrightarrow u(x) \geq u(y)$.

Now let us build the functions $f$ and $s$: $f(x) = u(x)$ for all $x$ belonging to $X$. If $xP_y$ then let us take $s(y, x) \in [0, \alpha]$, if $xIy$ then let us take $s(y, x) \in [\alpha, +\infty]$, where $\alpha = f(x) - f(y)$. This procedure yields a representation of $R$ by variable intervals.

(a) $xP_y \Leftrightarrow f(x) > f(y) + s(y, x), (\Rightarrow)$ Suppose that $xP_y$, then either $f(x) < f(y)$, $f(x) > f(y)$ or $f(x) = f(y)$. If $f(x) < f(y)$ then $f(x) < f(y) + s(y, x)$ and $f(y) \leq f(x) + s(y, x)$. If $f(x) > f(y)$ then $f(x) > f(y) + s(y, x)$ and $f(y) \leq f(x) + s(y, x)$. If $f(x) = f(y)$ then $f(x) = f(y) + s(y, x)$ and $f(y) = f(x) + s(y, x)$. (\Leftarrow) Let $x, y \in X$ with $f(x) \leq f(y) + s(y, x)$ and $f(y) \leq f(x) + s(x, y)$. Either $xP_y$, $yP_x$, or $xIy$. According to point (a) above, $xP_y$ or $yP_x$ are impossible because they lead, respectively, to $f(x) > f(y) + s(y, x)$ and $f(y) > f(x) + s(x, y)$. Hence $xIy$. \hfill $\Box$

When $X$ is finite, Proposition 1 gives Abbas and Vincke’s Theorem 4 (Abbas and Vincke, 1993).

**Definition:** A preference structure on a set $X$ is a triplet \{P, I, J\} of binary relations such that:

1. $P$ is asymmetric
2. $J$ is irreflexive
3. $\forall x \neq y$, $I$ is symmetric
4. $\forall x, y \in X$, $xy \Leftrightarrow \neg(yPx)$ and $\neg(xPy)$ and $\neg(xJy)$.

**Corollary** (Abbas and Vincke, 1993): Let $P, I$ and $J$ be three binary relations defined on the same finite set $X$. The following three conditions are equivalent.

1. \{P, I, J\} is a preference structure such that $J = \emptyset$ and $P$ is without circuit.
2. \exists $F: X \rightarrow H$ and \exists $S: X \times X \rightarrow H$, such that $\forall x, y \in X$
   2.1 $1xPy \Leftrightarrow f(x) - f(y) > s(x, y)$
   2.2 $2xy \Leftrightarrow f(x) - f(y) \leq s(x, y)$
   2.3 $3s(x, y) = s(y, x)$.
3. $\exists \tilde{f}: X \rightarrow \mathbb{R}$ and $\exists \tilde{s}: X \times X \rightarrow \mathbb{R}$, such that $\forall x, y \in X$
   \begin{align*}
   &1.xP_y \Rightarrow \tilde{f}(x) > \tilde{f}(y) + \tilde{s}(x, y) \\
   &2.xty \Leftrightarrow \tilde{f}(x) \leq \tilde{f}(y) + \tilde{s}(y, x) \quad \text{and} \quad \tilde{f}(y) \leq \tilde{f}(x) + \tilde{s}(x, y).
   \end{align*}

**Proof:** The binary relations $P$, $I$ and $J$ are defined on the same set $X$. Since $P$ is asymmetric and $I$ is symmetric, let us build a binary relation $R$ on $X$ such that $P_n = P$ and $I_n = I$. Since $J_n = J$ then $R$ is complete. Since $P$ is without circuit then $R$ is acyclic. Thus Condition 1 is equivalent to ‘$R$ is complete acyclic’. The equivalence of Conditions 1 and 3 comes directly from the proof of Proposition 1. To show the equivalence of Conditions 1 and 2, the proof of Proposition 1 still holds but with the requirement that the threshold function $s$ is symmetric, that is $s(x, y) = s(y, x)$ $\forall x, y \in X$. □

**Remark 6:** In the variable intervals model, $s(x, y) = \delta(x) \forall y \in X$ gives the Intervals model (Fishburn, 1970) which characterizes the interval duorders. $s(x, y)$ is interpreted as a perception threshold. In Fishburn’s intervals model, the perception threshold depends on each alternative $x$. Here, it depends on the compared alternatives and on the direction of the comparison: $s(x, y)$ is a priori different from $s(y, x)$ since in the proof of Proposition 1 we do not need to make any explicit assumption about the behavior of the function $s$. Of course one can assume, as in Abbas and Vincke (1993), that the function $s$ is symmetric. This assumption is, however, restrictive from a practical point of view as it means that the agents have the same behavior when comparing $x$ to $y$ as they do comparing $y$ to $x$.

Another corollary of Proposition 1 is Bridges’ characterization of complete acyclic relations on countable sets (Theorem 1, Bridges, 1983): if $R$ is a complete relation on a countable set $X$ then $R$ is acyclic if and only if there exists a function $f$ defined from $X$ onto $\mathbb{R}$ such that $\forall x, y \in X$, $xP_y \Rightarrow f(x) > f(y)$. This theorem leads to Abbas and Vincke’s Lemma 3 in Abbas and Vincke (1993) when the set $X$ is finite: let $R$ be a complete relation on a finite set $X$. $R$ is acyclic if and only if there exists a function $f$ defined from $X$ onto $\mathbb{R}$ such that $\forall x, y \in X$, $xP_y \Rightarrow f(x) > f(y)$.

Before we set out the next theorem, let us recall some definitions.

**Jaffray-extension:** Let $R$ and $R^*$ be two binary relations on $X$. $R^*$ is an extension of $R$ in the sense of Jaffray (Jaffray, 1975b) if for any $x, y$ belonging to $X$:

1. $xP_{R^*} y \Rightarrow xP_y y$.
2. $xE_{R^*} y \Rightarrow xE_y y$.

**Perfect density:** Let $R$ be a binary relation on a set $X$, and let $A$ be a subset of $X$. $A$ is said to be perfectly dense in $(X, R)$ if for all $x, y \in X$, $xP_{R^*} y \Rightarrow \exists a \in A$ such that $xRaRy$. This notion of density corresponds to the one used by Birkhoff (1948); Debreu (1954). The term ‘perfect’ comes from Jaffray (1975a).

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4 The intervals model characterizes completely the interval duorders when the set $X$ is countable (Fishburn, 1970); in the uncountable case one must add a ‘separability’ condition (Doignon et al., 1984).
Perfect separability: \((X,R)\) is perfectly separable if it includes a countable set \(A\) which is perfectly dense in \((X,R)\). This notion of separability is weaker than the one used by Subiza (1994).

Upper topology: Let \(F\) be a complete quasi-transitive relation on \(X\) and \(\mathcal{B}\) be the set of all upper sections with respect to \(P_R\) (the asymmetric part of \(F\)). The topology \(\mathcal{F}\) generated by \(\mathcal{B}\) is called the upper topology associated with \((X,F)\).

Finally the following result comes from Jaffray (1975b) (Theorem 1).

**Lemma 2** (Jaffray, 1975b): Let \(F\) be a complete quasi-transitive relation on \(X\), let \(A\) be a set perfectly dense in \((X,F)\) and let \(\mathcal{B}\) be the upper topology associated with \((X,F)\). There exists a preorder \(F^*\), a Jaffray-extension of \(F\) such that:

1. The upper topology associated with \((X,F^*)\) is included in \(\mathcal{B}\).
2. There exists a set \(D\) perfectly dense in \((X,F^*)\) with \(\text{Card } D = \text{Card } A\) or \(D\) is finite.

**Proof**: See Jaffray (1975b) for the proof. One remark should be made. In Jaffray (1975b), \(F\) was a strict partial order and \(F^*\) a strict weak order. It should not be forgotten, however, that strict partial orders and complete quasi-transitive relations are equivalent in duality, as are strict weak orders and preorders. 

**Theorem 1**: Let \(R\) be a complete acyclic relation on \(X\). The following three conditions are equivalent.

1. \((X,R)\) satisfies the variable intervals model.
2. There exists a preorder \(H\) such that \(P_R \subseteq P_H \subseteq R\) and \((X,H)\) is perfectly separable.
3. \((X,F)\) is perfectly separable, where \(F = TP_R \cup J_{TP_R}\), \(TP_R\) is the transitive closure of \(P_R\) and \(J_{TP_R}\) is the incomparability relation with respect to \(TP_R\).

**Proof**:

\((1) \Rightarrow (2)\): \(R\) is complete acyclic and \((X,R)\) satisfies the variable intervals model. We build a preorder \(H\) as follows: \(xHy \Leftrightarrow f(x) \geq f(y)\). \(xP_Ry \Leftrightarrow f(x) > f(y) + s(y,x) \Rightarrow f(x) > f(y) \Rightarrow xP_Hy\). \(H\) is obviously included in \(R\), so \(P_R \subseteq P_H \subseteq R\). It is easily checked that \(f\) is a numerical representation of \(H\), hence according to Birkhoff (1948), \((X,H)\) is perfectly separable.

\((2) \Rightarrow (1)\): Condition 2 is fulfilled. As \((X,H)\) is perfectly separable then \((X,H)\) has a numerical representation (Birkhoff, 1948). We can then build a representation of \(R\) by variable intervals as in Proposition 1.

\((2) \Rightarrow (3)\): \(F\) is a complete quasi-transitive relation (equivalently, \(TP_R\) is a strict partial order). \(TP_R\) is the asymmetric part of \(F\). \(TP_R\) is included in \(P_H\) and \(H\) is included in \(F\), so \(xTP_Ry \Rightarrow xP_Hy\), since \(TP_R\) is included in \(P_H\); this implies \(\exists a \in A\) (countable subset of \(X\))
such that $xHaHy$, since $(X,H)$ is perfectly separable. Finally, since $H$ is included in $F$, we obtain $xFaFy$. Hence $(X,F)$ is perfectly separable.

(3)$\Rightarrow$(2): $F$ is a complete quasi-transitive relation. $(X,F)$ being perfectly separable means that there exists a countable set $A$ perfectly dense in $(X,F)$. From Lemma 2, there exists a Jaffray-extension preorder $F^*$ such that there exists a set $D$ perfectly dense in $(X,F^*)$. Since $\text{Card } D = \text{Card } A$ and $A$ is countable then $D$ is also countable. Hence $(X,F^*)$ is perfectly separable. $F^*$ obviously belongs to $\mathcal{H}$. □

4. Latent preorders

Let an agent exhibit a complete acyclic preference relation $R$. According to the variable intervals model, this agent has an underlying preorder preference but his imperfect capacity of discrimination between alternatives and the nature of the alternatives compared make him deviate from his true preference. The purpose of this section is to approximate this unknown underlying true preorder preference by a preorder $L$. Remember that $T_f$ is the lower section partial preorder associated with $R$ and $T_i$ is the upper section partial preorder. $T_r = P_r + E_r$ and $T_i = P_i + E_i$. Let $J_f$ and $J_i$ denote, respectively, the incomparability relations with respect to $T_f$ and $T_i$. We want the preorders $L$ which approximate the underlying true preference to fulfill the following properties.

**Axiom 1:** $P_r \subseteq P_L \subseteq L \subseteq R$.

This axiom requires that we restrict ourselves to the set $\mathcal{H}$. Since $L$ is a preorder underlying $R$ (the preference relation displayed by the agent), it is normal to require that $L$ be included in $R$. Moreover, the asymmetric part of a preference relation is generally considered to represent the agent’s strict preference. $xP y$ means for the agent: ‘I strictly prefer (with respect to $R$) $x$ to $y$’. Therefore, it is normal to require that the agent strictly prefers $x$ to $y$ (with respect to $L$): $P_r \subseteq P_L$.

**Axiom 2:** Conservation of Equivalence from the true preference to $R$: $I_r(=E_r) = E_e$.

Let $L$ be the agent’s true underlying preorder preference, $L = P_L + I_L$. Since $L$ is a preorder, $L = P_i + E_i$, $x_{E_L}y$ means that $x$ and $y$ are equivalent, thus from the agent’s point of view, they have the same utility. **Axiom 2 says that the transfer of $E_f$ from $L$ (the true preference) to $R$ (the preference displayed) was without error.** We have now to justify the use of $E_e$ as the equivalence relation with respect to $R$. An alternative, used in the literature, is the following equivalence relation $S_R$: $\forall x,y \in X$, $xS_Ry \Leftrightarrow I(x) = I(y)$; where $I(x) = \{b \in X : xI_nb\}$ is the intermediate section associated with $x$. $xS_Ry$ means that $x$ and $y$ are indifferent to the same elements. It is easy to see that in general, $E_R$ is included in $S_R$. However, the binary relations generally used (semiduorder, interval duorder and their variants) are at least complete quasi-transitive, therefore $E_R = S_R$. Hence, for these relations, the distinction between $E_R$ and $S_R$ does not matter. Example 1 shows why in general $E_R$ is better than $S_R$. 
Example 1: Graph of a complete acyclic relation $R$ on $X=\{1,2,3,4\}$ (the loops are omitted).

\[\begin{array}{c}
1 & \rightarrow & 2 \\
2 & \rightarrow & 3 \\
3 & \rightarrow & 1 \\
1 & \rightarrow & 4 \\
4 & \rightarrow & 2 \\
\end{array}\]

\[\rightarrow\text{ represents } P_R\text{ and }\longrightarrow\text{ represents } I_R.\]

We have $1, 3: S_y$. Hence they must have the same utility. This is not acceptable since 1 is strictly preferred to 2 and 2 is strictly preferred to 3.

Axiom 3: $P_f$ or $P_e$ is included in $P_L$ (equivalently: $L$ is included in $T_f \cup I_f$ or in $T_e \cup I_e$). Here, we want $L$ to be rationalized by $T_f$ or $T_e$.

Axiom 4: No Contradiction of $L$ with $T_f \cap T_e$; that is, $L \supseteq T_f \cap T_e$.

Remark 7: By definition, $T_f \cap T_e = P_f \cap P_e + P_0 \cap E_f + P_f \cap E_e + E_f \cap E_e$. Axioms 2 and 4 then imply that $P_L \supseteq P_f \cap P_e + P_0 \cap E_f + P_f \cap E_e$.

Proposition 2: Let $R$ be a complete acyclic relation on a set $X$. The class $\mathcal{L}$ of preorders satisfying the system of axioms $\{A1,A2,A3,A4\}$ is not empty.

Proof: Let $P$ be the set of preorders built from $T_f \cup T_e$ in the following way: $Q=T_f \cup T_e$, $T_e=P_e + E_e$ and $T_f=P_f + E_f$.

Partition of $Q$ into an asymmetric component $P_0$ and a symmetric component $I_0$. $x\rightarrow y \iff xQ y$ and not$(yQ x)\rightarrow x(T_f \cup T_e)y$ and not$(y(T_f \cup T_e)x)\rightarrow (xT_f y$ or $xT_e y$) and not$(yT_f x)\rightarrow xT_f y$ and not$(yT_e x)\rightarrow xT_e y$ and not$(yT_f x)\rightarrow xT_f y$ and not$(yT_e x)\rightarrow xT_e y$ and not$(yT_f x)\rightarrow xT_f y$. Hence, $P_0 = (P_f \cap T_e \cup (P_e \cap T_f))$. $xQ y \iff xQ y$ and $yQ x \iff x(T_f \cup T_e)y$ and $y(T_f \cup T_e)x \iff (xT_f y$ and $xT_e y$) or $(xT_f y$ and $xT_e y$) or $(xT_f y$ and $xT_e y$). We obtain $I_0 = E_f \cup (T_f \cap T_e \cup (T_e \cap T_f)) \cup E_e$. Therefore, $I_0 = [E_f \cup E_e] \cup [(T_f \cap T_e) \cup (T_e \cap T_f)]$. We now partition $I_0$. First note that $T_f = P_e + E_e$ and $T_e = P_f + E_f$. Denote $T_f \cap T_e$ by $Z$ and $T_f \cap T_e$ by $Y$. $Z = (P_f + E_f) \cap (P_e + E_e)$ and $Y = (P_f + E_f) \cap (P_e + E_e)$. $Z = P_f \cap P_e + P_f \cap E_e + P_e \cap E_f + E_f \cap E_e$. $Z \cup Y = (Z \cap Y) + (Z \cap Y) + (Y \cap Y)$, $Z \cup Y = E_f \cap E_e$. So $Z \cup Y = (Z \cap E_f \cap E_e) + (E_f \cap E_e) + (Y \cap E_f \cap E_e)$. $Z \cup Y = P_f \cap P_e + P_f \cap E_e + P_e \cap E_f + E_f \cap E_e$. $Z \cup Y = P_f \cap P_e + P_f \cap E_e + P_e \cap E_f + E_f \cap E_e$. Say that $M = Z \cup Y$, then $I_0 = (E_f \cup E_e) \cup M, E_e \cup E_e = (E_f \cap E_e) + E_f \cap E_e + (E_f \cap E_e) + (E_f \cap E_e) = E_f \cap E_e + E_f \cap E_e + E_f \cap E_e + E_f \cap E_e$.
$E_c \cap P_f + E_c \cap P_f^\circ + E_c \cap J_c E_c \cap P_f E_c \cap P_f^\circ$ and $E_c \cap E_c$ are included in $M$. $I_0 = E_c \cap J_c + E_c \cap J_c + M I_0 = P_f \cap P_f^\circ + P_f \cap P_f^\circ + P_f \cap E_c + P_f \cap E_c + P_f \cap E_f + P_f \cap E_f^\circ$ and $E_c \cap E_c$ are included in $M$.

**Construction of the preorders:** $Q + J_0$ is, by definition, complete. We are interested in $J_0$ only by $J_0 \cap R$, that is, by $J_0 \cap P_R + J_0 \cap I_R$. Since $R$ is complete, $L1 = Q + (J_0 \cap R)$ is also complete. We have $L1 = [P_f \cap T_f \cup (P_f \cap T_f^\circ)] + P_f \cap E_c + P_f \cap E_c + P_f \cap E_f + P_f \cap E_f + E_c + P_f \cap P_f^\circ + E_c \cap J_c + E_c \cap J_c + E_c \cap E_c + J_0 \cap P_R + J_0 \cap I_R$. From $L1$, which is complete, we build a relation $A$ defined as follows:

- If $xP_f y$ then $xAy$.
- If $xP_f \cap E_c y$ then $xAy$.
- If $xP_f \cap E_c y$ then $xAy$.
- If $xP_f \cap E_f y$ then $xAy$.
- If $xP_f \cap E_f y$ then $xAy$.
- If $xP_f \cap E_f y$ then $xAX$.

We obtain the following relation $L2$ which is still complete. $L2 = A + J_0 \cap I_R + P_f \cap P_f^\circ + E_c \cap J_c + E_c \cap J_c + E_c \cap E_c$. However, if $A$ is asymmetric, $A$ is not necessarily transitive. So we have to transitivity close $A$. Let $J$ be the ‘incomparability’ relation of $A$. $J = L2A$. $J = A^2 \cap J_a + (J_0 \cap A^2 \cap J_0)$. If we add $A^2 \cap J_a$ to $A$, we obtain $A^*$ which is asymmetric and transitive. Let $L3 = A^* + (J_0 \cap A^2 \cap J_0)$. $J = (J_0 \cap I_R + P_f \cap P_f^\circ + E_c \cap J_c + E_c \cap J_c + E_c \cap E_c) \cup A^2 \cap J_c$. However, $(E_c \cap E_c) \cap A^2 = \emptyset$. Hence $J^* = (J_0 \cap I_R + P_f \cap P_f^\circ + E_c \cap J_c + E_c \cap J_c + E_c \cap E_c) \cup A^2 \cap J_c$. Say, finally, that $B = (J_0 \cap I_R + P_f \cap P_f^\circ + E_c \cap J_c + E_c \cap J_c + E_c \cap E_c) \cup A^2 \cap J_c$. Then $L3 = A^* + B + E_c \cap E_c$. $L3$ is still complete. We will arbitrarily add $B$ to $A^*$ but so as to preserve the transitivity and asymmetry of $A^*$. That is, if $(x, y)$ and $(y, x) \in B$ then we will add to $A^*$ either $(x, y)$, or $(y, x)$ such that we preserve the transitivity of $A^*$. The relation $A^* \cap E_c$ is still asymmetric and transitive. Let $L = A^* + (E_c \cap E_c)$ be a binary relation. $E_c \cap E_c$ is transitive because it is the symmetric component of $T$, the section partial preorder associated with $R$. Therefore, $L$ is a preorder.

**We complete the proof by remarking that $\mathcal{P}$ obviously coincides with $L$. □**

We call the class $\mathcal{X}$ of such preorders $L$, the class of latent preorders. These are the ‘best’ approximations of agents’ underlying preorder preferences in the sense that they are the only preorders satisfying the above axioms. Hence, when representing a complete acyclic relation by variable intervals, the function $f$ should be a utility function representing a latent preorder.

Before setting out the next remark, let us state the following definition.

**Definition** (Compatibility of a binary relation with a preorder): In the sense of Monjardet (1984), a preorder $V$ and a binary relation $R$ are **compatible** when the following three conditions hold for all $x, y, z$ in $X$. 

\[ E_c \cap P_f + E_c \cap P_f^\circ + E_c \cap J_c E_c \cap P_f E_c \cap P_f^\circ, E_c \cap P_f^\circ, E_c \cap J_c \text{ and } E_c \cap E_c \text{ are included in } M. \]
1. \( V \subseteq R \cup J_R \).
2. \( xVy \land xI_R z \Rightarrow xI_R y \land yI_R z \).
3. \( xVy \land xJ_R z \Rightarrow xJ_R y \land yJ_R z \).

This definition is the dual counterpart of Roberts’ definition in Roberts (1971) of compatibility between an asymmetric relation and a partial preorder. The meaning is that: (i) \( V \) never contradicts \( R \cup J_R \) (in the sense that \( xVy \) implies \( xRy \) or \( xJ_R y \)); it means also that \( P_R \) is included in \( P_Y \); (ii) any element belonging to a \( V \)-interval between two \( R \)-indifferent elements, is \( R \)-indifferent to these elements; (iii) any element belonging to a \( V \)-interval between two \( R \)-incomparable elements, is \( R \)-incomparable to these elements.

If the relation \( R \) is complete, then \( J_R = \emptyset \); therefore, the above definition yields: (i) \( V \subseteq R \); (ii) \( xVy \land xI_R z \Rightarrow xI_R y \land yI_R z \).

Remark 8: When \( R \) is a semiduorder, the section partial preorder is a preorder. \( T \) is called the latent preorder in the literature and is considered to be the underlying preorder associated with \( R \). Roberts (Roberts, 1971) has shown that \( T \) is compatible with \( R \) and includes any preorder compatible with \( R \). It is easy to see that when \( R \) is a semiduorder, our class \( L \) includes only \( T \). Therefore, both approaches give the same result.

Example 2: \( X = \{1,2,3,4,5\} \); \( R \) is a complete acyclic relation on \( X \).
5. Conclusion

The variable intervals model provides an interesting interpretation of the behavior of agents who have complete acyclic preferences. These agents have underlying preorder preferences, but due to their imperfect capacity of discrimination and the nature of the alternatives compared, they have complete acyclic preferences. Proposition 2 suggests a method for finding out about these underlying preferences. If the set of alternatives is uncountable, the necessary and sufficient Conditions 2 and 3 (in Theorem 1) for representation by variable intervals seem simpler than Subiza’s condition \([P]\) in Subiza (1994).

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