Choice functions and abstract convex geometries

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Abstract

A main aim of this paper is to make connections between two well – but up to now independently – developed theories, the theory of choice functions and the theory of closure operators. It is shown that the classes of ordinally rationalizable and path independent choice functions are related to the classes of distributive and anti-exchange closures. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A main aim of this paper is to make connections between two well – but up to now independently – developed theories, the theory of choice functions and the theory of closure operators.

A choice function maps a set to some of its subsets, an operator maps a set to a set which contains the set. For any choice function on a finite set of social alternatives, we define an operator being the union of all set with the same choice set. If, given an operator, any non-empty set has a non-empty set of extreme points with respect to the operator, then the operator defines a choice function via mapping a set to its extreme points.

A closure operator with anti-exchange property defines a choice function which satisfies path independence. The anti-exchange property is a combinatorial abstraction of the usual convex closure in Euclidean spaces. The concept of “path independence” of a choice function was suggested by Plott (1973) as a means of weakening the condition of

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rationality in a manner which preserves one of the key properties of rational choice, namely that choice over any subset should be independent of the way the alternatives were initially divided up to consideration.

The operator defined by a choice function, which satisfies path independence, is a closure operator and the corresponding set of closed sets is an abstract convex geometry.

A choice function is called ordinally rationalizable if it is rationalizable by a partial order. It is shown that a choice function is ordinally rationalizable if and only if the choice function is "consistent" with the operator, the operator is a closure operator and the set of closed sets forms a distributive lattice with respect to union and intersection.

The lattice of closed sets of an abstract convex geometry is a meet-distributive lattice (Edelman, 1986). For a convex geometry, it is established a necessary and sufficient condition that the corresponding lattice of closed sets is distributive.

Our approach gives a new insight on well known results (see Aizerman, 1985; Moulin, 1985) that path independence and the extension property (Plott, 1973), or path independence and the concordance property (Aizerman, 1985), are necessary and sufficient for ordinal rationalizability of a choice function.

2. Choice functions, posets, closure operators and distributive lattices

Let \( P \) be a finite set of social alternatives and let \( 2^P \) be the power set of \( P \) (the set of all subsets of \( P \)). A choice function \( C: 2^P \to 2^P \), assigns a non-empty subset \( C(A) \) of \( A \) to every \( A \in 2^P \setminus \emptyset \) and \( C(\emptyset) = \emptyset \). Let \( P \) be a partially ordered set (poset), i.e. \( P \) is a finite set endowed with some order \( \leq \), i.e. \( \leq \) is a reflexive (\( p \leq p \forall p \in P \)), transitive (for all \( p, q, \) and \( r \in P, \) if \( p \leq q \) and \( q \leq r, \) then \( p \leq r \)), antisymmetric (for all \( p \) and \( q \in P, \) if \( p \leq q \) and \( q \leq p, \) then \( p = q \)) binary relation. Then, for all \( A \in 2^P \setminus \emptyset \), define \( C(A, \leq) \) as the set of maximal elements in \( A \) with respect to the order \( \leq \), \( C(A, \leq) = \{ x \in A \mid \exists y \in A: x \leq y \} \). We wish to say that a choice function \( C \) is ordinally rationalizable if for any \( A \in 2^P \setminus \emptyset, \) \( C(A) = C(A, \leq) \) with some order \( \leq \) on \( P \).

Recall some definitions.

A map \( K: 2^P \to 2^P, \) \( K(\emptyset) = \emptyset, \) is said to be a closure operator if the following properties hold:

0. \( A \subseteq K(A) \) (extensivity)
1. \( K(K(A)) = K(A) \) (idempotence)
2. \( A \subseteq B \) implies \( K(A) \subseteq K(B) \) (monotonicity)

A set is said to be closed if \( A = K(A) \). The intersection of closed sets is closed. In general, union of closed sets is not a closed set. The set of closed sets endowed with operations \( A \cap B = A \cap B, A \cup B = K(A \cup B), \) \( A, B \) closed sets, becomes a lattice.

A lattice \( \mathcal{L} \) is a set endowed with two operations: the meet \( a \wedge b \in \mathcal{L} \) and the join \( a \vee b \in \mathcal{L} (a, b \in \mathcal{L}) \) which are idempotent: \( a \wedge a = a (a \vee a = a) \), commutative: \( a \wedge b = b \wedge a (a \vee b = b \vee a) \), and associative: \( (a \wedge b) \wedge c = a \wedge (b \wedge c) ((a \vee b) \vee c = a \vee (b \vee c)) \). A lattice is said to be distributive if for all \( a, b, c \in \mathcal{L} \) the following equality holds:
Obviously, such a relation is reflexive and transitive.

The fundamental result of Birkhoff says that every distributive lattice is isomorphic to the lattice of ideals of some ordered set (see, for example, Gratzer, 1978). An ideal of a poset \((P, \leq)\) is a subset \(I \subseteq P\) such that \(y \in I, x \leq y \implies x \in I\).

Given a choice function \(C\): \(2^P \to 2^P\), \(C(\emptyset) = \emptyset\), define the following map

\[
\tilde{C}(A) = \bigcup_{(A', C(A') = C(A))} A'.
\] (1)

We call \(\tilde{C}\) an operator, because it satisfies extensivity property \(A \subseteq \tilde{C}(A)\) for any \(A \subseteq P\).

Let \(C\) be an ordinally rationalizable choice function with respect to an order \(\leq\) on \(P\). Then \(\tilde{C}(A) = \{p \in P | p \leq a \text{ with some } a \in C(A)\}\) is an ideal of the poset \((P, \leq)\), and the following properties hold:

R1. For any \(A \subseteq P\) there holds \(C(\tilde{C}(A)) = C(C(A)) = C(A)\).

R2. The operator \(\tilde{C}\) defined by (1) is a closure operator.

R3. The union of closed sets of operator \(\tilde{C}\) is closed (then the lattice of closed sets is a distributive).

Observe that R1 and R2 imply that for any \(A \subseteq P\) the following equality holds:

\[
\tilde{C}(A) = \tilde{C}(C(A)).
\] (2)

In fact, because of monotonicity of \(\tilde{C}\) and \(C(A) \subseteq A\), we have the inclusion \(\tilde{C}(C(A)) \subseteq \tilde{C}(A)\). Because of R1 and the definition of \(\tilde{C}\), we have the opposite inclusion \(\tilde{C}(A) \subseteq \tilde{C}(C(A))\), and, hence, (2).

The following theorem shows that any choice function satisfying properties R1–R3 is ordinally rationalizable.

**Theorem 1.** Let a choice function \(C\): \(2^P \to 2^P\) satisfy properties R1–R3. Then \(C\) is ordinally rationalizable.

**Proof.** Define a binary relation \(\leq\) on \(P\) by the rule

\[
p \leq q \quad \text{if} \quad \tilde{C}(p) \subseteq \tilde{C}(q).
\] (3)

Obviously, such a relation is reflexive and transitive.

Assume \(C\) satisfies properties R1–R3. Then, because of property R1, the relation \(\leq\) is an order. In fact, with different \(p\) and \(q\), we have \(\tilde{C}(p) \neq \tilde{C}(q)\) (\(C(\tilde{C}(p)) = p \neq q = C(\tilde{C}(q))\)). Therefore, this relation is antisymmetric, hence, it is an order.

Because \(\tilde{C}\) is a closure operator (R2), for any \(A \subseteq P\), the set \(\tilde{C}(A)\) is an ideal of the poset \((P, \leq)\). Indeed, let \(a \in \tilde{C}(A)\), \(b \leq a\). Then the following inclusions hold \(\tilde{C}(b) \subseteq \tilde{C}(a) \subseteq \tilde{C}(\tilde{C}(A)) = C(A)\). Hence, \(b \in \tilde{C}(A)\). So, we have \(\tilde{C}(A) = \bigcup_{a \in \tilde{C}(A)} \tilde{C}(a)\).

Owing that union of closed sets is closed (R3), the following equality holds

\[
\tilde{C}(A) = \bigcup_{a \in \tilde{C}(A)} \tilde{C}(a).
\] (4)

Indeed, (4) is true because \(A \subseteq \bigcup_{a \in \tilde{C}(A)} \tilde{C}(a) \subseteq \tilde{C}(\bigcup_{a \in \tilde{C}(A)} \tilde{C}(a)) = \bigcup_{a \in \tilde{C}(A)} \tilde{C}(a)\) (the last equality holds because the union of closed set is closed).
Because of (2) and (4), there holds
\[ \tilde{C}(A) = \tilde{C}(\tilde{C}(A)) = \bigcup_{a \in \tilde{C}(A)} \tilde{C}(a). \] 
(5)

Show that \( C \) is ordinally rationalizable with respect to the order \( \preceq \) defined by (3), i.e. show that \( C(A) \) coincides with the set of maximal elements of \( A \) with respect to \( \preceq \).

Let \( A \) be an antichain, i.e., for any \( p \not= q \in A \) non \( [p \preceq q \lor q \preceq p] \) holds. Then \( C(A) \) is also an antichain \( (C(A) \subseteq A) \). Therefore, from (4) and (5) we conclude the equality \( A = C(A) \).

Given an arbitrary \( A \subseteq P \), consider the set of maximal elements in \( A \). Denote \( A' \) such a set. \( A' \) is an antichain, therefore we have \( C(A') = A' \). Because of (4) and the set \( \tilde{C}(A) \) is an ideal of \( (P, \preceq) \), there holds \( \tilde{C}(A) = \tilde{C}(A') \). Then, because of \( C(\tilde{C}(A)) = C(A) \), we conclude
\[ C(A) = C(\tilde{C}(A)) = C(\tilde{C}(A')) = C(A') = A'. \]

This yields that \( C(A) \) coincides with the set of maximal elements in \( A \), and, hence, \( C(A) \) is ordinally rationalizable. Q.E.D.

The following examples shows that a choice function \( C \) which satisfies only either properties R1–R2 or R2–R3 but not all properties R1–R3 could be not ordinally rationalizable.

**Example 1.** Let \( P \) be a set of 6 points in the plane (Fig. 1). For a subset of points \( A \subseteq P \) define the choice function \( C(A) \) being the set of extreme points to the convex hull of \( A \), \( \text{co}(A) \). Then \( \tilde{C}(A) = \text{co}(A) \) and \( C \) satisfies properties R1 and R2, but not R3. Such a choice function is not ordinally rationalizable. In fact, assume that \( C \) is ordinally rationalizable with respect to an order \( \preceq \) on \( P \). Then \( p_3 < p_4 \), because \( C(\{p_1, p_2, p_3, p_4\}) = \{p_1, p_2, p_4\} \), and \( p_4 < p_3 \), because \( C(\{p_3, p_4, p_5, p_6\}) = \{p_3, p_5, p_6\} \), a contradiction.

Fig. 1.
Example 2. Consider $P = \{a, b, c\}$ and the following choice function $C(\{a, b\}) = a, C(\{a, c\}) = a, C(\{b, c\}) = b, C(\{a, b, c\}) = \{b, c\}$. Such a function is not ordinally rationalizable because $C(\{a, b, c\}) \neq a$, and it satisfies properties R2 and R3, and does not satisfy R1. In fact, $\{a, b, c\} = \bar{C}(a) = \bar{C}(\{a, b\}) = \bar{C}(\{a, c\}) = \bar{C}(\{a, b, c\})$, $\{b, c\} = \bar{C}(\{b\}) = \bar{C}(\{b, c\}$ and $c = \bar{C}(c))$. Such a defined operator is a closure operator and the lattice of closed sets is the chain $\{a, b, c\} \supseteq \{b, c\} \supseteq \{c\}$, i.e. R2 and R3 holds. With the set $\{a\}$, we have $C(\bar{C}(a)) = C(\{a, b, c\}) = \{b, c\} \neq C(a) = a$. So, R1 is violated.

Example 2 also shows that the operator $\bar{C}$ with properties R2 and R3 does not define a choice function uniquely. The choice function $C'$ which coincides with $C$ on all sets, except $\{a, b, c\}$, where $C'(\{a, b, c\}) = a$, is ordinally rationalizable and for any subset $A \subseteq P$ there holds $\bar{C}'(A) = \bar{C}(A)$.

3. Path independence and abstract convexity

Here, we show that if a choice function satisfies path independence, then the operator defined by (1) satisfies properties R1 and R2, and the set of closed sets comes out as an abstract convex geometry. It is shown that under path independence a choice function is implemented as the set of the extreme points of the closure operator defined by (1). Moreover, every closure operator with the set of closed sets being convex geometry defines (via mapping a set to its extreme points) a choice function which satisfies path independence.

The set of closed sets of some abstract convex geometry is a meet-distributive lattice, which is in general not distributive. It is provided a necessary and sufficient condition when convex geometries define distributive lattices of closed sets.

Recall some notions of abstract convexity (see especially Edelman, 1986 and Edelman and Jamison, 1985).

Let $P$ be a finite set and let $K: 2^P \rightarrow 2^P$, $K(\emptyset) = \emptyset$, be a closure operator. Denote $\mathcal{K}(P, K)$ the set of closed sets of $P$, subsets of $P$ of the form $K(A)$. Let $A$ be a subset of $P$. A basis for $A$ is a minimal subset $S \subseteq A$ such that $K(S) = K(A)$. A priori there may be many bases for a particular set. A point $a \in A$ is said to be an extreme point of $A$ if $a \in K(A\{a\})$. For closed $A$ this is equivalent to $A\{a\}$ is a closed set. The set of extreme points of $A$ is denoted $\text{ex}(A)$. Extreme points may or may not exist. The set $\text{ex}(A)$ is contained in every basis of $A$.

The set $\mathcal{K}(P, K)$ is said to be anti-exchange (or abstract convex geometry) if, given any closed set $A$, and two different elements $p$ and $q$ in $P$, not in $A$, then $q \in K(A \cup p)$ implies that $p \in K(A \cup q)$.

The anti-exchange axiom is a combinatorial abstraction of a property of the usual convex closure in Euclidean spaces. Namely, for two points $p$ and $q$ not in the convex hull of the set $A$, if $q$ is in the convex hull of $A \cup p$ then $p$ is outside the convex hull of $A \cup q$ (see Fig. 2).

Thus every finite subset in a Euclidean space gives rise to a convex geometry.

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Thus every finite subset in a Euclidean space gives rise to a convex geometry.

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This example was suggested to the author by the referee.
Fig. 2.

An ordered set gives rise to another example of a convex geometry where closed sets are the ideals of the ordered set and so form a distributive lattice.

Some equivalent characterizations of convex geometries are summarized in the following:

**Theorem 2.** (Edelman and Jamison, 1985) Let \( \mathcal{K}(P, K) \) be the set of closed subsets of \( P \) with respect to a closure operator \( K \). Then the following are equivalent:

a) \( \mathcal{K}(P, K) \) is anti-exchange.

b) Every subset \( A \subseteq P \) has a unique basis.

c) For every closed set \( A \), \( A = \text{ex}(A) \).

d) For every \( A \subseteq P \), \( \text{ex}(A) \) is the (unique) basis of \( A \).

Plott (1973) has suggested the concept of *path independence* of a choice function \( C \): For any \( A, B \in 2^P \setminus \emptyset \),

\[
C(A \cup B) = C(C(A) \cup C(B)),
\]

as a means of weakening the condition of rationality in a manner which preserves one of the key properties of rational choice, namely that choice over any subset should be independent of the way the alternatives were initially divided up to consideration.

Show that a choice function \( C \) which satisfies path independence also satisfies properties R1 and R2, and the set \( \mathcal{K}(P, \hat{C}) \) is a convex geometry.
Proposition 1. Let a choice function $C$ satisfy path independence. Then the operator $\tilde{C}$ defined by (1) is a closure operator and $C(\tilde{C}(A)) = C(C(A)) = C(A)$ for any $A \subseteq P$.

Proof. Let $A_1, \ldots, A_m$ be a collection of all sets with $C(A_i) = C(A)$. Then, because of path independence, there holds
\[
C(\tilde{C}(A)) = C(A_1 \cup (A_2 \cup (\ldots \cup (A_{m-1} \cup A_m)))) = C(A_1)
\]
\[
\cup (C(A_2) \cup (\ldots \cup (C(A_{m-1}) \cup C(A_m)))) = C(C(A)).
\]
Since a choice function satisfying path independence satisfies also idempotence, the property R1 is met:
\[
C(\tilde{C}(A)) = C(C(A)) = C(A).
\]
Because of this, there holds $\tilde{C}(\tilde{C}(A)) = \tilde{C}(A)$. That yields property 1 of a closure operator. 

Show monotonicity, i.e., $A \subseteq B$ implies $\tilde{C}(A) \subseteq \tilde{C}(B)$. Suppose $A \subseteq B$, then, because of path independence and $C(\tilde{C}(A)) = C(A)$, we have
\[
C(\tilde{C}(A) \cup \tilde{C}(B)) = C(C(A) \cup C(B)) = C(A \cup B) = C(B).
\]
Therefore, there holds $\tilde{C}(B) \supseteq \tilde{C}(A) \cup \tilde{C}(B)$, hence $\tilde{C}(A) \subseteq \tilde{C}(B)$. Therefore, $\tilde{C}$ is a closure operator, and, hence, property R2 holds. Q.E.D.

Proposition 2. Let a choice function $C$ satisfy path independence. Then for any $A \subseteq P$, $C(A) = \text{ex}(A)$, where $\text{ex}(A)$ is the set of extreme points of the set $A$ with respect to the closure operator $\tilde{C}$.

Proof. Check that $\text{ex}(A) \subseteq C(A)$, i.e., if $a \in A$, $a \in C(A)$, then $a \in \tilde{C}(A\{a\})$.

In fact, let $a \in A$, $a \in C(A)$, then $A\{a\} = C(A) \cup Z$ with some subset $Z \subseteq A$. Because of properties R1 and R2 of the operator $\tilde{C}$ (Proposition 1), there holds
\[
A \subseteq \tilde{C}(A) = \tilde{C}(C(A)) \subseteq \tilde{C}(C(A) \cup Z) = \tilde{C}(A\{a\}).
\]
That yields the inclusion $a \in \tilde{C}(A\{a\})$, i.e. $a \in \text{ex}(A)$. Hence the inclusion $\text{ex}(A) \subseteq C(A)$ is established.

Show the reverse inclusion $\text{ex}(A) \supseteq C(A)$, i.e. if $a \in C(A)$, then $a \in \tilde{C}(A\{a\})$. In fact, let $a \in C(A)$, then $C(A) \not\subseteq A\{a\}$, and hence $C(A) \not\subseteq C(A\{a\})$. Because of $C(A) = C(A\{a\} \cup a)$, $a = C(a)$, $\forall a \in P$, and path independence, there holds
\[
C(A) = C(A\{a\} \cup a) = C(C(A\{a\} \cup C(a))) = C(\tilde{C}(A\{a\}) \cup a).
\]
Assuming $a \in \tilde{C}(A\{a\})$, we would have $C(\tilde{C}(A\{a\}) \cup a) = C(\tilde{C}(A\{a\})) = C(A\{a\})$. Thus, $C(A) = C(A\{a\})$, a contradiction. Therefore, there holds $C(A) \subseteq \text{ex}(A)$, and hence $C(A) = \text{ex}(A)$. Q.E.D.

Propositions 1 and 2, show that $\text{ex}(\tilde{C}(A)) = C(A)$ and so property c) of Theorem 2 is satisfied, i.e. we have the following:
**Theorem 3.** Let a choice function $C$ satisfy path independence. Then the set of closed sets of the closure operator $\mathcal{C}$ defined by (1) is a convex geometry.

We now show that any abstract convex geometry induces a choice function $C(A) = \text{ex}(A)$ satisfying path independence.

**Theorem 4.** Let $\mathcal{K}: 2^P \to 2^P$ be a closure operator with anti-exchange property. Then for any $A, B \subseteq P$ there holds

$$\text{ex}(A \cup B) = \text{ex}(\text{ex}(A) \cup \text{ex}(B)).$$

**Proof.** Because $\text{ex}(A)$ is the unique basis of $A, A \subseteq P$, we can rewrite (6) as follows

$$K(A \cup B) = K(\text{ex}(A) \cup \text{ex}(B)).$$

Check that (7) is valid: The inclusion $K((\text{ex}(A) \cup \text{ex}(B)) \subseteq K(A \cup B)$ holds due to monotonicity of the closure operator $K$.

Show $K(A \cup B) \subseteq K(\text{ex}(A) \cup \text{ex}(B))$: Pick an extreme point $a \in \text{ex}(A \cup B)$, and check that if $a \in A$, then $a \in \text{ex}(A)$, i.e., $a \in K(A \alpha)$, and if $a \in B$, then $a \in \text{ex}(B)$.

Suppose $a \in \text{ex}(A)$, $a \in A$ for some $a \in \text{ex}(A \cup B)$. Then, because of monotonicity of $K$, we have $a \in K(A \alpha) \subseteq K((A \cup B) \alpha)$, a contradiction. By the same arguments in the case with $a \in \text{ex}(A \cup B)$, $a \in B$, there holds $a \in \text{ex}(B)$.

So, we have $\text{ex}(A \cup B) \subseteq \text{ex}(A) \cup \text{ex}(B)$. Because of monotonicity of $K$ and Theorem 2c) the reverse inclusion $K((\text{ex}(A) \cup \text{ex}(B)) \supseteq K(A \cup B)$ holds. Therefore, (7) holds and hence we yield (6). Q.E.D.

So, it is shown that if a choice function satisfies path independence then properties R1 and R2 hold, and the set of closed subsets with respect to the operator (1) forms an abstract convex geometry. Any such a choice function is implemented as sets of extreme points with respect to the operator (1). Moreover, any closure operator with anti-exchange property induces, via taking extreme points, a choice function which satisfies path independence.

The lattice of closed sets of a convex geometry is a *meet-distributive* lattice (Edelman, 1986). Distributive lattices are meet-distributive, but not vice versa. For example, the lattice which corresponds to sets of the convex geometry of a finite set of points in a Euclidean space is, in general, not distributive.

Convex geometries with distributive lattices of closed sets are characterized as follows.

**Proposition 3.** Let $\mathcal{K}: 2^P \to 2^P$ be a closure operator with anti-exchange property. Then the lattice of closed sets is distributive if and only if for any $A \subseteq P$, there holds

$$K(A) = \bigcup_{a \in \text{ex}(A)} K(a).$$

**Proof.** Assume (8) holds. Then

$$K(A \cup B) = (\bigcup_{a \in \text{ex}(A)} K(a)) \cup (\bigcup_{b \in \text{ex}(B)} K(b)) = \bigcup_{z \in \text{ex}(A \cup B)} K(z) = K(A \cup B).$$

(9)
Inclusions \( K(A) \subseteq K(A \cup B) \) and \( K(B) \subseteq K(A \cup B) \) hold by monotonicity of a closure operator. Hence, \( K(A) \cup K(B) \subseteq K(A \cup B) \) holds. So, we have \( K(A) \cup K(B) = K(A \cup B) \), that yields that \( K(A) \cup K(B) \) is a closed set with any \( A \) and \( B \subseteq P \). Therefore, the set \( \mathcal{K}(P, K) \) is a distributive lattice with respect to union and intersection.

Let \( K \) be an anti-exchange closure operator with the set \( \mathcal{K}(P, K) \) being a distributive lattice. Then the choice function \( C(A); = \text{ex}(A) \) satisfies properties R1–R3 (Theorem 4, Proposition 1 and the set \( \mathcal{K}(P, K) \) is a distributive lattice). Therefore, because of Theorem 1, sets \( \text{ex}(A) \) are sets of maximal elements in \( A \) with respect to an order \( \leq \) on \( P \) defined by \( x \leq y \) if \( K(x) \subseteq K(y) \), \( x, y \in P \). So, (8) holds. Q.E.D.

Observe that the inclusion \( K(A) \supseteq \bigcup_{a \in \text{ex}(A), K(a)} \) holds with any closure operator.

From Theorems 1 and 3, and Proposition 3 we have the following:

**Corollary 1.** Let a choice function \( C: 2^P \rightarrow 2^P \) satisfy path independence. Then \( C \) is ordinally rationalizable if and only if for any \( A \subseteq P \) there holds

\[
\tilde{C}(A) = \bigcup_{a \in \text{ex}(A)} \tilde{C}(a). \tag{10}
\]

A consequence of this Corollary is that in a case of a single element choice function, i.e. when a choice over any set is always a single element, path independence is a necessary and sufficient condition for rationalizability (Plott, 1973). In fact, let \( C \) be a single element choice function which satisfies path independence. In such a case, \( \tilde{C}(A) = \tilde{C}(C(A)) \), that yields (10).

The condition (10) can be established in terms of the initial choice function by several ways.

**Proposition 4.** Let a closure operator \( K: 2^P \rightarrow 2^P \) satisfy anti-exchange property.

Assume that for any subset \( A \subseteq P \) and any \( z \in P \setminus A \) there holds

\[
\text{if } z \in \text{ex}(z \cup a) \forall a \in A, \text{ then } z \in \text{ex}(z \cup A). \tag{11}
\]

Then the lattice of closed sets is distributive.

**Proof.** Because of Proposition 3, show validity (8). Assume (8) does not hold with some \( A \subseteq P \). Then there exists \( z \in K(A) \setminus \bigcup_{a \in \text{ex}(A)} \text{K}(a) \). Hence, \( z \notin K(a) \) for any \( a \in \text{ex}(A) \). This implies that \( z \in \text{ex}(\{z \cup a\}) \) for any \( a \in \text{ex}(A) \), because if \( z \notin \text{ex}(\{z \cup a\}) \), then \( z \in K(\{z \cup a\}) \). But this is not the case. Hence, because of (11), we have \( z \notin \text{ex}(\{z \cup \text{ex}(A)\}) \).

Because in a convex geometry, for any \( A \subseteq P \), the set \( \text{ex}(A) \) is the unique basis of \( A \), there holds

\[
z \notin K(\{z \cup \text{ex}(A)\}) = K(\text{ex}(A)) = K(A). \tag{12}
\]

A contradiction. Q.E.D.

**Corollary 2.** Let a closure operator \( K: 2^P \rightarrow 2^P \) satisfy anti-exchange property. Assume that for any subsets \( A, B \subseteq P \), there holds

\[
\text{ex}(A) \cap \text{ex}(B) \subseteq \text{ex}(A \cup B). \tag{13}
\]
Then the lattice of closed sets is distributive.

Proof. Show that (8) holds. Assume (8) does not hold with some \( A \subseteq P \). Then, because of the proof of Proposition 4, with any \( z \in K(A) \setminus \bigcup_{a \in \text{ex}(A)} K(a) \), there holds \( z \in \bigcap_{a \in \text{ex}(A)} \text{ex}(z \cup a) \). Because of (13), we have \( z \in \text{ex}(z \cup \text{ex}(A)) \). Hence (12) holds. A contradiction. Q.E.D.

Observe that in presence of path independence, (11) comes out as the extension property (E) (Plott, 1973) and (13) comes out as Property \( \gamma \) (expansion consistency) (Sen, 1977) called also concordance (Aizerman, 1985). So one can reobtain the well known results that path independence and the extension property, or path independence and the concordance property, are necessary and sufficient for ordinal rationalizability of a choice function (see Aizerman, 1985; Moulin, 1985).

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