Three remarks on the many-to-many stable matching problem
Dedicated to David Gale on his 75th birthday

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Abstract

We propose a general definition of stability, setwise-stability, and show that it is a stronger requirement than pairwise-stability and core. We also show that the core and the set of pairwise-stable matchings may be non-empty and disjoint and thus setwise-stable matchings may not exist. For many labor markets the effects of competition can be characterized by requiring only pairwise-stability. For such markets we define substitutability and we prove the existence of pairwise-stable matchings. The restriction of our proof to the College Admission Model is simple and short and provides an alternative proof for the existence of stable matchings for this model.

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1. Introduction

In the many-to-many matching problem there are two sets of agents, \( P \) and \( Q \), and each agent may form partnerships with members of the opposite set. It is assumed that each agent has a quota, giving the maximum number of partnerships he or she may enter into and that he or she has a preference order among all allowable sets of partners. A set of allowable partnerships is called a matching. Roughly speaking, a matching is said to be stable if no subset \( S \) of agents, by choosing new partnerships only among themselves,
can obtain sets of partners which all members of \( S \) prefer. In the literature there are two
different ways of making this precise.

(1) A matching \( x \) is pairwise-stable if there are no agents \( p \) and \( q \) who
are not partners, but by becoming partners, possibly dissolving some of their partnerships
given by \( x \) to remain within their quotas and possibly keeping other ones, can both obtain a
strictly preferred set of partners.

(2) A matching \( x \) is in the core (corewise-stable) if there is no subset of agents who
by forming all their partnerships only among themselves, can all obtain a strictly preferred set of partners.

We propose here an obvious generalization of pairwise-stability:

(3) A matching \( x \) will be called setwise-stable if there is no subset of agents who by
forming new partnerships only among themselves, possibly dissolving some partnerships
of \( x \) to remain within their quotas and possibly keeping other ones, can all obtain a
strictly preferred set of partners.

From these definitions it is clear that (1) and (2) are special cases of (3). For a better
understanding of these three concepts see Fig. 1.

In Fig. 1a \( p_1 \) is matched to \( \{q_1,q_2\} \), but prefers \( q_3 \) to \( q_2 \); \( q_3 \) is matched to \( \{p_2,p_3\} \), but
prefers \( p_1 \) to \( p_2 \). The matching \( x_1 \) is not pairwise-stable because \( p_1 \) and \( q_3 \) can make a
better arrangement under the matching \( x'_1 \) shown in Fig. 1b. The matching \( x_1 \) is not
strong corewise-stable because the players in the coalition \( A = \{p_1,p_2,q_1,q_3\} \) weakly
prefer the matching \( x'_1 \) to the matching \( x_1 \) and, at least, one of the players strictly prefers
\( x'_1 \) to \( x_1 \). Indeed, \( p_1 \) and \( q_3 \) strictly prefer their new set of partners to that given by \( x_1 \).
Since \( q_1 \) and \( p_3 \) are indifferent between \( x_1 \) and \( x'_1 \), the coalition \( A \) does not block \( x_1 \) and
so \( x_1 \) is in the core. In Fig. 1c, \( p_1 \) and \( p_2 \) prefer \( \{q_1,q_2\} \) to \( \{q_1,q_4\} \); and \( q_1 \) and \( q_2 \) prefer
\( \{p_1,p_3\} \) to \( \{p_3,p_4\} \). In this case \( x_2 \) is not in the core since it is blocked by \( B = \{p_1,p_2,q_1,q_3\} \). In fact, there is a matching \( x'_2 \) at which every one in the coalition \( B \) is
better off and forms partnerships only with members of \( B \). Consequently \( x_2 \) is not
setwise-stable and is not in the strong core. In Fig. 1e \( p_1 \) prefers \( \{q_1,q_2,q_3\} \) to \( \{q_2,q_4,q_3\} \);
\( p_2 \) prefers \( \{q_1,q_3\} \) to \( \{q_5,q_6\} \); \( q_1 \) prefers \( \{p_1,p_2\} \) to \( \{p_3,p_4\} \) and \( q_4 \) prefers \( \{p_1,p_2\} \) to
\( \{p_5,p_6\} \). In this case \( x_3 \) is not setwise-stable. In fact the players in the coalition
\( C = \{p_1,p_2,q_1,q_3\} \) can make a better arrangement for all of them under the matching \( x'_3 \)
of Fig. 1f. They do this by forming their new partnerships only among themselves. Observe that in this example \( C \) does not weakly block the matching \( x'_3 \); the players in \( C \)
do not form all their partnerships only among themselves. With a convenient profile of
preferences (see Example 2) matching \( x_3 \) is corewise-stable, although it is not setwise-
stable.

Historically, definitions (1) and (2) above are standard and a version of (3) was
defined by Roth in Roth (1985) in the context of the College Admissions problem with

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1 A matching \( x \) is in the strong core (strong corewise-stable) if there is no subset of agents who by forming all
their partnerships only among themselves, can all obtain a weakly preferred set of partners and, at least one of
them, can obtain a strictly preferred set of partners. In the Marriage model with strict preferences the set of
stable matchings coincides with the core, which is equal to the strong core. For the College Admission model
of Gale and Shapley the set of stable matchings coincides with the strong core. However the strong core may
not coincide with the core (see Roth and Sotomayor, 1990).
responsive preferences. Roth called it group-stability. In Roth (1984) the concept of stability is presented as an extension of the group-stability concept in a many-to-many matching problem with money, where preferences are strict and satisfy a condition called substitutability. Substitutability was introduced in Kelso and Crawford (1982) and will be defined later. Regarding questions of existence, in the one-to-one case stable matchings (in all three senses) always exist. This is the basic result of Gale and Shapley (1962) which gave birth to the theory. In the negative direction, Example 2.7 of Roth and Sotomayor (1990) showed that without substitutability there may be no pairwise-stable matching even in the many-to-one case. The same result was reached earlier in Kelso and Crawford (1982), by assuming that agents’ preferences depend on a monetary variable which ranges continuously. For the case with substitutable preferences the existence theorem of Roth (1984) proves the existence of pairwise-stable matchings and not, as the author claims, that of stable matchings. In Blair (1988) Blair presented a version of Roth’s model, keeping the strictness and the substitutability property of the
preferences. He observed, through an example, that the core and the set of outcomes, called ‘stable’ by Roth, might be disjoint.

Of course there are no setwise-stable matchings in Blair’s example, since any setwise-stable matching is pairwise-stable and must be in the core. In his example, however, preferences do not satisfy the natural condition of being responsive. Thus the example of Blair does not show that setwise-stable matchings may not exist in general.

Without strictness of preferences over individuals, the strong core may be empty, even in the Marriage market. This can be easily seen by a simple example with one man and two women, where the man is indifferent between the two women. As for the core, whether or not core matchings always exist is apparently still an open question.

In the present paper we show that:

(1) Setwise-stability is a strictly stronger requirement than the other two concepts of stability. We prove this by giving an example of a case where there is a matching which is both pairwise-stable and strong corewise-stable but is not setwise-stable. In this example the preferences are not only substitutable and responsive but they satisfy the stronger condition of being separable and the even stronger condition of being representable by a separable cardinal utility function.

(2) Using this same kind of preference order, we give an example with the property that the core and the set of pairwise-stable matchings are non-empty and disjoint and thus setwise-stable matchings do not exist.

For empirical purposes, some markets are better modeled by restricting agents’ preferences to be substitutable rather than separable. This is the case of a variety of labor markets. For such markets we present a discrete model with substitutable and not necessarily strict preferences. Therefore, it is more general than Roth’s model which assumes strict preferences. In these markets firms negotiate with workers or groups of workers, but not with other firms; and workers negotiate with firms or groups of firms, but not with other workers. Thus the essential coalitions are restricted to one firm and a group of workers or to one worker and a group of firms. In this case Roth and Sotomayor (1990) proves that the concept of stability is equivalent to the concept of pairwise-stability when preferences are responsive. Blair (1988) proves the same result for the case where preferences are substitutable. Therefore, the effects of competition can be characterized by requiring only pairwise-stability.

It is then worth asking if pairwise-stable matchings always exist. For the case with strict preferences the answer is affirmative and the proof is given in Roth (1984) by means of an algorithm. For the general case substitutability needs to be redefined. In the present paper we do the following:

(3) We give a general definition of substitutability. Then we consider a many-to-many matching model with substitutable and not necessarily strict preferences. For this model we give a direct and short proof of the existence of pairwise stable matchings. Thus, our proof applies to all discrete matching models (one-to-one, many-to-one and many-to-many), with any kind of preferences: substitutable or responsive, strict or non-strict. It is an instructive exercise to verify that the restriction of our proof to the Marriage Model coincides with the existence proof of Sotomayor (1996) which is extremely short. When restricted to the College Admission Model with responsive preferences, our proof is also very simple and short and provides an alternative proof for the existence of stable matchings for that model.
This paper is organized as follows. Section 2 describes the many-to-many matching model with cardinal separable preferences and defines setwise-stability. Section 3 shows, through examples, that setwise-stability is strictly stronger than pairwise-stability plus core. Section 4 presents an example which shows that setwise-stable matchings may not exist. Section 5 redefines substitutability and describes a model in which preferences are substitutable and not necessarily strict. We then present a general proof of the existence of pairwise stable matchings. Section 6 concludes the paper.

2. Description of the model with separable preferences

There are two disjoint and finite sets of players, $P$ and $Q$. $P$ has $m$ elements and $Q$ has $n$ elements. Each player from one set is allowed to form partnerships with players from the opposite set. Each player $i \in P$ may form at most $r_i$ partnerships and each player $j \in Q$ may form at most $s_j$ partnerships. Denote by $|A|$ the number of elements of the set $A$. We say that a set of players $A$ is an allowable set of partners for $i \in P$ (resp. $j \in Q$) if $|A| \leq r_i$ (resp. $s_j$) and $A \subseteq Q$ (resp. $P$). For each pair $(i,j)$ there are two numbers $a_{ij}$ and $b_{ij}$. The preferences of the players $i$ and $j$ over allowable sets of partners are determined by these numbers. Therefore, say, $i$ prefers $j$ to $k$ if and only if $a_{ij} > a_{ik}$ and $i$ is indifferent between $j$ and $k$ if and only if $a_{ij} = a_{ik}$. We can interpret $a_{ij}$ (resp. $b_{ij}$) as being the payment $i$ (resp. $j$) can get in case $i$ and $j$ form a partnership. Under this interpretation it is natural to require that player $i$ prefers the set $S \subseteq Q$ to the set $S' \subseteq Q$ if and only if $\sum_{i \in S} a_{ij} > \sum_{i \in S'} a_{ij}$, where $S$ and $S'$ are allowable sets of partners for $i$. Thus, for example, if player $i$ has quota $r_i = 2$ and $a_{i1} > a_{i2} + a_{i3}$ this means that player $i$ prefers to form a partnership with player 1 and to have one unfulfilled position than to have both positions filled with players 2 and 3. For simplicity we will consider that the reservation utilities of $i \in P$ and $j \in Q$ are 0. For our purposes a player will only compare allowable sets of partners. We say that a set of players $A$ is acceptable to player $i$ (resp. $j$) if it is allowable to $i$ (resp. $j$) and $\sum_{j \in A} a_{ij} \geq 0$ (resp. $\sum_{i \in A} b_{ij} \geq 0$). If $i$ and $j$ are acceptable to each other we say that they are mutually acceptable.

**Definition 1.** A feasible matching $x$ is an $m \times n$ matrix $x_{ij}$ of 0’s and 1’s, defined for all pairs $(i,j) \in P \times Q$, such that $\sum_{i \in P} x_{ij} = r_i$, for all $i \in P$ and $\sum_{j \in Q} x_{ij} = s_j$, for all $j \in Q$. Furthermore, for all $i \in P$ and $j \in Q$, if $x_{ij} = 1$ then $i$ and $j$ are mutually acceptable. If $x_{ij} = 1$ (resp. $x_{ij} = 0$) we say that $i$ and $j$ are (resp. are not) matched at $x$.

Thus, a feasible matching is individually rational. We will denote by $C(i,x)$ the set of $j \in Q$ such that $x_{ij} = 1$. Thus, if player $i$ has quota five, the expression $C(i,x) = \{q_1,q_2,q_3\}$ denotes that player $i$ forms partnerships with $q_1$, $q_2$, and $q_3$ and has two unfulfilled positions. Similarly we will denote by $C(j,x)$ the set of $i \in P$ such that $x_{ij} = 1$. It is clear that $C(i,x) \subseteq r_i$ and $C(j,x) \subseteq s_j$ for all $(i,j) \in P \times Q$ and all feasible matching $x$.

Setwise-stability was defined in Section 1. The formal definition is the following:

**Definition 2.** A matching $x$ is setwise-stable if it is feasible and there are no feasible

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Footnote: For stability purposes, Definition 1 could require individual rationality instead of mutual acceptance. The reason is that under stability, if two players form a partnership then they are mutually acceptable.
matching \(x'\) and coalitions \(R \subseteq P\) and \(S \subseteq Q\), with \(R \neq \emptyset\) and \(S \neq \emptyset\), such that for all \(i \in R\), and for all \(j \in S\):

(i) if \(x'_{iq} = 1\) then \(q \in S \cup C(i, x)\) and if \(x'_{ip} = 1\) then \(p \in R \cup C(j, x)\);

(ii) \(\sum_{q \in Q} a_{iq} x'_{iq} > \sum_{q \in Q} a_{iq} x_{iq}\) and \(\sum_{p \in P} b_{pj} x'_{pj} > \sum_{p \in P} b_{pj} x_{pj}\).

That is, every new partner of any player in the coalition \(R \cup S\) belongs to this coalition. Every player \(y\) in \(R \cup S\) may continue to be matched with some old partners from \(C(y, x)\). Furthermore, everyone in the coalition strictly prefers the new arrangement given by \(x'\).

If there are such coalitions \(R\) and \(S\) we say that \(R \cup S\) causes an instability in the matching \(x\) via \(x'\).

When the essential coalitions are pairs \((i, j) \in P \times Q\), the coalitions which cause instabilities are restricted to these pairs of players. In this case the concept of stability is called pairwise-stability. Thus pairwise-stability is the concept of stability for the Marriage and the College Admission games. Formally we have:

**Definition 3.** The matching \(x\) is pairwise-stable if it is feasible and there is no pair \((i, j)\), with \(x_{ij} = 0\), such that \(i\) and \(j\) prefer each other to some of their partners or to an unfilled position, if any.

### 3. Stability and counterexamples for the case with separable preferences

This section shows that the concept of pairwise-stability and strong core are not equivalent to the concept of setwise-stability in the many-to-many matching game, although they are equivalent in the one-to-one and many-to-one cases. Instead, setwise-stability is strictly stronger than pairwise-stability plus core. Example 1 shows that there may be pairwise-stable matchings which are not in the core, so they are not setwise-stable. Example 2 presents a situation in which the set of setwise-stable matchings is a proper subset of the intersection of the core with the set of pairwise-stable matchings.

**Example 1.** (A pairwise stable matching which is not in the core). Let \(P = \{p_1, \ldots, p_4\}, Q = \{q_1, \ldots, q_4\}\), \(r_i = s_j = 2\) for all \(i, j = 1, \ldots ,4\). The pairs of numbers \((a_{ij}, b_{ij})\) are given in Table 1.

<table>
<thead>
<tr>
<th>(P)</th>
<th>(10,1)</th>
<th>((1,10))</th>
<th>((4,10))</th>
<th>((2,10))</th>
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<td>((10,1))</td>
<td>((4,4))</td>
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<tr>
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<td>((2,2))</td>
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<tr>
<td>(10,2)</td>
<td>((4,2))</td>
<td>((2,1))</td>
<td>((1,1))</td>
<td></td>
</tr>
</tbody>
</table>

Table 1

Number pairs \((a_{ij}, b_{ij})\) for Example 1 (the payoffs of each matched pair are shown in bold)
Consider the matching \( x \) where \( p_1 \) and \( p_2 \) are matched to \( \{q_3,q_4\} \) and \( p_1 \) and \( p_4 \) are matched to \( \{q_1,q_2\} \). (The payoffs of each matched pair are bold faced in Table 1 above). This matching is pairwise stable. In fact, \( p_3 \) and \( p_4 \) do not belong to any pair which causes an instability, because they are matched to their two best choices: \( q_1 \) and \( q_2 \); \( (p_1,q_3) \) and \( (p_1,q_4) \) do not cause instabilities since \( p_1 \) is the worst choice for \( q_3 \) and \( q_4 \) is the worst choice for \( q_2 \). \( (p_2,q_1) \) and \( (p_2,q_2) \) do not cause instabilities since \( q_1 \) is the worst choice of \( p_2 \) and \( p_2 \) is the worst choice of \( q_2 \). Nevertheless \( p_1 \) and \( p_2 \) prefer \( \{q_1,q_2\} \) to \( \{q_3,q_4\} \) and \( q_1 \) and \( q_2 \) prefer \( \{p_1,p_2\} \) to \( \{p_3,p_4\} \). Hence this matching is not in the core, since it is blocked by \( \{p_1,p_2,q_1,q_2\} \).

**Example 2. (A strong corewise-stable matching which is pairwise-stable and is not setwise-stable).** Consider \( P = \{p_1, \ldots, p_6\} \), \( Q = \{q_1, \ldots, q_6\} \), \( r_1 = 3 \), \( r_2 = r_3 = 2 \), \( r_4 = r_5 = r_6 = 1 \), \( s_1 = s_2 = s_4 = 2 \) and \( s_3 = s_5 = s_6 = s_7 = 1 \). The pairs of numbers \( (a_{ij},b_{ij}) \) are given in Table 2 below.

Consider the matching \( x \) at which \( p_1 \) is matched to \( \{q_2,q_3,q_4\} \); \( p_2 \) is matched to \( \{q_5,q_6\} \); \( p_3 \) and \( p_4 \) are matched to \( q_1 \); \( p_5 \) is matched to \( \{q_2,q_3\} \) and \( p_6 \) is matched to \( q_4 \). This is the matching \( x_5 \) of Section 1. (The payoffs of each matched pair are bold faced in Table 2 above). This matching is strong corewise-stable. In fact, if there is a matching \( y \) which weakly dominates \( x \) via some coalition \( A \), then, under \( y \), no player in \( A \) is worse off and at least one player in \( A \) is better off. Furthermore, matching \( y \) must match all players in \( A \) among themselves. By inspection we can see that the only players that can be better off are \( p_1 \), \( p_2 \), \( q_1 \) and \( q_4 \), for all remaining players are matched to their best choices. However, if \( A \) contains one player of the set \( \{p_1,p_2,q_1,q_4\} \) then \( A \) must contain all four players. In fact, if \( p_1 \in A \) then \( p_1 \) must form a new partnership with \( q_1 \); if \( q_1 \in A \) then \( q_1 \) must form a new partnership with \( p_2 \), and if \( q_4 \in A \) then \( q_4 \) must form a new partnership with \( p_1 \). As in Section 1, if \( q_4 \in A \) then \( q_4 \) must form a new partnership with \( p_1 \), so \( p_1 \) must be in \( A \). Then, if \( y \) weakly dominates \( x \) via \( A \), then \( p_1 \), \( p_2 \), \( q_1 \) and \( q_4 \) are in \( A \) and \( p_3 \) and \( p_4 \) form new partnerships with \( q_1 \) and \( q_4 \). Nevertheless, \( p_1 \) must keep his partnership with \( q_2 \), his best choice. Then \( q_2 \) must be in \( A \), so \( y \) cannot be worse off and so \( p_3 \) must also be in \( A \). But \( p_3 \) requires the partnership with \( q_4 \), who has quota of 2 and has already filled her quota with \( p_1 \) and \( p_2 \). Hence \( p_3 \) is worse at \( y \) than at \( x \) and then \( y \) cannot weakly dominate \( x \) via \( A \). The matching \( x \) is clearly pairwise-stable. Nevertheless, the coalition \( \{p_1,p_2,q_1,q_4\} \) causes an instability in \( x \). In fact, if \( p_1 \) is matched to \( \{q_1,q_2,q_3\} \)

### Table 2

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
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<tr>
<td>(0,0)</td>
<td>(10,2)</td>
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</tbody>
</table>

The matching \( x \) is clearly pairwise-stable. Nevertheless, the coalition \( \{p_1,p_2,q_1,q_4\} \) causes an instability in \( x \). In fact, if \( p_1 \) is matched to \( \{q_1,q_2,q_3\} \).
and \( p_2 \) is matched to \( \{q_1, q_3\} \) then \( p_1 \) gets 28 instead of 21 and the rest of the players in the coalition gets 11 instead of 6. Hence the matching \( x \) is not setwise-stable.

4. Existence of stable matchings for the case with separable preferences: A negative result

This section addresses the problem of the existence of stable matchings. In the negative direction, one-to-one matching markets with one or three sides may have no stable matching, as in the roommates problem (Gale and Shapley, 1962) and the man–woman–child marriage problem (Alkan, 1986). The non-existence of stable matchings in the roommates problem, for example, is due to the existence of odd cycles in the preferences of the players. Hence, since this fact cannot be observed in the one-to-one matching market with two sides, the Marriage model always has a stable matching. Likewise markets with many-to-one matching and non-substitutable preferences may have no stable matching (Kelso and Crawford, 1982 and Example 2.7 of Roth and Sotomayer, 1990). However, when preferences are substitutable the many-to-one matching market always has stable matchings (Kelso and Crawford, 1982). Example 3 below shows that existence theorems, which hold for two-sided one-to-one and many-to-one matching markets, may fail to hold in the discrete many-to-many cases. In these cases the substitutability property is not enough to guarantee the existence of stable matchings. Even the separability condition, which in continuous matching games with transferable utility makes existence of stable outcomes possible for the three kinds of matchings is not sufficient to preserve the existence of stable outcomes in the many-to-many case with non-transferable utilities. (The interested reader can see Sotomayor, 1992; Sotomayor, 1998, where a continuous model with separable utilities is discussed).

Example 3. (Nonexistence of stable matchings). Consider Example 1 again. We will show that this example does not have any setwise-stable matching. First, observe that \( p_3 \) prefers \( \{q_1, q_2\} \) to any other set of players and \( p_3 \) is the second choice for \( q_1 \) and \( q_2 \); \( q_3 \) prefers \( \{p_1, p_2\} \) to any other set of players and \( q_3 \) is the second choice for \( p_1 \) and \( p_2 \). Then, in any stable matching \( x \), \( p_3 \) must be matched to \( q_1 \) and \( q_2 \) while \( q_3 \) must be matched to \( p_1 \) and \( p_2 \). Separate the cases by considering the possibilities for the second partner for \( q_3 \), under a supposed stable matching \( x \):

Case 1. (\( q_3 \) is matched to \( \{p_2, p_3\} \)). Then \( x_{24} = 0 \) and we have that \( (p_2, q_4) \) causes an instability in the matching, since \( p_2 \) prefers \( q_4 \) to \( q_1 \) and \( p_2 \) is the second choice for \( q_4 \).

Case 2. (\( q_3 \) is matched to \( \{p_3, p_3\} \)). The following possibilities may occur:

(i) (\( q_2 \) is matched to \( \{p_3, p_4\} \)). Then \( (p_1, p_2, q_1, q_2) \) causes an instability in the matching. This matching is pairwise-stable but is not in the core.

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3 In that model, for each pair \( (i,j) \in P \times Q \) there is a non-negative number \( a_{ij} \) which can be divided between \( i \) and \( j \) in any way they want.
(ii) \((q_2 \text{ is matched to } \{p_1, p_3\})\). Then \((p_1, q_4)\) causes an instability in the matching, since \(p_1\) is the first choice for \(q_4\) and \(p_1\) prefers \(q_4\) to \(q_2\).

(iii) \((q_2 \text{ is matched to } \{p_2, p_3\} \text{ or } \{p_2\})\). Then \((p_4, q_4)\) causes an instability in both cases, since \(q_2\) is the second choice for \(p_4\) and \(q_2\) prefers \(p_4\) to \(p_2\) and prefers \(p_4\) to have an unfilled position.

Case 3. \((q_4 \text{ is matched to } \{p_1, p_2\} \text{ or } \{p_3\})\). Then \((p_3, q_4)\) causes an instability in both cases, since \(q_4\) is the second choice for \(p_3\) and \(q_4\) prefers \(p_3\) to \(p_2\) and prefers \(p_3\) to have an unfilled position.

Hence there is no possible arrangement among the players which forms a stable matching.

In the Blair example players’ preferences do not satisfy the condition of being responsive. Since separability implies responsiveness and consequently substitutability, Example 3 holds for these three kinds of preferences.

In Example 3 the core is non-empty and it is disjoint from the set of pairwise-stable matchings. In fact, the matching described in Case 2(i) is the only pairwise stable matching and it is not in the core. The core is non-empty because the matching where \(\{p_1, p_3\}\) is matched to \(\{q_1, q_2\}\) and \(\{p_3, p_4\}\) is matched to \(\{q_3, q_4\}\) is in the core.

5. The case with substitutable and non-necessarily strict preferences

5.1. Description of the model

This section describes the many-to-many matching model when players’ preferences are substitutable but not necessarily strict. This is an extension of the many-to-one matching model with substitutable and strict preferences presented (Roth and Sotomayor, 1990)\(^1\).

There are two disjoint and finite sets of players, \(P\) and \(Q\). \(P\) has \(m\) elements and \(Q\) has \(n\) elements.

Let \(y \in P \cup Q\). If \(A\) and \(B\) are any sets of potential partners for player \(y\) we write \(A >_y B\) to mean \(y\) prefers \(A\) to \(B\); \(A \geq_y B\) to mean \(y\) likes \(A\) at least as well as \(B\) and \(A \equiv_y B\) to mean \(y\) is indifferent between \(A\) and \(B\). Faced with a set \(A\) of possible partners, player \(y\) can determine which subsets of \(A\) he/she likes best. We denote the set of all such subsets of \(A\) by \(Ch_y(A)\). That is:

\[ P_1: T \in Ch_y(A) \text{ if and only if } T \subseteq A \text{ and } T \geq_y T' \text{ for all } T' \subseteq A. \]

\(^1\)In the many-to-many firm-worker models of Roth (1984) and Blair (1988) the preferences are strict. Furthermore, wages are negotiated and are modeled as a discrete variable. In the model we are describing here we treat salaries as an implicit part of the job description and we allow non strict preferences. When preferences are strict, our definition of substitutability (Definition 4) is equivalent to the definition presented in Roth (1984) which is a reformulation of the definition proposed in Kelso and Crawford (1982).
Proposition 1. Suppose the preferences are substitutable.

**P2**: If \( T \in Ch_y(A) \) and \( T \subseteq B \subseteq A \), then \( T \in Ch_y(B) \).

In particular, if \( y \) chooses \( T \) from \( A \) and \( j \in A \), so the choices of \( y \) from \( T \cup \{j\} \) contains \( T \).

The general notion of substitutable preferences is given in Definition 4 below. When preferences are strict, to say that \( y \) has substitutable preferences has the following meaning:

Let \( F \) and \( G \) be sets of potential partners of \( y \) such that \( F \cap G = \emptyset \). Then:

(A) If \( y \) chooses \( w \in F \) when \( G \) is available, then \( y \) still chooses \( w \) when \( G \) is not available.

Equivalently:

(B) If \( y \) does not choose \( w \in F \) when \( G \) is not available then \( y \) does not choose \( w \) when \( G \) is available.

(A) and (B) can also be stated in the following way:

(A) Let \( F' \subseteq F \). If \( F' \subseteq Ch_y(F \cup G) \) then \( F' \subseteq Ch_y(F) \).

(B) Let \( F' \subseteq F \) and \( S = Ch_y(F) \). If \( F' = F - S \) then \( Ch_y(F \cup G) \cap F' = \emptyset \).

A natural extension of (A) and (B) to the case with non-strict preferences is:

(A') Let \( F' \subseteq F \) and \( S' \in Ch_y(F \cup G) \). If \( F' \subseteq S' \) then \( F' \subseteq S \) for some \( S \in Ch_y(F) \).

(B') Let \( F' \subseteq F \) and \( S \in Ch_y(F) \). If \( F' = F - S \) then \( S' \cap F' = \emptyset \) for some \( S' \in Ch_y(F \cup G) \).

(A) and (B) are equivalent under strict preferences. However, \( (A') \) is not necessarily equivalent to \( (B') \) under non-strict preferences. Then \( (A') \) and \( (B') \) are necessary to characterize the substitutability of \( y \)'s preference. That is:

**Definition 4.** Player \( y \in P \cup Q \) has substitutable preferences if (i) for all \( S' \in Ch_y(F \cup G) \) there is some \( S \in Ch_y(F) \) such that \( S' \cap F \subseteq S \) and (ii) for all \( S \in Ch_y(F) \) there is some \( S' \in Ch_y(F \cup G) \) such that \( S' \cap F \subseteq S \).

When preferences are strict, conditions (i) and (ii) are equivalent to require that if \( Ch_y(F \cup G) = S' \) then \( S' \cap F \subseteq Ch_y(F) \), so Definition 4 is equivalent to Definition 6.2 of Roth and Sotomayor (1990). This is the sense in which \( y \) regards the players in \( S' \cap F \) more as substitutes than complements: \( y \) continues to want to work with \( S' \cap F \) even if some of the other possible partners become unavailable.

The proposition below extends Proposition 2.3 of Blair, 1988 to the case where preferences are not necessarily strict.

**Proposition 1.** Suppose the preferences are substitutable. Let \( F \) and \( G \) be sets of possible partners for player \( y \in P \cup Q \). Let \( S \in Ch_y(F) \). If \( T \in Ch_y(S \cup G) \) then \( T \in Ch_y(F \cup G) \).

**Proof.** By substitutability [Definition 4(ii)] there is some \( S' \in Ch_y(F \cup G) \) such that \( S' \cap F \subseteq S \). Since \( S' \cap G \subseteq G \) we may write that \( S' \subseteq S \cup G \subseteq F \cup G \), where the second inclusion follows from the assumption that \( S \in Ch_y(F) \). By **P2**, \( S' \subseteq Ch_y(S \cup G) \). Now let \( T \in Ch_y(S \cup G) \). Then \( S' \equiv y \), and so \( T \in Ch_y(F \cup G) \).
**Definition 5.** The matching \( x \) is feasible if for every player \( y \in P \cup Q \) we have that \( C(y,x) \in Ch_j(C(y,x)) \).5

Note that if the preferences are separable and player \( y \) has quota \( q \), then \( Ch_j(C(y,x)) \) contains either the set of the \( q \) most preferred acceptable partners in \( C(y,x) \) or all the acceptable players in \( C(y,x) \), whichever set has the smaller size. Hence, Definitions 1 and 5 are equivalent when preferences are separable.

**Definition 6.** The matching \( x \) is pairwise-stable if it is feasible and there is no pair \( (i,j) \) with \( x_{ij} = 0 \) such that if \( T \in Ch_i(C(i,x) \cup \{j\}) \) and \( S \in Ch_j(C(j,x) \cup \{i\}) \) then \( T > x \), \( i \) and \( S > x \).

If there is such a pair \( (i,j) \), P1, P2 and the feasibility of \( x \) imply that \( j \in T \) and \( i \in S \). In fact, by P1, \( T \) is a subset of \( C(i,x) \cup \{j\} \); if \( j \in T \) then \( T \subseteq C(i,x) \); P2 applied to \( T \subseteq C(i,x) \subseteq C(i,x) \cup \{j\} \) implies that \( T \in Ch_i(C(i,x)) \); by the feasibility of \( x \), \( T \equiv C(i,x) \), which contradicts that \( T > x \). Hence \( j \in T \). Analogously we show that \( i \in S \). We say that \( (i,j) \) causes an instability in \( x \).

When preferences are separable, one verifies that Definitions 3 and 6 are equivalent.

### 5.2. Existence of pairwise-stable matching

For every \( j \in Q \) and every feasible matching \( x \) set:

\[
A(j,x) = \{ i \in P; (i,j) \text{ causes an instability in } x \}.
\]

**Remark 1.** If \( A(j,x) \neq \emptyset \), then for all \( \phi \neq S \subseteq A(j,x) \) and for all \( T \in Ch_i(C(j,x) \cup S) \) we must have that \( T > x \). In fact, if not we would have that \( T \equiv C(j,x) \), by P1. Now let \( i \in S \). Then using that \( C(j,x) \subseteq C(j,x) \cup \{i\} \subseteq C(j,x) \cup S \) and applying P2, we get that \( C(j,x) \subseteq Ch_i(C(j,x) \cup \{i\}) \), which contradicts the fact that \( (i,j) \) causes an instability at \( x \).

Set:

\[
E = \{ x \text{ feasible}; \text{ if } A(j,x) \neq \emptyset \text{ for some } j \in Q, \text{ then for all set } S \text{ with } \phi \neq S \subseteq A(j,x), \text{ there is some } B \subseteq S \text{ such that } C(j,x) \cup B \in Ch_i(C(j,x) \cup S) \}.
\]

**Remark 2.** It follows from Remark 1 that the set \( B \) in the definition of \( E \) is non-empty.

Thus, if a feasible matching \( x \) is in \( E \) and \( (i,j) \) causes an instability for all \( i \in S \), then \( j \) will be better off by forming new partnerships with some elements of \( S \), without dissolving any of her old partnerships. Observe that if \( x \) is pairwise-stable then \( x \) is in \( E \). When preferences are separable then \( E \) is the set of feasible matchings \( x \) such that \( x \) is unstable and \( j \) causes an instability with some firm \( i \), then \( j \) has at least one unfilled position and \( p \geq i \) for all \( p \in C(j,x) \).

**Lemma 1.** Let \( x \) be a feasible matching. Consider \( j \in Q \). Suppose that for all \( i \in B \subseteq P \)

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5 If \( x \) is feasible then \( x \) is individually rational, but the converse is not true. In fact, a matching \( x \) may not be feasible without being individually irrational, since it might still be that \( C(y,x) \not> \phi \).
we have that $x_{ij} = 0$ and $(i,j)$ does not cause an instability in $x$. Furthermore, for all $i \in B$ and all $T_i \in \text{Ch}_i(C(i,x) \cup j)$ we have that $T_i \succ_i C(i,x)$. Then $C(j,x) \in \text{Ch}_j(C(j,x) \cup B)$.

**Proof.** It follows from Definition 6 that $C(j,x) \in \text{Ch}_j(C(j,x) \cup \{i\})$, for all $i \in B$. If $|B| = 1$ we are done. If not take $\{i_1,i_2\} \subseteq B$. By Proposition 1, making $F = C(j,x) \cup \{i_1\}$, $S = C(j,x)$, $G = \{i_2\}$ and $T = S$ then $C(j,x) \in \text{Ch}_j(C(j,x) \cup \{i_1\} \cup \{i_2\})$. By repeated application of Proposition 1 we get that $C(j,x) \in \text{Ch}_j(C(j,x) \cup B)$. 

**Lemma 2.** Let $x \in E$. Consider $j \in Q$. Suppose that for all $i \in B \subseteq P$ we have that $x_{ji} = 0$. Furthermore, for all $i \in B$ and all $T_i \in \text{Ch}_i(C(i,x) \cup j)$ we have that $T_i \succ_i C(i,x)$. Then there is some $T \in \text{Ch}_j(C(j,x) \cup B)$ such that $C(j,x) \subseteq T$.

**Proof.** Set $B = B_1 \cup B_2$, where no $i \in B_1$ causes an instability with $j$ in $x$ and $B_2 \subseteq A(j,x)$. By Lemma 1 $C(j,x) \in \text{Ch}_j(C(j,x) \cup B_1)$. If $B_2 = \emptyset$ then we are done. Otherwise, by definition of $E$ there is some $T \in \text{Ch}_j(C(j,x) \cup B_2)$ such that $C(j,x) \subseteq T$. We claim that $T \in \text{Ch}_j(C(j,x) \cup B_1 \cup B_2)$. But this is immediate from Proposition 1 applied to $F = C(j,x) \cup B_1$, $G = B_2$ and $S = C(j,x)$.

It is clear that the set $E$ is non-empty, since it contains the matching where every one is unmatched. Since $E$ is finite and the preferences are transitive, there is a matching $x^*$ (there might be more than one) which is maximal for the players in $P$ among all matchings in $E$. That is, if $x \in E$ and some $i \in P$ prefers $x$ to $x^*$, then there is some $i' \in P$ such that $i'$ prefers $x^*$ to $x^*$.

**Theorem.** The matching $x^*$ is pairwise-stable.

**Proof.** Suppose, by contradiction, that $x^*$ is unstable via some pair $(i,j) \in P \times Q$. Then $A(j,x^*) \neq \emptyset$. By definition of $E$ and Remark 2, by making $S = A(j,x^*)$, there is some $\phi \neq B \subseteq A(j,x^*)$ such that $C(j,x^*) \cup B \in \text{Ch}_j(C(j,x^*) \cup A(j,x^*))$. Now construct a new matching $x'$ as follows: match $j$ with $C(j,x^*) \cup B$ and if $i \in B$ choose $C_i \in \text{Ch}_i(C(i,x^*) \cup \{j\})$ and match $i$ to $C_i$. [There might be some dissolved partnerships $(i,k)$ where $i \in B$ and $k \in C(i,x^*)$]. It follows from Definition 6 that $j \in C_i \succ_i C(i,x^*)$ for all $i \in B$. Now keep the partnerships of the remaining players. Therefore, $x'$ is weakly preferred by all players in $P$ and strictly preferred by all players in $B$. We are going to show that $x' \in E$ which contradicts the definition of $x^*$. First observe that $j$ cannot cause an instability with any player $p$ in $P$. In fact, it suffices to verify the instabilities caused by pairs $(p,j)$, with $p \in A(j,x^*)$. Then let $p \in A(j,x^*)$. But it is immediate from $P_j$ applied to $(C(j,x^*) \cup B) \subseteq (C(j,x^*) \cup B \cup \{p\}) \subseteq (C(j,x^*) \cup A(j,x^*))$ that $C(j,x^*) \cup B \in \text{Ch}_j(C(j,x^*) \cup B \cup \{p\})$. Thus, $C(j,x') \in \text{Ch}_j(C(j,x') \cup \{p\})$ and $(p,j)$ does not cause an instability. Now we only need to check that in all dissolved partnerships $(i,k)$, with $A(k,x') \neq \emptyset$, if $\phi \neq B' \subseteq A(k,x')$ then there is some $S \in \text{Ch}_k(C(k,x') \cup B')$ such that $C(k,x') \subseteq S$. Then let $k$ belong to some dissolved partnership, with $A(k,x') \neq \emptyset$, and let $\phi \neq B' \subseteq A(k,x')$. First observe that if $x_{ik} = 0$ and $x^*_{ik} = 1$ then $i \in B'$. In fact, if $(i,k)$ was dissolved then

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6 When preferences are separable we can choose the matching $x$ which maximizes $\sum_i \rho_i x_{i,j}$, for all $x \in E$. 

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(There is a missing reference to the proposition number in the text.)
by construction of \(x'\). If \(S_i \subseteq C_h(C(i,x') \cup \{k\})\) then \(S_i \supseteq C(i,x')\) and \(S_i \subseteq C(i,x') \cup \{k\} \subseteq C(i,x') \cup \{j\}\), where the last inclusion follows from assertion (1). So \(S_i \subseteq C(i,x*) \cup \{j\}\), and so \(C(i,x') \supseteq S_i\) by assertion (1). Hence, \(S_i \supseteq C(i,x')\). Therefore, \((i,k)\) does not cause an instability at \(x'\) and so \(i \in B'\).

Hence, \(x^*_k = 0\) for all \(i \in B'\). Therefore, we only need to establish that for all \(i \in B'\) and all \(T_i \in Ch_i(C(i,x*) \cup \{k\})\) we have that \(T_i \supseteq C(i,x*)\). If this is done, then Lemma 2 applied to \(B'\) and \(k\) implies that there exists some \(T \in Ch_i(C(k,x*) \cup B')\) such that \(C(k,x*) \subseteq T\). Since \(C(k,x*) \cup B' = [C(k,x') \cup B'] \cup [C(k,x*) \setminus C(k,x')]\) we can use substitutability property [Definition 4(ii)], making \(F = C(k,x') \cup B', G = C(k,x*) \setminus C(k,x')\), to get that there is some \(S \in Ch_i(C(k,x') \cup B')\) such that \(T \cap [C(k,x*) \cup B'] \subseteq S\). But \(C(k,x*) \subseteq C(k,x*) \subseteq T\). Furthermore, \(C(k,x') \subseteq C(k,x') \cup B'\), so \(C(k,x') \subseteq T \cap [C(k,x') \cup B'] \subseteq S\) and so \(C(k,x') \subseteq S\), and we have the desired result. Then suppose by contradiction that there exists some \(i \in B'\) and some \(T_i \in Ch_i(C(i,x*) \cup \{k\})\) such that \(T_i \supseteq C(i,x*)\). Then

\[
C(i,x*) \subseteq Ch_i(C(i,x*) \cup \{k\})
\]

by the definition of \(Ch_i\). Assertion (2) and the fact that \(i \in A(k,x')\) imply that \(C(i,x*) \neq C(i,x')\), so \(C(i,x^*) \subseteq Ch_i(C(i,x*) \cup \{j\})\) by construction of \(x'\). Using assertion (2) and making \(F = C(i,x*) \cup \{k\}, G = \{j\}\) and \(S = C(i,x*)\), we can apply substitutability property [Definition (4ii)] to get that there is some \(S' \in Ch_i(C(i,x*) \cup \{k\} \cup \{j\})\) such that \(S' \cap [C(i,x*) \cup \{j\}] \subseteq C(i,x*)\). Hence, \(x^*_k = 0\) implies \(k \in S'\). Thus, \(S' \supseteq C(i,x*) \cup \{j\} \subseteq C(i,x') \cup \{j\} \cup \{k\}\). \(P_2\) can be applied yielding that \(S' \in Ch_i(C(i,x*) \cup \{j\})\). But \(C(i,x') \in Ch_i(C(i,x*) \cup \{j\})\), so \(S' \supseteq C(i,x')\) and hence \(C(i,x') \in Ch_i(C(i,x*) \cup \{j\})\). Also \(C(i,x') \subseteq [C(i,x') \cup \{j\}] \subseteq [C(i,x*) \cup \{j\}] \cup \{k\}\). Then \(P_2\) can be used and it implies that \(C(i,x') \in Ch_i(C(i,x*) \cup \{j\})\), which contradicts the fact that \(i \in A(k,x')\).

Therefore, the matching \(x'\) is in \(E\), which contradicts the definition of \(x^*\).

Hence, \(x^*\) is pairwise-stable and the proof is complete. 

6. Final conclusions

The concept of stability presented by Gale and Shapley in their seminal paper is that a matching is stable if it cannot be upset by a coalition of players acting together, in a way that benefits all of them. Thus stable matchings are the matchings which we can expect to occur and hence they belong to the core.

Since the essential coalitions for the Marriage game are pairs of players of opposite sides, the instabilities that can arise in this game can be identified by examining only such small coalitions. Therefore, pairwise-stability is the adequate concept of stability for the Marriage game.

When the players may have more than one partner, we should expect that the
instabilities might be caused by coalitions of any size. This suggests that the appropriate concept of stability for the multiple-partners models is that of setwise-stability.

In Roth (1984) was proved that the essential coalitions for the College Admission model of Gale and Shapley are still pairs of players of opposite sides. Thus setwise-stability is equivalent to pairwise-stability in that model. Because of this, the College Admission model can be reduced to a Marriage model for the purposes of proving the existence of stable matchings (Gale and Shapley, 1962). Another result of Roth (1984) is that the set of stable matchings for the College Admission model, with strict preferences, coincides with the strong core, which may be a proper subset of the core. This kind of inclusion is illustrated by the following simple example. Consider one college \( C \) with quota 2 and three students: \( s_1, s_2 \) and \( s_3 \). College \( C \) prefers \( s_1 \) to \( s_2 \) and \( s_2 \) to \( s_3 \) and all the students prefer \( C \) to be unmatched. The only stable matching, which will be called \( x \), matches \( C \) to \( s_1 \) and \( s_2 \) and leaves \( s_3 \) unmatched. Matching \( x \) is also the only strong corewise-stable matching. However, the core also contains the unstable matching \( x' \) where \( C \) is matched to \( s_1 \) and \( s_3 \) and \( s_2 \) is unmatched. In fact, the coalition \((C, s_2)\) causes an instability in \( x' \) but it does not dominate the matching \( x' \): \( C \) cannot be better off by only being matched to \( s_3 \); the coalition \((C, s_1, s_2)\) weakly dominates \( x' \) but it does not dominate \( x' \), since \( s_1 \) cannot be better off. Hence \( x' \) is not dominated by any coalition and so it is in the core.

It would be nice if Roth results extended from the many-to-one to the many-to-many matching models. Nevertheless, this is not the case: the essential coalitions of the discrete many-to-many matching game are not always pairs of players and so the equivalence between pairwise-stability and setwise-stability is lost. Also strong corewise matchings may be setwise unstable when both sides of the market may have multiple partners.

It becomes apparent from our results that setwise-stability is a general concept of stability. Moreover, it is a strictly stronger requirement than pairwise-stability plus corewise-stability.

For a variety of markets, setwise-stability is a more natural economic concept than corewise-stability. This is the case of job markets, where salaries are treated as an implicit part of the job description. Thus they are not negotiated as part of the agreement between each firm and worker. Because of this, salaries are simply one of the factors that determine the preferences that workers have over the firms. Workers are indifferent to which other workers are employed by the same firm. Similarly, firms are indifferent to which other firms employ the same worker. For such markets the core property of a matching is not enough to guarantee that this matching occur. To illustrate this fact, consider two firms \( p_1 \) and \( p_2 \) and two workers \( q_1 \) and \( q_2 \). Each firm may employ and want to employ both workers; worker \( q_1 \) may take, at most, one job and prefers \( p_1 \) to \( p_2 \); worker \( q_2 \) may work and want to work for both firms. The rules of the market are that any worker and firm may sign an employment contract with each other if they both agree. If the agents can communicate with each other, the outcome that we expect to observe in this market is obvious: \( p_1 \) hires both workers and \( p_2 \) hires only worker \( q_2 \). Of course this outcome is in the strong core. There still is another strong corewise-stable outcome which we do not expect to observe: \( p_1 \) hires only \( q_2 \) and \( p_2 \) hires both workers. That is, both outcomes are in the strong core but only the first one is expected to
occur. Our explanation for this is that only the first outcome is setwise-stable. It is this property, not the corewise-stability, that makes the first outcome to occur.

The example above explains why the existence of corewise-stable outcomes is not treated in this paper. However, mathematically, it is worthy to know if the strong core is empty or non-empty. We think that this subject deserves to be explored in a future investigation.

The negative result about the existence of setwise-stable matchings also opens space to new investigations in the matching theory: to find a sufficient condition for the existence of setwise-stable matchings and then to establish which properties of the many-to-one model extend to the general case.

In the context of firms and workers, the intrinsic properties of the market make it natural that the essential coalitions be formed by one firm and a group of workers or one worker and a group of firms. Then, if we rule out the possibility of two or more firms to collude with two or more workers, this concept of stability is equivalent to the concept of pairwise-stability. For these markets we presented a many-to-many matching model with substitutable and not necessarily strict preferences. A general proof of the existence of pairwise-stable matching was then provided.

Regarding setwise-stability concept as a new solution concept may be an interesting challenge for research in cooperative game theory. The extension of the concept of setwise-stability to a general coalitional game is the following: An outcome is setwise-stable if there is no subset of agents who by forming new coalitions only among themselves, possibly dissolving some of their current coalitions to remain within their quotas and possibly keeping other ones, can all obtain a higher total payoff. In a general coalitional game each player has a quota, representing the maximum number of coalitions she can enter into. (The models in which players have no quota are special cases where the quotas are equal to the total number of players). A set of coalitions, which contain a given player \( p \) in each coalition, is an allowable set of coalitions for \( p \) if the number of coalitions in the set is not greater than the quota of \( p \). For each coalition \( S \) there is a number \( C_s \) representing the value of \( S \). The players in \( S \) can split \( C_s \) among themselves in any way they want. (In some contexts \( C_s \) can be interpreted as the amount of money the players in the coalition \( S \) can do if they work together. The quota of a player is the maximum number of coalitions she can contribute with her labor). A feasible allocation is any set of coalitions which respects the quotas of the players. An outcome \((u,x)\) is an allocation \( x \) and a set of numbers \( u_{ps} \), defined only for \( p \) and \( S \) such that \( p \in S \). The outcome is feasible if \( \sum_{p \in S} u_{ps} = C_s \) and \( u_{ps} \geq 0 \) for all \( S \in x \) and all \( p \in S \). Then saying that an outcome is setwise-stable is equivalent to requiring that there are no coalition \( A \) and no feasible outcome \((u',x')\) such that for all \( p \in A \): (a) if \( p \in S \in x' \) then \( S \not\subseteq A \); and (b) \( \sum_{s \in x'} u'_{ps} \geq \sum_{s \in x} u_{ps} \), for all \( p \in A \).

The general model described above was not yet explored in the literature. The new idea is that a corewise stable outcome may be setwise-unstable.

When the set of players is formed by two disjoint and finite sets, and the players of one side form partnerships with players of the opposite side, we obtain a special game studied in Sotomayor (1992) and Sotomayor (1998). In this game the set of setwise-stable outcomes coincides with the set of pairwise-stable outcomes and may be a proper subset of the core.
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