Optimal job search in a changing world

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Abstract

This paper studies a job search model. Wage offers arrive in a Poisson fashion from a known distribution. Absent the option to quit, or in a stationary world, it is well-known that the reservation wage for an impatient worker is the flow return on the asset ‘being unemployed’. I assume: (1) the worker may hold a job at his leisure, quitting at will; and (2) the wage distribution evolves deterministically. I show that the reservation wage is the flow return on the ‘unemployment asset’ with an option to renew at the same rate. I consider the effect of this renewal option, and investigate the continuity properties of the reservation wage. © 1999 Elsevier Science B.V. All rights reserved.

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There is a time for everything, ... a time to search and a time to give up, a time to keep and a time to throw away. – Ecclesiastes 3:1,6

1. Introduction

That one must search for the lowest price, best paying job, or optimal mate is both economically realistic and theoretically compelling. For instance, a standard continuous time formulation of the job search model goes like this. An unemployed worker awaits job offers, whose arrival process is Poisson. Jobs differ only by wage, whose distribution is known; employment precludes further search. The cost of search is an explicit flow payment (perhaps for a referral service) plus a pure time cost: The impatient worker prefers to earn money sooner rather than later. He then will accept any job paying

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weakly more than a reservation wage, equal to his expected value from further search. Think of unemployment as a financial security, affording the right to search further, and secure any wage found. In a stationary world, it is well-known that the reservation wage is the net flow return to this asset.

In this decision theory paper, I venture that the known wage distribution evolves deterministically. While this assumption is best seen as a stepping stone to a more general analysis, it aptly captures the temporary employment dynamics of Smith (1995). In the standard steady-state theory, quits do not occur, for if a job is optimal to accept, it is optimal to retain forever. If the worker is indentured to his job, the above financial intuition obtains. Instead I assume that the worker may retain any position at his pleasure, quitting at will. Optimal exercise of this quit option adds a nice twist to the standard analysis. I consider the resulting link between the reservation wage and the unemployment value, underscoring that only a non-local relationship exists. I then observe that the worker’s reservation wage becomes the flow return on the unemployment ‘asset’ with an option to renew at the same rate.

The problem can be more generally cast as an individual searching for many-dimensional ‘prizes’, rather than jobs. While this does not enrich the above analysis, a continuity result is best understood in this setting. I show that the optimal acceptance set is a lower hemicontinuous correspondence of time. As a result, the reservation wage is necessarily an upper semi-continuous function of time. The paper concludes with a simple example of how some well-known comparative statics may fail in a nonstationary environment, and discuss some new ones that arise.

2. The model

Consider a continuous time setting, where an individual either has a job, or is searching for one while paying a flow search cost $c$. The income flow $I_t$ at any time $t$ is thus either $-c$ or some positive wage, generically denoted $w$. The word ‘cost’ is loosely applied here, since $c \geq 0$ or $c < 0$ is possible: In a macroeconomic context, $-c$ might well measure unemployment compensation. Even $c = 0$ is permissible, in which case there is a pure time cost of search, owing to worker impatience: He discounts future payoffs at the interest rate $r > 0$.

Job arrivals follow a Poisson process with parameter $\rho > 0$. Thus, an offer arrives with chance about $\rho dt$ in a small $dt$ time span. The distribution of wage offers that are tendered deterministically evolves with time, with the c.d.f. process $\{F_t(w)\mid t, w \geq 0\}$. For each $t$, the distribution $F_t$ has support on $[0, \infty)$ with bounded finite mean; moreover, $t \mapsto F_t(w)$ is a measurable function of time for every wage $w$. For instance, the wage distribution may be fixed, or shift finitely or countably many times, or perhaps continuously with the passage of time.

For simplicity, the individual cannot search while working, and so cannot hold more than one job at a time. This reflects the opportunity cost inherent in accepting a job—a tradeoff at the heart of wage search models. Otherwise, the problem is trivial: The first positive wage is accepted, and thereafter any better-paying offer. (More generally, a tradeoff arises so long as there is a lower flow arrival rate of offers while working than
searching.) Any job may be retained forever if desired, or freely dropped at will. Search costs are assumed low enough that one will never stop searching altogether, but not so low (and negative) as to render working unprofitable.

The individual chooses which wages to accept while searching, and ongoing jobs to quit, to maximize the expected present value of future wages less search costs borne. Since the costs and outside options of current and proposed jobs are identical, they must be treated symmetrically. Further, the deterministic evolution of \( F_t \) implies that an optimal strategy is open loop (known at time 0) and not in feedback form: It cannot depend on the current wage or wage history, for instance. Thus, the individual accepts or retains any wage \( w \) belonging to an acceptance set \( \mathcal{A}_t \).

3. The main results

3.1. Optimal values

Recall the time-\( t \) (average) value of being unemployed \( V_t \) given in Eq. (1). Let \( V_t(w) \) be the employed (average) value with wage \( w \) in hand. Given the time-path \( \{V_t\} \), it is immediate that \( V_t(w) = \max_{T \geq 1} \{w(1 - e^{-r(T - t)}) + e^{-r(T - t)}V_T\} \). Since one may always decline an offer (choose \( T = t \)), the inequality \( V_t(w) \geq V_t \) is necessarily always true. To ensure a standard closed acceptance set, I proceed indirectly. Call a wage \( w \) at time \( t \) strictly agreeable if \( V_t(w) > V_t \); this inequality obviously requires that the job optimally be retained for a positive time span \( T - t > 0 \). A wage is agreeable iff \( w = \lim_{n \to \infty} w_n \) for a sequence of strictly agreeable wages \( w_n \). The closed set \( \mathcal{A}_t \) of agreeable wage is then uniquely defined.

The conclusion afforded by dynamic programming, specialized to the job search context, is that the value can be described recursively:

**Lemma 1.** There exists a unique continuous solution \( v_t \) of the Bellman equation

\[
v_t = \int_0^\infty r e^{-r(s-t)} \left( -c(1 - e^{-r(T - t)}) + e^{-r(T - t)} \int_0^s v_s(w) dF_s(w) \right) ds
\]

where \( v_t(w) = \max_{T \geq 1} \{w(1 - e^{-r(T - t)}) + e^{-r(T - t)}v_T\} \). Moreover, the solution satisfies \( v_t = V_t \), and is continuous and a.e. differentiable.

The functional form of Eq. (2) fundamentally differs from that in Van Den Berg (1990), on account of my allowing jobs to be dropped. On the other hand, the proof in Van Den
Berg (1990) that a continuous value function uniquely exists, by means of a standard contraction-mapping argument, applies here with few changes,\footnote{His proof, for instance, assumed stationarity after some finite time \( \tau \); however, a continuous function on \([0,T]\) is uniformly continuous, and so a simple limiting exercise completes the argument.} and is thus omitted.

### 3.2. Optimal strategies

Since the employed value \( v_i(w) \) is continuous and increasing in any wage \( w \in \mathcal{A}_i \), we have the first standard result: An optimal strategy entails monotonic preferences.

**Lemma 2 (Monotonicity).** The optimal strategy has the form \( \mathcal{A}_i = [\theta_i, \infty) \), where \( \theta_i \) is the reservation wage—the (lower) threshold of agreeable wages.

Given the optimal reservation wage time-path \((\theta_t)\), we may then substitute out for the employed value \( v_i(w) \). At the first moment after \( t \) that the threshold \( \theta_t \) equals or exceed \( w \), namely \( T_t(w) = \inf\{s > t | \theta_s \geq w \} \), the individual quits a job paying \( w \). Since the value is scaled to the same units as the wage, this yields:

\[
v_i(w) = \begin{cases} 
w(1 - e^{-r(T_t(w) - t)}) + e^{-r(T_t(w) - t)}v_{T_t(w)} & \text{if } w \geq \theta_t \\
v_i & \text{else}
\end{cases}
\]  

(3)

Recalling Lemma 1, observe that \( T_t(w) = \max_{T \geq t}\{w(1 - e^{-r(T - t)}) + e^{-r(T - t)}v_T\} \).

I now partially characterize optimal behavior, and compute \( \langle \theta \rangle \) given \( \langle v \rangle \). There are many approaches to this problem. For instance, one could treat \( \langle \theta \rangle \) as an infinite-horizon control, and use the calculus of variations.\footnote{The analysis in Van Den Berg (1990) of a similar nonstationary job search problem curiously disallowed quitting. Thus, he did not have to derive separate laws of motion for \( \langle \theta \rangle \) and \( \langle v \rangle \).} But a less ambitious route is much more transparent. For ease of notation, define \( \theta^{-1}(t) = T_t(\theta_t) \), namely the next time that the reservation wage will be at least as high as its current level. This is the moment that a maximizing individual will drop a job paying \( \theta_t \).

**Theorem 1 (Reservation Wages).** For any real cost \( c \), and given time path of value functions \( \langle v \rangle \), the reservation wage satisfies \( \theta_t \leq v_t \) for all \( t \). Moreover, \( \theta_t = \lim_{s \rightarrow \theta^{-1}(t)}(v_t - s - r^{-1}(s - t)v_t)/(1 - e^{-r(s - t)}) \), or equivalently

\[
\theta_t = \begin{cases} 
v_t - v_i/r & \theta^{-1}(t) = t \\
v_t - e^{-r(\theta^{-1}(t) - t)}(v_{\theta^{-1}(t)} - \theta_t) & \theta^{-1}(t) > t \\
v_t & \theta^{-1}(t) = \infty
\end{cases}
\]

Proof. The inequality \( \theta_t \leq v_t \) follows from Eq. (3) and the identity \( v_t = v_i(\theta_t) \).

The result for \( \theta^{-1}(t) = \infty \) is essentially that of Van Den Berg (1990). Next, suppose \( \theta^{-1}(t) = T \in (t, \infty) \). Then optimal behavior demands that the job paying \( \theta_t \) be dropped at time \( T \). Hence, \( v_t = \theta_t + e^{-r(T - t)}[v_T - \theta_T] \), yielding the result for \( \theta^{-1}(t) > t \). Finally,
suppose that $\theta^{-1}(t) = t$. By monotonicity, if $w > \theta_1$ then it must be optimal to hold the job paying $w$ for some positive period of time, i.e. $v(t) < w + e^{-(s-s)}(v_t - w)$ for some $s > 0$; conversely, if $w < \theta_1$ then it is strictly better to stay unemployed, and the reverse inequality holds for all $s > 0$. Thus, $\theta_1 = \lim_{s \to \theta_1}(v_t - e^{-(s-s)}v_t)/r$. In that case, $\theta_1 = v_t - v_t/r$, by l'Hôpital's rule.

In words, when the reservation wage is rising, so that $\theta^{-1}(t) = t$, its relationship to the unmatched value is purely local (and has a natural financial interpretation, as we shall see); when it is weakly monotonically decreasing on $[t, \infty)$, then $\theta_1 = v_t$ for all $s \geq t$, again a purely local relationship arises. But when it is falling, and later to rise higher, global optimality comes into play.

### 3.3. Some financial intuition

Define the spot market wage system $\langle \psi_t \rangle$ implicitly through $v_t = \int_t^{\infty} r e^{-(s-t)} \psi_t \ ds$. Interpret $\psi_t$ as the reservation wage at time $t$ in a hypothetical spot market. An individual is always indifferent between remaining unemployed and accepting her spot market reservation wage for an arbitrarily short period of time, during which she cannot search. By the Fundamental Theorem of Calculus, we have the more explicit formula $\psi_t = v_t - v_t/r$. Rewriting this as $r(\psi_t/r) = v_t + v_t/r$ makes it clear that $\psi_t$ is flow value or the flow return and $v_t$ the associated instantaneous capital gain of the asset ‘being unemployed’, whose present value is $v_t/r$.

But Theorem 1 establishes that $\psi_t \neq \theta_1$ in general. It turns out that $\theta_1$ is the flow return with an option to renew at the same rate. Intuitively, this follows from the fact that an individual taking a job does so knowing that she may remain in it for as long as she wishes at the same wage. That is, at time $t_0$, she is indifferent about exchanging the variable return on the asset $\langle v_t, t \geq t_0 \rangle$ for the flow return $w = \theta_{t_0}$ that is fixed (for as long as she wishes). If the option is never exercised, i.e. $\theta^{-1}(t) = t$, then $\theta_1 = \psi_t$. Otherwise, the renewal option has value, and so $\theta_1 < \psi_t$. When it is forever exercised, in the case $\theta^{-1}(t) = \infty$, then the unemployed asset $v_t$ is effectively sold, and so $\theta_1 = \psi_t$. If the renewal option is not exercised at some future date, then no purely local relationship between $\theta_1$ and $v_t$ can exist.

### 3.4. A general continuity result

I temporarily abstract away from the model discussed above, in order to derive a somewhat more penetrating result. Treat the jobs as prizes $x$ providing flow utility $u(x)$ to the owner, where $x$ is a vector of prize attributes. For instance, in the job search problem, there may be a multidimensional job characteristics space, with wage not the only consideration. Modify the job search model to suppose that the individual is constrained in the number of prizes she can simultaneously enjoy. Suppose that there is given a Borel space of prizes $x \in \mathbb{R}^n$ whose distribution follows a known deterministic evolution. As above, one either possesses a prize or must search for one.

I turn to a simple property of the optimal acceptance sets $\langle x_t \rangle$ over time.
Theorem 2 (Characteristic Space). If the utility function $u$ is nonnegative, continuous, and satisfies non-satiation, then $\mathcal{A}$ is a lower hemicontinuous correspondence of time: $t_n \to t$ and $x \in \mathcal{A}_t$ implies $\exists x_n \to x$ with $x_n \in \mathcal{A}_{t_n}$ $\forall n$.

Proof. Without loss of generality assume $c=0$. Then $\mathcal{A} \neq \emptyset$, for the empty strategy is dominated by $\mathcal{A}_t = \mathbb{R}^n$, since $u \geq 0$ for all utilities.

Suppose that $x \in \mathcal{A}_t$. We must show that for all sequences of times $t_n \to t$, there is a sequence of prizes $x_n \to x$ such that $x_n \in \mathcal{A}_{t_n}$ for all large enough $n$. For definiteness, it suffices to argue that this limit separately holds for both increasing and decreasing sequences of times. Consider the first case, with $t_n \downarrow t$. If $v_i(x) = v_j$ even though $x \in \mathcal{A}_{t_n}$, then $x \in \mathcal{A}_{t_n}$ for all large enough $n$ by definition. So suppose that $v_i(x) > v_j$. I shall employ the sequence $x_n = x$. Due to the accounting identity Eq. (3), prize $x$ must be held until some time $T_i(u(x)) > t$ if $v_i(x) > v_j$. So $v_i$ and $v_j(x)$ are both right continuous in time, yielding $v_i(x) > v_j$ for $s$ close enough to $t$, $s > t$. Hence, $\mathcal{A}_{t_n}$ for all large enough $n$.

Next, consider the second case, with $t_n \downarrow t$. Here we must appeal to simple optimality considerations. Suppose that $v_i(x) > v_j$, and consider the sequence $x_n = x$. For $s < t$, let $\tilde{v}_i(x)$ be the not necessarily optimal average value of holding onto prize $x$ until time $t$ and then behaving optimally. Then $v_i(x) \geq \tilde{v}_i(x)$ for $s$ close to $t$, and so $x \in \mathcal{A}_{t_n}$ for all large enough $n$.

Finally, suppose that $v_i(x) = v_j$ even though $x \in \mathcal{A}_{t_n}$ (so that $x \in \mathcal{A}_{t_n}$ for $s > t$ close enough to $t$, by definition). Here we must appeal to the properties of $u$. By local nonsatiation, there exists a prize $y$ close to $x$ yielding utilities $u(y) > u(x)$. Write

$$v_{i_n}(y) - v_{i_n}(x) = [v_{i_n}(y) - \tilde{v}_{i_n}(y)] + [\tilde{v}_{i_n}(y) - \tilde{v}_{i_n}(x)] + [\tilde{v}_{i_n}(x) - v_{i_n}(x)]$$

Then the outer terms on the RHS must vanish as $n \to \infty$, while the inner one has a positive limit because both $x_n \in \mathcal{A}_{t_n}$ for $s > t$ close enough to $t$ (by assumption for $x$ and by monotonicity for $y$). Thus, for large enough $n$, we have $v_{i_n}(y) - v_{i_n}(x) > 0$. By continuity and local nonsatiation, there must exist prizes $x_n \to x$ yielding utilities $u(x_n) > u(x)$ and such that $v_{i_n}(x_n) - v_{i_n}(x) > 0$, i.e. $x_n \in \mathcal{A}_{t_n}$, as required. □

Corollary (Wage Space). In the job search problem, the reservation wage $\theta_0$ is an upper semicontinuous function of time.

This follows from Theorem 2 and the definition of what an u.s.c. function is; namely, that for all $\varepsilon > 0$, $\theta_0 > \theta_0 - \varepsilon$ on some small enough neighborhood of $t_0$. So while the average unmatched value $v_i$ is a continuous function of time, Theorem 1 illustrates how the reservation wage can discontinuously jump up when $v_i$ jumps.

Intuitively, the acceptance set may suddenly ‘implode’ but can never ‘explode’, and the reservation wage can ‘jump up’ but can never ‘jump down’.

4. An illustrative example

I conclude with an example showing how nonstationary analysis along with a quit option yields some nonstandard comparative statics insights (cf. Mortensen, 1986). Suppose, as in Fig. 1, that wages are uniform on (0,1) until time $\tau > 0$ and thereafter
uniform on \((0, ae^{-\gamma(t-\tau)})\), for some \(a>1\) and \(\gamma \geq 0\). This corresponds to a foreseen wage distribution 'spike' when \(\gamma > 0\), and a foreseen one-shot upward shift in the wage distribution when \(\gamma = 0\). Search costs are \(c \geq 0\).

**Case 1 Analysis.** \(t \geq \tau\)

Since the prospects are deteriorating over time after \(\tau\), we have \(v_i = \theta_i\), and a job once accepted will never be dropped. Thus, Eqs. (2) and (3) reduce to

\[
v_i = -c + \int_\tau^\infty p e^{-(r + \rho) (s-\tau)} \left( c + \frac{ae^{-\gamma(s-\tau)} - \theta_i}{ae^{-\gamma(s-\tau)}} \right) \frac{ae^{-\gamma(s-\tau)} + \theta_i}{2} + \frac{\theta_i}{ae^{-\gamma(s-\tau)}} ds
\]

Since Theorem 1 asserts that \(\theta_i = v_i\) for \(t \in [\tau, \infty)\), we have

\[
\theta_i = -c + \int_\tau^\infty p e^{-(r + \rho) (s-\tau)} \{c + (a^2 e^{-\gamma(s-\tau)} + \theta_i^2 e^{\gamma(s-\tau)})/2a\} ds
\]

Thus, so long as \(\theta_i > 0\), we have

\[
\dot{\theta}_i = - (\rho/2a) [a^2 e^{-\gamma(t-\tau)} + \theta_i^2 e^{\gamma(t-\tau)}] + (\rho + r) \theta_i + rc
\]

For reasons of tractability, I confine attention to two cases. First, if \(\gamma = 0\) and \(c \geq 0\), then the problem is stationary, and so optimally \(\dot{\theta}_i = 0\). Hence,

\[
\theta_i = \max \left( 0, a(\rho + r)/\rho - \sqrt{a^2 (\rho + r)^2 / \rho^2 - (a^2 - 2acr/\rho)} \right)
\]
for all $t \geq \tau$. Observe that $\theta > 0$ (and thus the equations are well-defined) precisely when $c < a \rho / 2 r$, i.e. when flow search costs are not too high.

Second, if $\gamma > 0$ and $c = 0$, then Eq. (4) admits the solution $\theta_1 = e^{-\gamma(t - \tau)} \theta_0$, where

$$\theta_0 = \max \left( \frac{a (\rho + r + \gamma)}{\rho - \sqrt{a^2 (\rho + r + \gamma)^2 / \rho^2 - a^2}} \right)$$

**Case 2:** Analysis: $t < \tau$.

If $\theta_0 \geq 1$, then the individual will drop any job agreed to in $[0, \tau)$; thus, she solves a de facto finite horizon problem on $[0, \tau)$, subject to $u \leq \theta_0$. Otherwise, if $\theta_0 < 1$, we can act as if the problem is finite horizon with a terminal wage $\max(0, w - \theta_0)$ if wage $w$ is held at time $\tau$. In general, the Bellman equation is

$$[1 - e^{-r(t-\tau)}] \theta_1 = \int_0^\tau pe^{-\theta(s-\tau)} \left( -c (1 - e^{-r(s-\tau)}) \right)$$

$$+ e^{-r(s-\tau)} (1 - e^{-r(s-\tau)}) (1 - \theta_0)(1 + \theta_0)/2 + \theta_0^2) ds$$

$$+ \max(0, e^{-r(t-\tau)} (1 - \theta_0)/2 \int_\tau^\tau \rho (1 - \theta_0)e^{-r(s-\tau)} ds)$$

(5)

For $\theta_0 \geq 1$, the last term is zero, and so

$$\theta_1 = \left( \frac{e^{-r(t-\tau)}}{1 - e^{-r(t-\tau)}} + \rho \right) \theta_0 - \rho (\theta_0^2 + 1)/2 + \frac{rc}{1 - e^{-r(t-\tau)}}$$

As might be guessed, this differential equation admits no neat closed form solution, but can be numerically evaluated. By the same token, only a numerical solution is available for $\theta_0 < 1$; however, we can apply l’Hôpital’s rule to Eq. (5), and deduce that $\theta_0 = \lim_{\tau \to \infty} \theta_0$, $\theta_1 = \rho (1 - \theta_0)^2 / 2 r > 0$. Thus, the greater is $\theta_0$, the lower is $\theta_1$.

(A) Improved Wages: FSD Shifts. Standard reservation wage monotonicity with respect to improvements with respect to first-order stochastic dominance (FSD) fails: Consider $\gamma = 0$. In that case, if the wage spike occurs at an earlier time $\tau' < \tau$, then $\theta$ shifts down on $[\tau' - \epsilon, \tau')$, for some $\epsilon > 0$. But it is still true that the unmatched value $\langle v \rangle$ uniformly increases on $[0, \tau)$, and in fact many stylized steady-state intuitions are best seen as applying to average rather than flow values.

(B) Increased Risk: SSD Shifts. While the example cannot illustrate this, it is also true that increases in risk (i.e. a mean-preserving spread) need not uniformly increase $\langle \theta \rangle$. This is intuitive if the increase is concentrated starting at some time.

(C) Policy Duration. Regarding the literature on permanent versus a transitory government policy shifts, a ‘more permanent’ change (smaller $\gamma$) cannot decrease and can increase reservation wage volatility around the regime shift. Indeed, if $\theta_0 < 1$, then a fall in $\gamma$ pushes up $\theta_1$, and hence down $\theta_2$.

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The non-zero maximand consists of the average terminal value if the wage that is accepted exceeds $\theta$, times (the integral) the chance that the first accepted wage exceeds $\theta$. 

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By extrapolation, if the wage distribution were cyclical, say uniform on \([0,1]\) at times \([0,1)\cup[2,3)\cup\ldots\) and uniform on \([0,a]\) at times \([1,2)\cup[3,4)\cup\ldots\), then an optimizing individual will also employ a cyclical reservation wage rule. This will intuitively lead to a stochastic number of voluntary unemployment spells and eventual ‘permanent’ position with a sufficiently high wage.

5. Conclusion

This paper aims for a richer decision theory foundation for equilibrium search theory. There are two modelling assumptions worth commenting on, and defending. First, I have assumed that it is the outside environment, and not the job characteristics, that is variable. For if one’s own wage were subject to change, then it might well be true that no current reservation wage rule alone could optimally serve as a quitting yardstick. In that case, or if one were slowly learning one’s preferences about the job, then there might exist an index (e.g. Gittins index) that would serve the same role as the wage in this paper. Second, there is no distributional uncertainty here: I assume that the wage distribution evolves deterministically. It is not clear how to relax this assumption, nor even of any equally compelling way to formalize a nonstationary but uncertain environment. Still, a proper resolution of that complication will hopefully build on the simple insights of this paper.

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