On comparing equilibrium and optimum payoffs in a class of discrete bimatrix games

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Abstract

In an \( m_1 \times m_2 \) bimatrix game, consider the case where payoffs to each player are randomly drawn without replacement, independently of payoffs to the other player, from the set of integers \( 1, 2, \ldots, m_1, m_2 \). Thus each player’s payoffs represent ordinal rankings without ties. In such ‘ordinal randomly selected’ games, assuming constraints on the relative sizes of \( m_1 \) and \( m_2 \) and ignoring any implications of mixed strategies, it is shown that payoffs to pure Nash equilibria (second-degree) stochastically dominate payoffs to pure Pareto optimal outcomes. Thus in such games where pure strategy sets do not differ much in size and payoffs conform with concave von Neumann-Morgenstern utility functions over ordinally ranked outcomes, players would prefer (ex ante) a ‘random pure strategy Nash equilibrium payoff’ to a ‘random pure Pareto optimal outcome payoff’.

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1. Introduction

Subject to important qualifications, average payoffs to pure Nash equilibria and average payoffs to pure Pareto optimal outcomes are compared in a reasonably large class of discrete bimatrix games. Perhaps surprisingly, despite the fact that noncooperative equilibria are often inefficient, we find that players would prefer (ex ante) a ‘random pure strategy Nash equilibrium payoff’ to a ‘random pure Pareto optimal outcome payoff’.
payoff’ in the class of games considered. (We use quotes here because we consider only one reasonable definition for these objects, while more than one is possible. We discuss this at some length in Section 2.)

One important qualification is the restriction to pure strategies. This qualification is not easily defended, though it may be somewhat less significant under the assumption we make below that players’ payoffs are independent of one another. In turn, this means we are largely concerned here with general-sum games, where mixed strategies may not be as crucial for the existence of equilibria as they are in constant-sum games. In any case, allowing mixed equilibria and optima probably leads to a much more difficult problem, not amenable to the simple combinatoric approach adopted here. While it might be possible to extend the analysis to mixed strategies and outcomes, those who believe strongly in the soundness of such strategies might best view these results as being merely suggestive.

Another qualification is the restriction to a class of games with a particular payoff structure: players have concave von Neumann-Morgenstern utility functions over ordinally ranked pure outcomes. This concavity assumption is maintained in all but the $2 \times 2$ case, where arbitrary monotonic vN-M utility functions are considered and first-degree dominance is proved. More generally, the results can probably be extended to cases with some convexity of utility, but it is not clear what the final form of those results would be.

In summary then, this note is intended to represent a small step in understanding some of the systematic relationships between various kinds of ‘optimality’ and solution concepts considered important in game theory and economics in general. For example, such relationships may be relevant in cooperative game theory, where solutions often choose Pareto optimal outcomes (subject to some form of individual rationality constraint) or maximize some welfare function, in preference to the alternative, which one might suppose to be a noncooperative equilibrium of the underlying game.

2. Distributions of equilibrium and optimum payoffs

We begin with a completely ordinal analysis. Let $m_1$ and $m_2$ be integers with $m_1 \geq 2$, $m_2 \geq 2$. In an ordinal randomly selected bimatrix game we have two payoff matrices $(A,B)$, where $A = (a_{ij})$ is $m_1 \times m_2$, $B = (b_{ij})$ is $m_1 \times m_2$ and the $a_{ij}$ (of which there are $M = m_1 m_2$) are the result of a random drawing from the set $S = \{1, 2, \ldots, M\}$ without replacement, as are the $b_{ij}$. Thus payoffs to each player form a discrete grid, that is, payoffs lie in a finite set of evenly spaced points. We adopt the convention that players prefer outcomes associated with higher numbers to those with lower numbers. Further, the payoffs to Player 1 (P1) are statistically independent of the payoffs to Player 2 (P2). These assumptions would hold if the variables $a_{ij}$ were drawn independently from a continuous cumulative distribution function $F_i$ and then only their order retained as information (and similarly for the $b_{ij}$). Equivalently, we may think of an ordinal randomly selected bimatrix game as an $m_1 \times m_2$ bimatrix selected according to the uniform probability over the $m_1 \times m_2$ bimatrices such that for each player, every payoff in $S$ appears exactly once.
We adopt the standard definition of pure strategy Nash equilibrium (abbreviation: PNE): the outcome \((i^*, j^*)\) is a PNE if \(a_{i^*j^*} \geq a_{ij}\) for \(i = 1, \ldots, m_1\) and \(b_{i^*j^*} \geq b_{ij}\) for \(j = 1, \ldots, m_2\). Also, we say the outcome \((i^*, j^*)\) is Pareto optimal (abbreviation: PO) if no other outcome \((i, j)\) is strictly preferred by both players.

The analysis is elementary. Our first task is to write down explicitly the probability function of the random variable \(X\) determined by P1 payoffs in a PNE \((i^*, j^*)\), with precise definition for \(X\) given below. To begin, we first define for each pair \((i, j)\) the random variable \(X(i, j)\) by:

\[
X(i, j) = \begin{cases} 
1 & \text{if } (i, j) \text{ is a PNE} \\
0 & \text{otherwise}
\end{cases}
\]

Our approach is to use the formula

\[
P(a_{i^*j^*} = k | X(i^*, j^*) = 1) = \frac{P(X(i^*, j^*) = 1 | a_{i^*j^*} = k)P(a_{i^*j^*} = k)}{P(X(i^*, j^*) = 1)}
\]

Now let \(\hat{X}\) represent the random variable with range \(S\) and probability function given by

\[
P(\hat{X} = k) = P(a_{i^*j^*} = k | X(i^*, j^*) = 1)
\]

Note that \(P(\hat{X} = k) = 0\) for \(k = 1, \ldots, m_1 - 1\).

We interject that, among other interpretations, we can think of \(\hat{X}\) as being determined by a process consisting of choosing a fixed pair \((i^*, j^*)\) and then repeatedly selecting a bimatrix according to the uniform probability over \(m_1 \times m_2\) bimatrices, stopping when \((i^*, j^*)\) is a PNE and observing its payoff to P1. We want to emphasize that \(\hat{X}\) is meant to represent selection of a ‘random PNE payoff’, but also that other reasonable interpretations of this phrase are possible. This fact becomes even more relevant in our discussion of ‘random pure PO payoffs’ below. Finally, and perhaps most simply, the probability function of \(\hat{X}\) is identical with that of the maximum number in a random sample of size \(m_1\) drawn without replacement from the set \(S\).

Returning now to the formula for \(P(a_{i^*j^*} = k | X(i^*, j^*) = 1)\), first, the unconditional probability \(P(a_{i^*j^*} = k) = 1/M\). Next, the two events \(\{a_{i^*j^*} = \max a_{ij}\}\) and \(\{b_{i^*j^*} = \max b_{ij}\}\) are independent since they involve disjoint sets of independent random variables. The first event has probability \(1/m_1\) and the second has probability \(1/m_2\). Thus \(P(X(i^*, j^*) = 1) = 1/M\) and so

\[
P(a_{i^*j^*} = k | X(i^*, j^*) = 1) = P(X(i^*, j^*) = 1 | a_{i^*j^*} = k)
\]

Now,

\[
P(X(i^*, j^*) = 1 | a_{i^*j^*} = k) = \frac{P(\{k = a_{i^*j^*} = \max a_{ij}\} \cap \{b_{i^*j^*} = \max b_{ij}\})}{P(a_{i^*j^*} = k)},
\]

and the events \(\{k = a_{i^*j^*} = \max a_{ij}\}\) and \(\{b_{i^*j^*} = \max b_{ij}\}\) are independent since they involve disjoint sets of independent random variables. It follows that

\[
P(X(i^*, j^*) = 1 | a_{i^*j^*} = k) = \frac{P(\max a_{ij} | a_{i^*j^*} = k)}{m_2}
\]
To evaluate the numerator in the RHS above, note that we are drawing a random sample of size \(m_1 - 1\) from a population of size \(M - 1\) without replacement, where there are \(M - k\) payoffs larger than \(k\) and \(k - 1\) payoffs smaller than \(k\). Thus we seek the probability of drawing all \(m_1 - 1\) members of the sample from the set of \(k - 1\) payoffs smaller than \(k\). This is just the corresponding hypergeometric probability

\[
\binom{k - 1}{m_1 - 1} / \binom{M - 1}{m_1 - 1}.
\]

Easy calculations now give

\[
P(a, i, j, = k|X(i*, j*) = 1) = \binom{k - 1}{m_1 - 1} / \binom{M}{m_1}
\]

for \(k = m_1, m_1 + 1, \ldots, M\). At this point it may be instructive to note that calculations also give

\[
E(\hat{X}) = \frac{m_1(M + 1)}{m_1 + 1}
\]

Thus in percentage terms and for large \(m_1\), expected PNE payoffs are quite near the maximum possible payoff in the ordinal case.

Next, we want to formulate the probability function of \(\hat{Y}\), where \(\hat{Y}\) is the random variable defined by P1 payoffs in a Pareto optimal outcome \((i^*, j^*)\), with precise definition for \(\hat{Y}\) given below. Proceeding as before, define for each pair \((i, j)\) the random variable \(Y(i, j)\) by

\[
Y(i, j) = \begin{cases} 
1 & \text{if } (i, j) \text{ is a PO outcome} \\
0 & \text{otherwise}
\end{cases}
\]

Now fix \(k \in S\). Again, we use the formula

\[
P(a, i, j, = k|Y(i^*, j^*) = 1) = \frac{P(Y(i^*, j^*) = 1|a, i, j, = k)P(a, i, j, = k)}{P(Y(i^*, j^*) = 1)}
\]

Let \(\hat{Y}\) represent the random variable with range \(S\) and probability function given by

\[
P(\hat{Y} = k) = P(a, i, j, = k|Y(i^*, j^*) = 1)
\]

Note that \(P(\hat{Y} = k) > 0\) for \(k = 1, \ldots, M\).

The variable \(\hat{Y}\) has an interpretation analogous to that of \(\hat{X}\) as discussed above, with the bimatrix selection process stopping when \((i^*, j^*)\) is a PO outcome. Again, \(\hat{Y}\) is meant to represent the selection of ‘random pure PO payoffs’, but other reasonable interpretations of this phrase are possible. For example, suppose instead of fixing \((i^*, j^*)\) and randomly selecting the bimatrix, we simply choose with uniform probability a PO

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\(^1\)It follows from independence that the distribution of payoffs in pure noncooperative equilibria of the corresponding Stackelberg game (which always exist) are also given as in Eq. (1) above for both leader and follower. In particular, there is no (ex ante) advantage or disadvantage to being either the Stackelberg leader or follower if there is no predictable relationship between P1 and P2 payoffs.
outcome from each bimatrix and then choose with uniform probability one of these, observing its payoff to P1. With this approach we can obtain a very different probability function for pure PO payoffs from the one considered here. Indeed, it is not clear if our results on stochastic dominance, given below, would survive this change in any meaningful way. In any case, we will not try to argue that our definition of ‘random pure PO payoffs’ (or ‘random PNE payoffs’) is ‘correct’, but only that it is interesting and reasonable to study as a first step. We also think the story presented here is quite incomplete.

Returning to the formula for \( P(a_{i,j} = k | Y(i^*, j^*) = 1) \), first, the unconditional probability \( P(a_{i,j} = k) = 1/M \). Next, given \( a_{i,j} = k \), we see that \((i^*, j^*)\) is a PO outcome if and only if for \( l = 1, \ldots, M - k \) the payoff pairs \((a_{ij} = k + l, b_{ij})\) satisfy \( b_{ij} < b_{i,j^*} \). So if \( b_{1,j^*} = n > M - k \), there are \( n - 1 \) payoffs \( b_{ij} < b_{i,j^*} \) and these may be chosen in \( \binom{n - 1}{M - k} \) ways. Summing over such \( n \) and using Total Probabilities, we have

\[
P(Y(j^*, j^*) = 1 | a_{i,j} = k) = \frac{1}{M} \sum_{n=M-k+1}^{M} \binom{n - 1}{M - k} / \binom{M - 1}{M - k}
\]

and calculations give

\[
P(Y(i^*, j^*) = 1 | a_{i,j} = k) = \frac{1}{M - k + 1}.
\]

Using this and Total Probabilities, it follows that

\[
P(Y(i^*, j^*) = 1) = \frac{1}{M} \sum_{n=1}^{M} 1/n.
\]

(See O’Neill (1981) for a different derivation of \( P(Y(i^*, j^*) = 1) \).) Finally, putting these facts together in Eq. (3), we have

\[
P(a_{i,j} = k | Y(i^*, j^*) = 1) = \left( \sum_{n=1}^{M} 1/n \right)^{-1} \frac{1}{M - k + 1}
\]

For comparison with Eq. (2), note that

\[
E(\hat{Y}) = \left( \sum_{n=1}^{M} 1/n \right)^{-1} \sum_{k=1}^{M} \frac{k}{M - k + 1}
\]

which (for large \( M \)) we approximate by

\[
E(\hat{Y}) \approx (\ln M)^{-1} \int_{1}^{M} \frac{y}{M - y + 1} \, dy \approx \frac{(\ln M) M}{\ln M}
\]

Again, in percentage terms and for large \( M \), expected PO payoffs are near the maximum possible payoff in the ordinal case. However, for large \( m_1 \) (and hence \( M \)) and moderate values of \( m_2 \), and given this estimate of \( E(\hat{Y}) \), we see that \( E(\hat{X}) \) is much larger than
and it seems plausible that a stochastic dominance relationship will exist between $\hat{X}$ and $\hat{Y}$. We investigate this in some detail in the next section. (In fact, see Theorem 2 below for a sufficient condition on the relative sizes of $m_1$ and $m_2$ that guarantees $E(\hat{X}) > E(\hat{Y})$.)

### 3. Stochastic dominance

We apply the standard notions of (first- and second-degree) stochastic dominance to the discrete case at hand. Thus given any two real valued random variables $X$ and $Y$, with the respective known cumulative probability distribution functions $F(x)$ and $G(y)$, we say that $\hat{X}$ first-degree dominates $\hat{Y}$, and write $\hat{X} \succ_{1} \hat{Y}$, if

$$E_{\mu}(U(x)) \geq E_{\nu}(U(y))$$

for every non-decreasing utility function $U$, and

$$E_{\mu}(U(x)) > E_{\nu}(U(y))$$

for some such $U$. Further, we say that $\hat{X}$ second-degree dominates $\hat{Y}$, and write $\hat{X} \succ_{2} \hat{Y}$ if Eq. (5) holds for every non-decreasing concave utility function $U$ and Eq. (6) holds for some such $U$; i.e. when

$$U(a x_1 + (1 - a)x_2) \geq a U(x_1) + (1 - a)U(x_2)$$

for any pair $x_1,x_2$ and positive fraction $a$. In our case, where $\hat{X}$ and $\hat{Y}$ are given as in Section 2 (with $m_1$, $m_2$ fixed), results from the literature imply:

$$\hat{X} \succ_{1} \hat{Y} \Leftrightarrow \sum_{k=1}^{M_1} P(a_{r,s} = k|Y(i^*, j^*) = 1) \geq \sum_{k=1}^{M_1} P(a_{r,s} = k|X(i^*, j^*) = 1)$$

for $M_1 = m_1, m_1 + 1, \ldots, M$. Also,

$$\hat{X} \succ_{2} \hat{Y} \Leftrightarrow \sum_{N=1}^{M_1} \sum_{k=1}^{N} P(a_{r,s} = k|Y(i^*, j^*) = 1) \geq \sum_{N=1}^{M_1} \sum_{k=1}^{N} P(a_{r,s} = k|X(i^*, j^*) = 1)$$

for $M_1 = m_1, m_1 + 1, \ldots, M$. (See, for example, Hanoch and Levy (1969) or, more directly for our discrete grid case, Fishburn and Lavalle (1995).) Note that first-degree dominance concerns 'utility functions' satisfying $U(1) \preceq U(2) \preceq \cdots \preceq U(M)$, while second-degree stochastic dominance concerns such functions with $U(2) - U(1) \succeq U(3) - U(2) \succeq \cdots \succeq U(M) - U(M - 1)$. Regarding first-degree dominance, for square games we have

**Theorem 1.** If $m_1 = m_2 = 2$, then $\hat{X} \succ_{1} \hat{Y}$. If $m_1 = m_2 > 2$, then $\hat{X} \nprec_{1} \hat{Y}$. 
Proof. Using Eq. (1) and Eq. (4), the inequalities in Eq. (7) are

$$\left(\sum_{n=1}^{M} \frac{1}{n}\right)^{-1} \sum_{k=1}^{M_1} \frac{1}{M - k + 1} \geq \frac{M_1!}{(M - m_1)!} \frac{(M - m_1)!}{M!}$$

(9)

for $M_i = m_1, m_1 + 1, \ldots, M$. If $m_1 = m_2 = 2$, these are easily established by calculations. If $m_1 = m_2 > 2$, let $M_1 = M - 1$. Then the LHS of Eq. (9) is $1 - \left(\sum_{n=1}^{M} \frac{1}{n}\right)^{-1}$ and the RHS is $1 - \frac{1}{M_1}$. Since $m_1 > \sum_{n=1}^{M_1} \frac{1}{n}$ if and only if $m_1 > 2$, Eq. (9) is violated for this value of $M_1$ and this completes the proof.

We regard Theorem 1 as giving sufficient reason to abandon further study of first-degree dominance and proceed to the second-degree case. For this, using Eq. (1) and Eq. (4), we note that the inequalities in Eq. (8) are

$$\left(\sum_{n=1}^{M} \frac{1}{n}\right)^{-1} \sum_{n=0}^{M_1 - n} \frac{M_1 - n}{M - n} \geq \frac{M_1 + 1}{m_1 + 1} \prod_{n=0}^{m_1 - 1} \frac{M_1 - n}{M - n}$$

(10)

for $M_i = m_1, m_1 + 1, \ldots, M$. We remark that in trying to establish Eq. (10), despite the mitigating leading factors on both sides, one might hope to make creative use of known results such as the arithmetic–geometric mean inequality, for example. Unfortunately, our efforts in this direction have not been successful, and so we must be content with our own devices. Thus our results will take the form of loose sufficient conditions on the relative sizes of $m_1$ and $m_2$ guaranteeing that the inequalities in Eq. (10) will hold. Our basic result is given in the following theorem, from which two corollaries follow easily.

**Theorem 2.** If $m_1 + 1 > 2 \sum_{n=1}^{M_1} \frac{1}{n}$, then $\hat{X} > \hat{Y}$.

Proof. The terms $\frac{M_1 - n}{M - n}$ are all between 0 and 1 for $M_i = m_1, \ldots, M$. It follows that

$$\left(\sum_{n=1}^{M} \frac{1}{n}\right)^{-1} \sum_{n=0}^{M_1 - n} \frac{M_1 - n}{M - n} \geq \left(\sum_{n=1}^{M} \frac{1}{n}\right)^{-1} \frac{1}{M} \sum_{n=0}^{M_1 - 1} (M_1 - n)$$

$$= \left(\sum_{n=1}^{M} \frac{1}{n}\right)^{-1} \frac{1}{M} \frac{(M_1 + 1)M_1}{2}$$

$$\geq \frac{(M_1 + 1)M_1}{(m_1 + 1)M}$$

$$\geq \frac{M_1 + 1}{m_1 + 1} \prod_{n=0}^{M_1 - 1} \frac{M_1 - n}{M - n}$$

for $M_i = m_1, \ldots, M$. This completes the proof.

In the square matrix case, we have

**Corollary 1.** If $m_1 = m_2 = 2$, then $\hat{X} > \hat{Y}$.
Proof. For \( m_1 = m_2 < 9 \), the inequalities in Eq. (10) can be established with a programmable calculator or equivalent. (Such case-by-case verification is certainly inelegant, but there seems to be no obvious alternative.)

If \( m_1 = m_2 \geq 9 \), we have \( m_1 + 1 > 2 \sum_{n=1}^{m_1^2} 1/n \) and Theorem 2 applies. This completes the proof.

As a second special case, we have

Corollary 2. If \( m_1 = m_2 \geq 9 \), then \( \hat{X} > \hat{Y} \).

Proof. Note that for \( m_1 = m_2 \geq 9 \), \( m_1 + 1 > 2 \sum_{n=1}^{m_1^2} 1/n \geq 2 \sum_{n=1}^{m_1 m_2} 1/n \), and Theorem 2 applies again. This completes the proof.

Returning to Theorem 2, we note that the inequality given there really places an upper bound on \( m_1 \) (depending on \( m_1 \)) that guarantees \( \hat{X} > \hat{Y} \). Perhaps this is best seen by considering a continuous approximation to the inequality

\[
m_1 + 1 > 2 \sum_{n=1}^{m_1 m_2} 1/n \tag{11}
\]

If \( C \approx 0.5772 \) is Euler’s constant, then we approximate \( \sum_{n=1}^{m_1 m_2} 1/n \) by \( \ln m_1 m_2 + C \) and Eq. (11) is replaced (after elementary manipulations) by

\[
m_2 < (e^{0.5 - C} \frac{e^{m_1/2}}{m_1}) \approx (0.9257) \frac{e^{m_1/2}}{m_1} \tag{12}
\]

The RHS of Eq. (12) grows rapidly for \( m_1 \geq 9 \), with approximate values 9.3, 13.7, 20.6 and 31.1 for \( m_1 = 9 \), 10, 11 and 12, respectively. On the other hand, for \( m_1 = 2, \ldots, 8 \), the RHS of Eq. (12) is less than \( m_1 \), and so Eq. (12) is not very powerful in the small-numbers cases. (This is why case-by-case verification was needed in the proof of Corollary 1.) In summary then, Theorem 2 essentially says that for \( m_1 \geq 9 \), if the cardinality of P1’s pure strategy set is not much smaller than that of P2, then \( \hat{X} > \hat{Y} \).

To illustrate the gross character of Theorem 2 for small-numbers, note that verification also shows \( \hat{X} > \hat{Y} \) if \( m_1 = 3 \) and \( m_2 = 8 \), even though Eq. (11) is violated, while \( \hat{X} > \hat{Y} \) if \( m_1 = 3 \) and \( m_2 = 9 \). Tighter sufficient conditions (particularly in small-numbers cases) for \( \hat{X} > \hat{Y} \) would certainly be desirable, but the arithmetic of these cases seems untidy.

References