Stable effectivity functions and perfect graphs

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Abstract

We consider the problem of characterizing the stability of effectivity functions (EFF), via a correspondence between game theoretic and well-known combinatorial concepts. To every EFF we assign a pair of hypergraphs, representing clique covers of two associated graphs, and obtain some necessary and some sufficient conditions for the stability of EFFs in terms of graph-properties. These conditions imply, for example, that to check the stability of an EFF is an NP-complete problem. We also translate some well-known conjectures of graph theory into game theoretic language and vice versa. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let us consider multiplayer games, in which \( I \) and \( A \) denote finite sets of players and outcomes, respectively. Subsets of players are called coalitions, while subsets of outcomes are called blocks. An effectivity function (or EFF in short) is a mapping \( \varepsilon : 2^I \times 2^A \mapsto \{0, 1\} \) representing the ‘power’ of the players, in very general terms. Namely, \( \varepsilon(K, B) = 1 \) for \( K \subseteq I \) and \( B \subseteq A \), i.e. \( K \) is said to be effective for \( B \), exactly when the coalition \( K \) is able to guarantee that an outcome from the block \( B \) will be realized, see, for example, Moulin (1983), Peleg (1984). (Obviously, the players of a
coalition $K$ may choose to cooperate in many different ways, hence the same coalition
may be effective for many different blocks of the outcomes.)

We shall consider monotone effectivity functions, i.e. for which $\mathcal{E}(K', B') \geq \mathcal{E}(K, B)$
holds for all $K' \supseteq K$ and $B' \supseteq B$. It is quite natural to assume both that a larger set of
players have a greater influence, and that it is easier to guarantee the final outcome to fall into a larger set. Hence monotonicity is a natural assumption for effectivity functions corresponding to games.

A simple example could be a voting game, i.e. in which players are voters, and outcomes are candidates. In this case the effectivity function describes the power of certain groups of voters (coalitions) to eliminate some of the candidates from the final race.

Another example is given by game forms $(I, A, X, g)$, where $X = \times_{i \in I} X_i$, and the
(finite) set $X_i$ represents the possible strategies of the player $i \in I$, and where the mapping $g:X \rightarrow A$ defines the game form, i.e. determines the outcome for each situation in which all the players have chosen a strategy. A coalition $K \subseteq I$ is effective for a block $B \subseteq A$ whenever the players of $K$ can choose a strategy vector $x_K \in \times_{i \in K} X_i$ such that the restriction $g(x_K, \#)$ takes values only from $B$, i.e. if the outcome of the game falls into the set $B$ no matter what strategy the players outside of $K$ follow.

Not all effectivity functions correspond, of course, to game forms. Besides the
monotonicity, there are other (fairly natural) properties satisfied by effectivity functions of game forms. One of them, the so called superadditivity states that if two disjoint coalitions $K_1$ and $K_2$ are, respectively, effective for the blocks $B_1$ and $B_2$, then, since the players of both coalitions can independently practice their power, the union coalition $K_1 \cup K_2$ must also be effective for the intersection $B_1 \cap B_2$. Moulin and Peleg (1982) proved that monotonicity and superadditivity together with some boundary conditions characterize the effectivity functions corresponding to game forms.

The individual preferences of the players over the different outcomes are represented by a utility function, i.e. by a mapping of the form $u: I \times A \rightarrow \mathbb{R}$, where the real value $u(i, a)$ for a player $i \in I$ and outcome $a \in A$ is interpreted as the profit of player $i$ in the case of outcome $a$ is realized. For a coalition $K \subseteq I$ and outcome $a \in A$ let $PR(K, a, u) = \{a' \in A | u(i, a') > u(i, a) \text{ for all players } i \in I\}$ denote the set of outcomes preferred over $a$ unanimously by all players of $K$. We shall say that a coalition $K$ can reject an outcome $a$ if $\mathcal{E}(K, PR(K, a, u)) = 1$, i.e. if the players in $K$ have the power guaranteeing that the outcome of the game will be preferred by all of them over $a$.

For a given effectivity function $\mathcal{E}$ and utility $u$, the core $\mathcal{C}(\mathcal{E}, u)$ is defined as the set of outcomes not rejected by any of the coalitions. One of the central problems of game theory is to find conditions guaranteeing that the core $\mathcal{C}(\mathcal{E}, u)$ is not empty. A somewhat surprising fact is that the non-emptiness of $\mathcal{C}(\mathcal{E}, u)$ can be guaranteed in certain cases, even if the utility function is not specified. In other words, one can find conditions based only on the structure of the effectivity function, which imply that $\mathcal{C}(\mathcal{E}, u) \neq \emptyset$ for all utility functions $u$. Effectivity functions for which this happens are called stable.

In this paper we study the stability of effectivity functions, present some necessary and some sufficient conditions for stability, and show, among other results that testing the stability of a given effectivity function is an NP-complete problem.
2. Results

The prime tools in our analysis are combinatorial. We show that an effectivity function $\mathcal{E}$ can equivalently be represented by a pair of hypergraphs, forming clique covers of two graphs on the same set of vertices. We shall start with the interesting special case when the two graphs are complementary to each other. (We follow standard graph theoretical notations, some of them, for completeness, included in Appendix A.)

Let us consider a graph $G = (V, E)$, and let us assign a player to every maximal clique and an outcome to every maximal stable set of $G$. Let us denote by $I$ the set of all the players (maximal cliques) and by $A$ the set of all the outcomes (maximal stable sets). For every vertex $v \in V$ let us denote by $K_v$ the family of maximal cliques containing vertex $v$, and similarly, let $B_v$ denote the family of maximal stable sets containing vertex $v$. Finally, let us associate to the graph $G$ an effectivity function $\mathcal{E}_G$ by defining $\mathcal{E}_G(K, B) = 1$ for a coalition $K \subseteq I$ and block $B \subseteq A$ if and only if $K_v \subseteq K$ and $B_v \subseteq B$ for some vertex $v \in V$ (and defining $\mathcal{E}_G(K, B) = 0$ otherwise).

This correspondence was introduced and several results were shown by Boros and Gurvich (1994, 1995), see also Boros et al. (1995). In order to state the new results, we shall recall first some of the properties shown earlier. (For the precise definitions see Appendix A and Sections 3–4, and for more information on perfect and on kernel-solvable graphs see Appendix A and also Berge (1970, 1975), Berge and Duchet (1983, 1987), Blidia et al. (1988) and Maffray (1988)).

**Proposition 1.** A graph $G$ is perfect if and only if the corresponding EFF $\mathcal{E}_G$ is balanced.

This follows from a characterization of perfect graphs given by Lovasz (1972a, Theorem 2).

**Proposition 2.** A graph $G$ is kernel-solvable if and only if the corresponding EFF $\mathcal{E}_G$ is stable.

This property is in fact a direct consequence of the criterion of stability stated by Keiding (1985), (see Section 3).

Scarf (1967) proved that balanced NTU-games have non-empty cores, and later Danilov and Sotskov (1987, 1991, 1992) reformulated this result in terms of EFFs, see also Gurvich (1997).

**Proposition 3.** Balanced EFFs are stable.

As a consequence of all above one can arrive to the following theorem.

**Theorem 4.** Perfect graphs are kernel-solvable.
This statement was conjectured by Berge and Duchet (1983). The converse direction, namely that

**Conjecture 5.** Kernel-solvable graphs are perfect

was also conjectured in the same paper, and it is still an open problem. Berge and Duchet (1983) observed that Conjecture 5 would result from the Strong Perfect Graph Conjecture. Boros and Gurvich (1994) proved that Conjecture 5 is equivalent to either of the following conjectures.

**Conjecture 6.** If a graph \( G \) is kernel-solvable then its complementary graph \( G^c \) is kernel-solvable, too.

**Conjecture 7.** If a graph \( G \) is kernel-solvable and a vertex of \( G \) is substituted by an edge (or by a clique) then the resulting graph \( G' \) is kernel-solvable, too.

Thus, assuming that the above conjectures hold true, for EFFs corresponding to graphs a reasonably simple criterion of stability exists, stated in Proposition 2. However, only some very special EFFs are generated by graphs in the above way.

Let us consider in the sequel the general case, and let us represent an arbitrary EFF \( \mathcal{E} \) by specifying explicitly the list of coalition-block pairs for which \( \mathcal{E} \) takes value 1. More precisely, let us consider \( \mathcal{E} = (K_i, B_j; J, I, A) \), where \( I \) denotes the set of players, and \( A \) stands for the set of outcomes, as above, and \( J \) denotes the set of indices of the pairs \( K_i \subseteq I, B_j \subseteq A \) for which \( \mathcal{E}(K_i, B_j) = 1 \), i.e. for which coalition \( K_i \) is effective for block \( B_j \). We can interpret the EFF \( \mathcal{E} \) as a pair of hypergraphs \( \mathcal{H} \) and \( \mathcal{R} \) on the sets of vertices \( I \) and \( A \), respectively, the edges of which have a common set of indices \( J \). Let us consider the dual (transposed) hypergraphs \( \mathcal{H}^d \) and \( \mathcal{R}^d \) on the common vertex set \( J \) defined by

\[
\mathcal{E} = \{C_j \subseteq J | j \in I\}, \quad \mathcal{F} = \{S_a \subseteq J | a \in A\}
\]

where \( C_j = \{j \in J | K_j \ni i\} \), and \( S_a = \{j \in J | B_j \ni a\} \).

An EFF \( \mathcal{E} \) can equivalently be specified by the hypergraphs \( \mathcal{E} \), and \( \mathcal{F} \), as well. In the sequel, we shall use both notations \( \mathcal{E} = (C_j, S_a; I, A; J) \) and \( \mathcal{E} = (K_i, B_j; J, I, A) \).

Given an EFF \( \mathcal{E} \), Peleg (1984) introduced the dual EFF \( \mathcal{E}^d \) by exchanging the roles of players and outcomes, i.e. \( \mathcal{E}^d = (S_a, C_j, A, I, J) = (B_j, K_i, J, A, I) \) using the notations above.

Let us observe that in the special case when an EFF \( \mathcal{E}_G \) is generated by a graph \( G = (V, E) \), we have \( J = V \) and the hyperedges of \( \mathcal{E} \) and \( \mathcal{F} \) are, respectively, the maximal cliques and maximal stable sets of \( G \). Hence we have the relation

\[
\mathcal{E}^d_G = \mathcal{E}_G^c,
\]

where \( G^c \) denotes the complementary graph to \( G \).

We can now state some necessary and some sufficient conditions for the stability of an arbitrary EFF \( \mathcal{E} \) in terms of the corresponding hypergraphs \( \mathcal{E} \) and \( \mathcal{F} \).
Theorem 8. If an EFF \( \mathcal{E} = (C_i, S_a; I, A; J) \) or its dual \( \mathcal{E}^d \) is stable then,

(i) for every subset \( J' \subseteq J \) such that \( |C_i \cap J'| \leq 1 \) for all \( i \in I \) there exists an \( a \in A \) such that \( J' \subseteq S_a \), and

(ii) for every \( J'' \subseteq J \) such that \( |S_a \cap J''| \leq 1 \) for all \( a \in A \) there exists an \( i \in I \) such that \( J'' \subseteq C_i \).

A simple consequence of this statement is the following corollary.

Corollary 9. If an EFF \( \mathcal{E} = (C_i, S_a; I, A; J) \) is stable then every pair \( j, j' \) must belong either to \( C_i \) for some \( i \in I \) or to \( S_a \) for some \( a \in A \) (or both). In other words, if \( \mathcal{E}(K_1, B_1) = \mathcal{E}(K_2, B_2) = 1 \) and \( K_1 \cap K_2 = B_1 \cap B_2 = \emptyset \) then the EFF \( \mathcal{E} \) cannot be stable.

Let us remark that the necessary conditions of stability (i) and (ii) of Theorem 8 can be reformulated in dual terms, namely if \( \mathcal{E} \) or \( \mathcal{E}^d \) are stable, and for some \( J' \subseteq J \) we have

(iii) \( \mathcal{E}^d(K_j, B_j) = 1 \) for all \( j \in J' \) and \( K_j \cap K_{j'} = \emptyset \) for all \( j \neq j' \in J' \), then \( \cap_{j \in J'} B_j \neq \emptyset \) must hold; and similarly

(iv) \( \mathcal{E}^d(K_j, B_j) = 1 \) for all \( j \in J' \) and \( B_j \cap B_{j'} = \emptyset \) for all \( j \neq j' \in J' \) must imply that \( \cap_{j \in J'} C_j \neq \emptyset \).

Remark 10. Gurvich and Vasin (1978) proved that condition (i) holds if the EFF \( \mathcal{E} \) is playing-minor (i.e. if \( \mathcal{E} \preceq \mathcal{E}_g \) holds for some EFF \( \mathcal{E}_g \) corresponding to a normal game form g). Later, Boros and Gurvich (1994) gave a shorter proof based on a theorem by Moulin and Peleg (1982) characterizing EFFs generated by game forms (as monotone and superadditive). Respectively, we can see that condition (iv) holds if the dual EFF \( \mathcal{E}^d \) is playing-minor. Thus, playing-minority of both EFFs \( \mathcal{E} \) and \( \mathcal{E}^d \) are necessary for the stability of either one of \( \mathcal{E} \) or \( \mathcal{E}^d \).

The main contribution of this paper is the following ‘perfect split criterion’.

Theorem 11. Both EFFs \( \mathcal{E} = (C_i, S_a; I, A; J) \) and its dual \( \mathcal{E}^d \) are stable if there exists a graph \( G = (J, E) \) such that

(i) \( G \) is perfect;

(ii) every (maximal) clique of \( G \) is a subset of some \( C_i, i \in I \);

(ii') every (maximal) stable set of \( G \) is a subset of some \( S_a, a \in A \).

From Theorems 8 and 11 we will be able to derive that

Theorem 12. To check the stability of an EFF, given as \( \mathcal{E} = (C_i, S_a; I, A; J) \) is an NP-complete problem.
Using Theorems 8 and 11, we can extend the list of equivalent Conjectures 5–7 on kernel-solvability and translate them into game theoretic language.

**Conjecture 13.** If an EFF $\mathcal{E} = (K, B; J, I, A)$ is stable while its dual $\mathcal{E}^d$ not, then there exist coalitions $K_1$, $K_2$ and blocks $B_1$ and $B_2$ such that $\mathcal{E}(K_1, B_1) = \mathcal{E}(K_2, B_2) = 1$, $K_1 \cap K_2 \neq \emptyset$, and $B_1 \cap B_2 \neq \emptyset$ hold.

In dual terms we can state this conjecture equivalently as

**Conjecture 14.** If an EFF $\mathcal{E} = (C, S; I, A; J)$ is stable, and its dual $\mathcal{E}^d$ not, then there must exist a player $i \in I$ and an outcome $a \in A$ such that $|C_i \cap S_i| \geq 2$.

In particular, we conjecture that such an EFF cannot be generated by a graph. The Strong Perfect Graph Conjecture (SPGC) can also be reformulated in terms of EFFs.

**Conjecture 15.** If an EFF $\mathcal{E}$, corresponding to a graph $G$ is not stable then the graph $G$ has an odd hole or an odd antihole.

In Section 5 we will prove that Conjectures 13 and 14 are equivalent to Conjecture 5 and that Conjecture 15 is equivalent to SPGC.

3. Keiding’s theorem and its dual

The following simple criterion of stability was obtained by Keiding (1985). Given an EFF $\mathcal{E} = (K, B; J, I, A)$ and a utility function $u : I \times A \to \mathbb{R}$, let us denote by $R_j$ for $j \in J$ a subset of the outcomes (not necessarily all of them) which are strictly worse than any one from $B_j$ for all the players of the coalition $K_j$. In other words, $R_j \subseteq \{a \in A | u(i, a) < u(i, b) \text{ for all } b \in B_j \text{ and } i \in K_j\}$ (R)

By this definition, $B_j \cap R_j = \emptyset$ for every $j \in J$, and hence the coalition $K_j$ being effective for $B_j$, can reject all the outcomes in $R_j$. Consequently, these outcomes cannot belong to the core $C(\mathcal{E}, u)$.

Let us consider now an extended list $T = \{(K_j, B_j, R_j) | j \in J\}$ such that $K_j$ is effective for $B_j$ and rejects $R_j$ for every $j \in J$. The core $C(\mathcal{E}, u)$ is empty if all the outcomes are rejected, i.e. $\cup_{j \in J} R_j = A$. In this case we shall call the list $T = \{(K_j, B_j, R_j) | j \in J\}$ a rejecting table for the pair $(\mathcal{E}, u)$. Let us denote the family of such rejecting tables by $RT(\mathcal{E}, u)$.

Let us consider next the case when we are given only an EFF $\mathcal{E}$ but no utility $u$ is specified. We shall call a list of the form $T = \{(K_j, B_j, R_j) | j \in J\}$ a rejecting table of the EFF $\mathcal{E}$, if $\mathcal{E}(K_j, B_j) = 1$ for every $j \in J$ and $\cup_{j \in J} R_j = A$. Let us denote the family of such lists by $RT(\mathcal{E})$. Let us call a rejecting table $T \in RT(\mathcal{E})$ utilizable if $T \in RT(\mathcal{E}, u)$ for some utility function $u$. By definition of stability, an EFF $\mathcal{E}$ is not stable iff it has a utilizable rejecting table $T \in RT(\mathcal{E})$. 


To have a combinatorial characterization of utilizability of rejecting tables, we shall need the following definitions. Let us call a subset $J_c = \{j_1, \ldots, j_r\} \subseteq J$, $r \geq 2$ of the rows of a rejecting table $T \in RT(\mathcal{E})$ a cycle, if $B_{j_1} \cap R_{j_2} \neq \emptyset$, $B_{j_2} \cap R_{j_3} \neq \emptyset$, ..., and $B_{j_r} \cap R_{j_1} \neq \emptyset$. Such a cycle $J_c$ will be called a common-player cycle if all the corresponding coalitions have a player in common, i.e. if $\bigcap_{j \in J_c} K_j \neq \emptyset$.

**Theorem 16.** (Keiding, 1985) An EFF $\mathcal{E}$ is stable if and only if every rejecting table $T \in RT(\mathcal{E})$ contains a common-player cycle.

The proof results immediately from the following lemma.

**Lemma 17.** A rejecting table $T \in RT(\mathcal{E})$ is utilizable if and only if it contains no common-player cycle.

**Proof.** If there exists a common-player cycle $J_c$ in $T \in RT(\mathcal{E})$, then $T$ can not be utilizable because the preference of any common player $i \in \bigcap_{j \in J_c} K_j$, according to Eq. (R), would be cyclic over the outcomes $a_1 \in B_{j_1} \cap R_{j_2}$, $a_2 \in B_{j_2} \cap R_{j_3}$, ..., $a_r \in B_{j_r} \cap R_{j_1}$.

Conversely, if there are no common-player cycles in $T \in RT(\mathcal{E})$, then the inequalities in Eq. (R) induce an acyclic partial preference over the outcomes for every player $i \in I$. Due to the acyclicity, we can choose a utility $u(i, \ast)$ for every player $i \in I$, which realizes the same preferences. For such a utility function $u$ we shall have $T \in RT(\mathcal{E}, u)$, completing the proof of the lemma. 

Keiding’s theorem can be reformulated in terms of the hypergraphs $\mathcal{E}$ and $\mathcal{F}$ corresponding to the effectivity function $\mathcal{E}$.

Let us begin with the special case of EFFs generated by graphs. Given a graph $G = (V, E)$, let us direct some of its edges. (We assume that some edges may remain non-directed but no edge can be bidirected.) The obtained partially directed graph $D$ is called a suborientation of $G$. We say that a vertex $v \in V$ of the suborientation $D$ rejects a subset $V' \subseteq V$ if $v \not\in V'$ and every edge $(v, v')$ for $v' \in V'$ is directed from $v'$ to $v$ in $D$. A suborientation $D$ of $G$ is called rejecting if every (maximal) stable set of $G$ is rejected in $D$ by a vertex. A directed cycle of $D$ whose vertices form a clique in $G$ is called a clique-cycle of $D$. A rejecting suborientation without clique cycles is called a clique-acyclic rejecting suborientation (or in short a CARS).

**Proposition 18.** (Boros and Gurvich, 1994) A graph $G$ has no CARS if the corresponding EFF $\mathcal{E}_G$ is stable.

**Proof.** In fact this proposition is ‘dual’ to Keiding’s theorem. Let us suppose that the EFF $\mathcal{E}_G$ is not stable and consider a rejecting table $T \in RT(\mathcal{E}_G)$ which has no common-player cycles. Every outcome $a \in A$ is rejected by some coalition according to $T$, hence there exists a corresponding row $(K, B, R)$ in $T$ for which $\mathcal{E}_G(K, B) = 1$, and $a \in R$. This implies, according to the definition of $\mathcal{E}_G$, the existence of a vertex $v \in V$ for
which $K_s \subseteq K$ and $B_v \subseteq B$. Then $v \in S_v$, where $S_v$ is the maximal stable set corresponding to the outcome $a$, since all such maximal stable sets are included in $B_v$ by the definition of $\mathcal{E}_G$, and $B \cap R = \emptyset$ by the definition of a rejecting table. Let us orient all the edges of $G$ between the stable set $S_v$ and vertex $v$ towards vertex $v$. By repeating the same for all the outcomes, we obtain an orientation $D$ of (some of) the edges of $G$, and we claim that it is a CARS. Indeed, $D$ is rejecting because $T \in RT(\mathcal{E}_G)$ is a rejecting table and $D$ is clique-acyclic (and in particular, no edge of $G$ is bidirected) because $T \in RT(\mathcal{E}_G)$ has no common-player cycles.

For the reverse direction, we can construct a rejecting table of the form $T = \{(K_s, B_v, R_v) | v \in V\} \in RT(\mathcal{E}_G)$ from any CARS $D$ of $G$, by defining $R_v$ as the set of maximal stable sets rejected in $D$ at vertex $v$. It is easy to verify that $T$ is indeed a rejecting table for $\mathcal{E}_G$, since $D$ is a rejecting suborientation, and that $T$ has no common-player cycles, since such a cycle in $T$ would correspond to a clique-cycle in $D$.

Thus, the above constructions provide a correspondence between CARS of a graph $G$ and rejecting tables without common-player cycles of the corresponding EFF $\mathcal{E}_G$, in both directions. □

There is a simple relation between CARS and kernel solvability of graphs. Given a graph $G = (V, E)$, let us direct all of its edges, (allowing some edges to be bidirected.) The obtained ‘overdirected’ graph $D^+$ is called a superorientation of $G$. Given a superorientation $D^+$, a subset $S \subseteq V$ is called a kernel if $S$ is a stable set of $G$ and every vertex outside of $S$ has a successor in $S$, according to $D^+$. In other words, if no edge of $G$ has both endpoints in $S$ and for every vertex $v \in S$ there exists a vertex $v' \in S$ such that the edge $(v, v')$ in $D^+$ is directed, from $v$ towards $v'$ (or bidirected.) Obviously, only a maximal stable set of $G$ can be a kernel. If the graph $G$ is complete (i.e. $G$ is a clique) then only a single vertex can be a kernel, and a vertex $v$ is a kernel iff $v$ is a sink in $D^+$, i.e. iff all the edges incident to $v$ are directed towards $v$ (or are bidirected.) Finally, a graph $G$ is called kernel-solvable if a superorientation $D^+$ of $G$ has a kernel whenever every clique of $G$ has a kernel.

There is a simple one-to-one correspondence between sub- and superorientations of a graph $G$, via interchanging unoriented edges with bidirected ones, and vice versa. It is easy to observe that a suborientation is a CARS iff the corresponding superorientation has no kernel. Thus, we obtain the following claim.

**Proposition 19.** A graph $G$ has no CARS if $G$ is kernel-solvable.

**Theorem 20.** (Boros and Gurvich, 1994) A perfect graph has no CARS.

By Proposition 19, this theorem implies Theorem 4. According to Conjecture 5, the converse, i.e. that graphs with no CARS are perfect, is conjectured to hold too.

Let us now consider a general EFF $\mathcal{E} = (C, S; I, A; J)$, and let us assign to the hypergraph $\mathcal{E} = \{C_i | i \in I\}$ a graph $G(\mathcal{E}) = (J, E(\mathcal{E}))$ on the vertex set $J$, whose edges are those pairs $j, j' \in J$, $j \neq j'$ for which $\{j, j'\} \subseteq C_i$ for some $i \in I$. Given a suborientation $D$ of $G(\mathcal{E})$, we shall call it $\mathcal{E}$-acyclic if there exists no directed cycle...
whose vertices all belong to a clique \( C_i \) for some \( i \in I \), and we shall say that \( D \) is \( \mathcal{I} \)-rejecting if the subsets \( S_a \subseteq J \) are rejected in \( D \) for all \( a \in A \). We will use the same abbreviation, CARS, for a \( \mathcal{C} \)-acyclic and \( \mathcal{I} \)-rejecting suborientation of the graph \( G(\mathcal{C}) \).

**Proposition 21.** An EFF \( \mathcal{E} = (C_i, S_a; I, A; J) \) is stable if the graph \( G(\mathcal{C}) \) has no CARS.

The proof goes along exactly the same lines as for EFFs generated by graphs, in Proposition 18, and we omit it here.

Kernel-solvability also generalizes for an arbitrary effectivity function \( \mathcal{E} = (C_i, S_a; I, A; J) \) as follows.

**Proposition 22.** A graph \( G(\mathcal{C}) \) has no CARS if for any linear order of the vertices in the hyperedges \( C_i, i \in I \), there exists always an \( a \in A \) such that the hyperedge \( S_a \) is dominating, i.e. for every vertex \( j \in J S_a \) there is an \( i \in I \) such that \( j \in C_i \), \( C_i \cap S_a \neq \emptyset \) and there exists a vertex \( j' \in C_i \cap S_a \) which is greater than \( j \) in the given order of \( C_i \).

The proof goes along the same lines as for Proposition 19 above, and we omit it here.

Let us consider an interpretation of the above. Let \( \mathcal{E} = (C_i, S_a; I, A; J) \) be an EFF. Then there exists a `winning’ team, i.e. a team \( I \) such that for every participant \( j \in S_a \), there exists a member of the team \( j' \in S_a \) who is better than \( j \) in some test(s) \( i' \in I \).

4. Monotonicity

The family of EFFs \( \mathcal{E}: 2^I \times 2^A \rightarrow \{0, 1\} \) admit a natural partial order, defined by \( \mathcal{E} \leq \mathcal{E}' \) iff \( \mathcal{E}(K, B) \leq \mathcal{E}'(K, B) \) for all \( K \subseteq I \), and \( B \subseteq A \), or in other words, iff \( \mathcal{E}(K, B) = 1 \) implies \( \mathcal{E}'(K, B) = 1 \). Thus, for two effectivity functions \( \mathcal{E} = (K_j, B_j; J; I, A) \) and \( \mathcal{E}' = (K_j', B_j'; J'; I', A') \), the relation \( \mathcal{E} \leq \mathcal{E}' \) holds iff for every \( j \in J \) there exists \( j' \in J' \) such that \( K_j \subseteq K_j' \) and \( B_j \subseteq B_j' \). In particular, an EFF \( \mathcal{E} = (K_j, B_j; J; I, A) \) will remain the same if we remove all the pairs \( (K_j, B_j) \) which are not inclusion-minimal.

Analogously, for EFFs \( \mathcal{E} = (C_i, S_a; I, A) \) and \( \mathcal{E}' = (C_i', S_a' ; I', A'; J') \) we have \( \mathcal{E} \leq \mathcal{E}' \) iff for every \( i \in I \) there exists \( i' \in I' \) such that \( C_i \subseteq C_i' \), and for every \( a \in A \) there is \( a' \in A' \) such that \( S_a \subseteq S_{a'} \). In particular, the EFF \( \mathcal{E} = (C_i, S_a; I, A; J) \) will remain the same if we remove all the players \( C_i \) and outcomes \( S_a \) which are not inclusion-minimal.

It is both obvious and well known that stability is antimonotone.

**Lemma 23.** If EFF \( \mathcal{E} \) is stable and \( \mathcal{E}' \leq \mathcal{E} \) then EFF \( \mathcal{E}' \) is stable, too.

Let us apply this to EFFs generated by graphs. Given a graph \( G \) and an induced subgraph \( G' \) in \( G \), the above definitions imply that \( \mathcal{E}_{G'} \leq \mathcal{E}_G \). Thus, according to Lemma...
23, $\mathcal{E}_G$ is stable whenever $\mathcal{E}_G$ is stable. In other words, if $G'$ has a CARS then $G$ also has one, i.e. $G'$ is kernel-solvable whenever $G$ is kernel-solvable.

Analogously, we can define ‘induced subEFEs’ of an arbitrary EFE. Given an EFE $\mathcal{E} = (C_i, S_a; I, A; J)$ and a subset $J' \subseteq J$, the subEFE $\mathcal{E}' = (C'_i, S'_a; I', A'; J')$ induced by $J'$ is defined by setting $I' = \{i \mid i \in I \cap C_i \neq \emptyset\}$, and $A' = \{a \in A \mid S_a \cap J' \neq \emptyset\}$, and defining $C'_i = C_i \cap J'$ for all $i \in I'$, and $S'_a = S_a \cap J'$ for all $a \in A'$. Obviously, $\mathcal{E}' \subseteq \mathcal{E}$ for any $J' \subseteq J$, thus the following is implied by Lemma 23.

**Lemma 24.** An induced subEFE $\mathcal{E}'$ is stable whenever the EFE $\mathcal{E}$ is stable.

As an application, let us consider an EFE $\mathcal{E}_0 = (C_i, S_a; I, A; J)$ for which $I = \{i_1, i_2\}$, $A = \{a_1, a_2\}$, and $J = \{1, 2\}$, and where $C_1 = S_1 = \{1\}$, and $C_2 = S_2 = \{2\}$. It is easy to verify that $T_0 = \{(i_1), \{a_1\}, \{a_2\}; \{(i_2), \{a_1\}, \{a_2\}\}$ is a rejecting table for $\mathcal{E}_0$, with no common player cycles, and hence $\mathcal{E}_0$ is not stable, by Theorem 16. Together with Lemma 24 this implies Corollary 9 of Theorem 8.

**5. Proofs of main theorems and their applications**

**Proof of Theorem 8.** Given an EFE $\mathcal{E} = (C_i, S_a; I, A; J)$, let us suppose, in contrast to (i) that there exists a subset $J' \subseteq J$ such that $|C_i \cap J'| \leq 1$ for every $i \in I$ but there exists no $a \in A$ for which $J' \subseteq S_a$. We shall prove that under these conditions $\mathcal{E}$ is not stable. We can assume that $J'$ is an inclusion-minimal subset with these properties, i.e. that for every $J'' \subseteq J'$ there exists an $a \in A$ such that $J'' \subseteq S_a$. This implies immediately that $|J'| \neq 1$, since otherwise $J' \subseteq S_a$ would hold for some $a \in A$. Let us consider the subEFE $\mathcal{E}' = (C'_i, S'_a; I', A'; J')$ induced by the set $J'$. By its definition and by the minimality of the set $J'$, the equation $\mathcal{E}_j, (\{j\}, J' - j) = 1$ holds for all $j \in J'$. Since such an EFE cannot be stable whenever $|J'| > 1$, we obtain that $\mathcal{E}$ is not stable, by Lemma 24. otherwise subEFE $\mathcal{E}'$, would be stable, too.

The second part (ii) of the theorem is dual to the first part. □

Let us note that in fact we have shown a bit more. The conditions of Theorem 8 are necessary not only for the stability of $\mathcal{E}$ but also for the stability of its dual $\mathcal{E}^d$.

**Proof of Theorem 11.** Let us note first that conditions (ii) and (ii') are equivalent with the relation $\mathcal{E} \leq \mathcal{E}_G$. Since $G$ is assumed to be perfect by (i), $\mathcal{E}_G$ is stable by Proposition 18 and Theorem 20 (see Boros and Gurvich, 1994), and thus $\mathcal{E}$ is stable, too, by Lemma 24. □

Let us remark that again we have shown a somewhat stronger result. The conditions of Theorem 11 are sufficient not only for the stability of $\mathcal{E}$ but also for the stability of its dual $\mathcal{E}^\circ$. Indeed, let us consider the complementary graph $G^\circ$ instead of $G$. Cliques of $G^\circ$ are stable sets of $G$ and vice versa, and by the definition, we have $\mathcal{E}^d = \mathcal{E}_{G^\circ}$. It is known that a graph $G$ is perfect iff its complement $G^\circ$ is perfect (see Lovasz, 1972b),
thus if conditions (i), (ii) and (ii’) of Theorem 11 hold for \( \mathcal{E} \) and \( G \) then they hold for \( \mathcal{E}^d \) and \( G^c \), as well.

Let us note that the conditions of Theorem 8 are necessary but not sufficient for stability and the conditions of Theorem 11 are sufficient but not necessary. This is so, for example, because otherwise the stability of \( \mathcal{E} \) would be equivalent to the stability of \( \mathcal{E}^d \), which is not the case, as we shall see in the next section. The following example will clarify these limits of Theorems 8 and 11.

**Example 25.** For every positive integer \( k \) let us define an EFF \( \mathcal{E}_k \) by setting \( J = \{1, \ldots, k\} \), and \( I = A = \{(p, q)|1 \leq p < q \leq k\} \), and by defining \( C_{(p,q)} = S_{(p,q)} = \{p, q\} \) for all \( (p, q) \in I = A \). Direct computations show that \( \mathcal{E}_k \) is stable if \( k \leq 6 \), but it is not stable for \( k = 7 \). A corresponding CARS for \( \mathcal{E}_7 \) is given by the following three directed cycles \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 1 \), \( 1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 1 \), and \( 1 \rightarrow 5 \rightarrow 2 \rightarrow 6 \rightarrow 3 \rightarrow 7 \rightarrow 4 \rightarrow 1 \). Since \( \mathcal{E}_7 \) is an induced subEFF of \( \mathcal{E}_k \) for any \( k > 7 \), it follows that \( \mathcal{E}_k \) is not stable for all \( k \geq 7 \).

Let us note that the conditions of Theorem 8 hold automatically for every \( k \), but \( \mathcal{E}_6 \) is not stable if \( k \geq 7 \). On their turn, the conditions of Theorem 11 hold only for \( k \leq 4 \), while \( \mathcal{E}_4 \) and \( \mathcal{E}_6 \) are still stable.

For example, if \( k = 5 \) the only graph satisfying (ii) and (ii’) of Theorem 11 is the hole \( C_5 \) but it is not perfect, while if \( k = 6 \) then either \( G \) or \( G^c \) must contain a triangle, consequently either (ii) or (ii’) must be violated.

Thus for \( \mathcal{E}_5 \) and \( \mathcal{E}_6 \) no ‘perfect split’ exist, but still these EFFs are stable.

Let us consider now an important class of EFFs for which Theorems 8 and 11 are applicable.

Given an EFF \( \mathcal{E} = \langle C_i; S_a; I, A; J \rangle = \langle K_i; B_j; I, A \rangle \), let us assume that

\[
|C_i \cap S_a| = 1 \quad \text{for all } i \in I \text{ and } a \in A \quad \text{(P)}
\]

or in dual terms,

\[
K_i \cap K_j = \emptyset \quad \text{or} \quad B_j \cap B_j = \emptyset \quad \text{(or both) for every } j \neq j' \in J \quad \text{(P\#)}
\]

Let us suppose first that there exists a pair \( j \neq j' \in J \) such that both sets \( K_i \cap K_j \) and \( B_j \cap B_j \) are empty, or in dual terms, that there exists a pair \( j, j' \in J \), \( j \neq j' \) which does not belong to the same sets \( C_i \), \( i \in I \) or \( S_a \), \( a \in A \). Then, according to Corollary 9 of Theorem 8, neither \( \mathcal{E} \) nor \( \mathcal{E}^d \) is stable.

So let us now assume that for every pair \( j \neq j' \in J \) exactly one of the following two options holds

(i) \( K_i \cap K_j \neq \emptyset \) or
(ii) \( B_j \cap B_j \neq \emptyset \).

or in dual terms,

\[ (i\#) \{j, j'\} \subseteq C_i \text{ for some } i \in I, \text{ or} \]

\[ (ii\#) \{j, j'\} \nsubseteq C_i \text{ for some } i \in I, \text{ or} \]
(ii$^a$) $\{j, j^*\} \subseteq S_a$ for some $a \in A$.

Let us denote by $G = (J, E)$ and $G^c = (J, E^c)$ the two complementary graphs having the same set of vertices $J$, and the edges of which are defined by (i) and (ii), respectively. Theorem 8 claims that neither $\mathcal{E}$ nor its dual $\mathcal{E}^d$ can be stable unless $\mathcal{E} = \mathcal{E}_G$ and $\mathcal{E}^d = \mathcal{E}^c_G$. So let us now assume that both these equalities hold.

Furthermore, Theorem 11 claims that both $\mathcal{E}$ and $\mathcal{E}^d$ are stable if the graph $G$ is perfect.

Finally, let us suppose that the graph $G$ is not perfect. This is the only open case. According to Conjecture 5, in this case the EFF $\mathcal{E} = \mathcal{E}_G$ is not stable. According to the results of Lovasz (1972a), the complementary graph $G^c$ is not perfect either, in this case, thus according to Conjecture 5, the dual EFF $\mathcal{E}^d = \mathcal{E}^c_G$ is not stable either.

**Proposition 26.** Conjectures 5 and 13 are equivalent.

**Proof.** The above arguments prove that if the assumption in Eq. (P) and Conjecture 5 hold then for the dual pair of EFFs $\mathcal{E}$ and $\mathcal{E}^d$, either both or neither of them are stable, i.e. Conjecture 5 implies Conjecture 13.

Let us suppose that Conjecture 5 does not hold then the equivalent Conjecture 6 fails too, i.e. there exists a complementary pair of graphs $G$ and $G^c$ such that $G$ is kernel-solvable while $G^c$ is not. Then, according to Propositions 1–3, the EFF $\mathcal{E}_G$ is stable while its dual $\mathcal{E}^d = \mathcal{E}^c_G$ is not. However, the assumption in Eq. (P) holds for EFFs generated by graphs, hence Conjecture 13 fails.

**Proposition 27.** SPGC and Conjecture 15 are equivalent.

**Proof.** (SPGC $\Rightarrow$ Conjecture 15) Let us suppose that the graph $G$ has neither odd holes nor odd antiholes. SPGC claims that $G$ is perfect. According to Theorem 4, $G$ is then kernel-solvable, and according to Propositions 1–3, the EFF $\mathcal{E}_G$ is stable.

Conjectures 5 and 15 $\Rightarrow$ SPGC. Let us suppose that the graph $G$ contains neither odd holes nor odd antiholes. According to Conjecture 15, the EFF $\mathcal{E}_G$ is stable. According to Propositions 1–3, the graph $G$ is then kernel-solvable, implying finally that, according to Conjecture 5, the graph $G$ is perfect.

Conjecture 15 $\Rightarrow$ Conjecture 5. Since Conjecture 5 is equivalent to Conjecture 6 (see Boros and Gurvich, 1994), we shall instead show that Conjecture 15 implies Conjecture 6.

Odd holes and odd antiholes are complementary. Thus the graph $G$ contains one of these iff the complementary graph $G^c$ contains the complementary one. Let us suppose that both graphs $G$ and $G^c$ contain an odd hole or an odd antihole, implying that they both are not kernel-solvable (see, for example, Berge and Duchet, 1983 or Boros and Gurvich, 1994). On the other hand, if both graphs contain neither odd holes nor odd antiholes then both graphs are kernel-solvable, according to Conjecture 14. Thus, in all
cases the graph $G$ is kernel solvable iff its complement $G^c$ is also kernel solvable, which is exactly Conjecture 6.

Now we will derive Theorem 12 from Theorems 8 and 11 and from Lemma 28 below.

Given an integer $b \in \mathbb{N}$, let us define a graph $G_n = (V_n, E_n)$ consisting of $2n$ vertices $\{1, 2, \ldots, 2n\}$, and $n$ pairwise non-adjacent edges $E_n = \{(i, n + i) | 1 \leq i \leq n\}$.

Lemma 28. Given a graph $G_n = (V_n, E_n)$, and given $k$ subsets $\mathcal{M} = \{M_1, \ldots, M_k\}$ of the vertex-set $V_n$, it is an NP-complete problem to check the validity of the following statement:

$$\text{Every (maximal) stable set of } G_n \text{ is contained in some subset } M_i \in \mathcal{M} \tag{Q}$$

Proof. We shall prove the lemma by reducing the well-known NP-complete problem of tautology for DNFs (see, for example, Garey and Johnson, 1979) to Eq. (Q).

Obviously, the problem of testing the validity of Eq. (Q) belongs to NP.

Let us consider an arbitrary DNF on $n$ variables consisting of $k$ terms

$$D(X_1, \ldots, X_n) = \bigvee_{i=1}^{k} \left( \bigwedge_{j \in P_i} X_j \wedge \overline{X}_j \right)$$

where $X_j$, $j = 1, \ldots, n$ denote the Boolean variables, and $\overline{X}$ denotes the negation of $X$.

The tautology problem for $D$ consists in deciding if $D$ evaluates to true for every true-false assignments to the variables $X_1, \ldots, X_n$.

Let us now associate to such a DNF a family $\mathcal{M} = \mathcal{M}_D$ of $k$ subsets of $V_n$, as in the lemma. Let us define

$$M_i = \{j | 1 \leq j \leq n, j \not\in P_i\} \cup \{j + n | 1 \leq j \leq n, j \not\in N_i\}$$

for $i = 1, \ldots, k$. It is then straightforward to verify that $D$ is a tautology (i.e. it evaluates always to true) iff Eq. (Q) holds true for the family $\mathcal{M}_D$.

Thus, the general problem of tautology for DNFs is polynomially reduced to testing the validity of Eq. (Q). □

Proof of Theorem 12. We shall prove the theorem by reducing the problem of testing the validity of Eq. (Q) to the stability testing of effectivity functions.

Obviously, testing the stability of an effectivity function belongs to NP by Theorem 16 (Keiding, 1985).

Let us consider the graph $G_n = (V_n, E_n)$ and a family $\mathcal{M} = \{M_1, \ldots, M_k\}$ of $k$ subsets of $V_n$, and let us associate an EFF $\mathcal{E} = \mathcal{E}(G_n, \mathcal{M}) = (C, S; I, A; J)$ to this pair, as follows. The EFF $\mathcal{E}(G_n, \mathcal{M})$ consists of $n$ players $I = \{1, \ldots, n\}$ and $k$ outcomes $A = \{1, \ldots, k\}$, and we have $J = V_n$, $C_i = \{i, i + n\}$ for $i = 1, \ldots, n$, and $S_a = M_a$, for $a = 1, \ldots, k$ (i.e. $\mathcal{E} = E_n$ and $\mathcal{F} = \mathcal{M}$).

For this EFF $\mathcal{E}(G_n, \mathcal{M})$ we have that if Eq. (Q) holds for $\mathcal{M}$, then it is stable by Theorem 11, since $G_n$ is a perfect graph, and if Eq. (Q) does not hold for $\mathcal{M}$, then it cannot be stable by Theorem 8.
Thus, we have reduced Eq. (Q) to the problem of testing the stability of EFFs, and thus the theorem follows by Lemma 28. □

Remark 29. Let us observe that the condition in Eq. (Q) is equivalent to condition (i) of Theorem 8 for the EFF \( \mathcal{E}(G, \mathcal{M}) \), and it is equivalent with the fact that \( \mathcal{E}(G, \mathcal{M}) \) is playing-minor, see Remark 10.

Thus in fact, it is an NP-complete problem to check condition (i) (i.e. playing-minority) even in a special case when it is sufficient (and not only necessary) for stability.

6. Stability is not selfdual

As we mentioned earlier, all the conditions of Theorems 8 and 11 hold (or do not hold) for the dual EFFs \( \mathcal{E}^d \) and \( \mathcal{E}^d \) simultaneously. However in general, it may happen that an EFF \( \mathcal{E} \) is stable while its dual EFF \( \mathcal{E}^d \) is not.

Example 30. (Peleg, 1984, Example 6.3.16 and Remark 6.3.17) Let us consider the EFF \( \mathcal{E} = (K_j, B_j; J, I; A) \) with \( I = A = J = \{1, 2, 3, 4\} \) defined by the effective pairs in the table below. It is easy to see that by defining

<table>
<thead>
<tr>
<th>( j )</th>
<th>( K_j )</th>
<th>( B_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 3</td>
<td>1 4</td>
</tr>
<tr>
<td>2</td>
<td>1 3</td>
<td>2 4</td>
</tr>
<tr>
<td>3</td>
<td>1 2</td>
<td>3 4</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1 2 3</td>
</tr>
</tbody>
</table>

\( R_1 = \{2\}, R_2 = \{3\}, R_3 = \{1\}, \text{ and } R_4 = \{4\} \) we obtain a rejecting table \( T = \{(K_j, B_j; R_j) | j = 1, \ldots, 4\} \), in which all the outcomes are rejected, and hence the EFF \( \mathcal{E} \) is not stable. It is not difficult to check, however, that its dual \( \mathcal{E}^d \) is stable.

In a sense this example is minimal. If \( \mathcal{E} \) is stable and \( \mathcal{E}^d \) is not then both the number of players and the number of outcomes cannot be less than 4.

For EFFs generated by graphs such examples probably do not exist. Let us consider an arbitrary pair of complementary graphs \( G \) and \( G^c \) and the corresponding dual EFFs \( \mathcal{E} = \mathcal{E}_G \) and \( \mathcal{E}^d = \mathcal{E}_{G^c} \). If graphs \( G \) and \( G^c \) are perfect then both EFFs are stable, according to Theorems 4 and 11. If these graphs are not perfect then probably both EFFs are not stable (see Conjectures 5 and 13), although this is still an open problem.

Given a pair of graphs \( G_1 = (J, E_1) \) and \( G_2 = (J, E_2) \) with the same set \( J \) of vertices, let us associate an EFF \( \mathcal{E}(G_1, G_2) = (C, S; I, A; J) \) to this pair of graphs by defining the hypergraphs \( \mathcal{E} \) and \( \mathcal{J} \) as the sets of all the maximal cliques of \( G_1 \) and \( G_2 \), respectively. In other words, \( \mathcal{E}(G_1, G_2) = \langle K_j, B_j; J, I, A \rangle \), where \( I \) is the set of maximal cliques of \( G_1 \), \( A \) is the set of maximal cliques of \( G_2 \), \( K_j = \{i \in I | j \supseteq i\} \) and \( B_j = \{a \in A | S_a \supseteq j\} \) for \( j \in J \), and \( \mathcal{E}(K, B) = 1 \) iff \( K_j \subseteq K \) and \( B_j \subseteq B \) for some \( j \in J \).
By this definition, the EFFs $\mathcal{E}(G_1, G_2)$ and $\mathcal{E}(G_2, G_1)$ are dual. Furthermore, if the edge sets of the graphs $G_1$ and $G_2$ cover all pairs $(j, j')$, $j, j' \in J$, $j \neq j'$, then the conditions (i) and (ii) of Theorem 8 are satisfied. Let us call an effectivity function bigraphic, if it can be represented as such an associated EFF $\mathcal{E}(G_1, G_2)$.

As we shall see, stability is not selfdual even for these bigraphic EFFs.

**Example 31.** Let $G_1 = C_9^+$ be the graph shown in Fig. 1 and let $G_2$ be the 9-antihole $C_9^-$. The graphs $G_1$ and $G_2$ have the common set of vertices $J = \{1, \ldots, 9\}$, their edges together contain all the pairs of $J$, and they share three common edges $(2,6)$, $(3,8)$ and $(5,9)$. It was verified by computer that the EFF $\mathcal{E}(G_1, G_2)$ is stable. Indeed, a case analyses shows that in any CARS of $C_9^+$ the 9-cycle must be cyclically directed. Without any loss of generality, we can assume that it is directed clockwise, i.e. all the edges $(j, j + 1)$ are directed from $j$ to $j + 1$, see Fig. 1. Then, in order to reject all nine 4-cliques of $C_9^+$ one must direct the three (shared) central edges of as $(6,2)$, $(3,8)$ and $(9,5)$. This still leaves us, with no more edges to direct, with the clique $\{2,5,8\}$ un-rejected. Thus indeed, $\mathcal{E}(G_1, G_2)$ is stable.

At the same time, the dual EFF $\mathcal{E}(G_2, G_1)$ is not stable since there exists a CARS of $C_9^-$ given by the directed edges $9 \rightarrow 7$, $8 \rightarrow 6$, $7 \rightarrow 5$, $6 \rightarrow 4$, $5 \rightarrow 3$, $4 \rightarrow 2$, $3 \rightarrow 1$, $2 \rightarrow 9$, $1 \rightarrow 8$, $2 \rightarrow 7$, $8 \rightarrow 4$, $5 \rightarrow 1$, while keeping all the other edges of $C_9^-$ un-directed.

Let us note that in this example the set of common edges of graphs $G_1$ and $G_2$ is critical, in the following sense. Let $G'_2 = C_9^-$ be the 9-antihole again, while let $G'_1$ contain the complementary 9-cycle and maybe some other edges. Then the EFF $\mathcal{E}(G'_1,$

![Fig. 1. An 'almost' CARS: All the maximal cliques of $C'_9$, except $\{2,5,8\}$, are rejected.](image-url)
$G_1^*$ is not stable whenever at least one of the three edges $(2,6), (3,8)$ and $(5,9)$ (up to a cyclic isomorphism) is missing. The corresponding critical CARS* are given in Fig. 2.

The stability of an EFF $\mathcal{E}_G$ generated by a graph $G$ is equivalent to the perfectness of this graph if Conjecture 5 is true, implying that in this case the stability of $\mathcal{E}_G$ could be expressed in terms of the clique and chromatic numbers of the graph $G$. The previous example shows that the stability of a bigraphic EFF $\mathcal{E}(G_1, G_2)$ cannot be expressed in such terms. Indeed, let us consider the following two pairs of graphs $(G_1, G_2)$: let $G_1 = C_9^+$ the 9-antihole and let $G_2 = C_9$ the 9-hole first, and let $G_1' = G_1$, and $G_2' = C_9^+$ the graph in Fig. 1, second. The graphs $C_9^+$ and $C_9$ have the same clique and chromatic numbers (2 and 3, respectively); moreover, their complements, the 9-antihole $C_9^+$ and $C_9^{+c}$, also have the same clique and chromatic numbers (4 and 5, respectively).

Fig. 2. CARS, all the maximal cliques of $C_9^+$ are rejected.
However, the EFF $\mathcal{E}(G_1^*, G_2^*)$ generated by the second pair is stable, while the EFF $\mathcal{E}(G_1, G_2)$ generated by the first pair is not.

Let us call an EFF $\mathcal{E}$ convex if $\mathcal{E}(K, B) = 1$ and $\mathcal{E}(K, B) = 1$ imply that either $\mathcal{E}(K_1 \cup K_2, B_1 \cup B_2) = 1$ or $\mathcal{E}(K_1 \cap K_2, B_1 \cup B_2) = 1$ (or maybe both) for every $K_1, K_2 \subseteq I$, and $B_1, B_2 \subseteq A$.

By this definition, the family of convex EFFs is selfdual, i.e. $\mathcal{E}$ is convex iff $\mathcal{E}^d$ is convex.

**Theorem 32.** (Peleg, 1984) Convex EFFs are stable.

One could conjecture that all the maximal stable EFFs are convex, or in other words, that for every stable EFF $\mathcal{E}$ there exists a convex EFF $\mathcal{E}'$ such that $\mathcal{E}' \geq \mathcal{E}$. This conjecture would provide a nice characterization of stable EFFs, if it were true. In fact, every EFF $\mathcal{E}$, such that it is stable and its dual $\mathcal{E}^d$ is not stable, provides a counterexample. To see this, let us suppose that $\mathcal{E}'$ is a stable EFF for which $\mathcal{E}^d$ is not stable, and that $\mathcal{E} \leq \mathcal{E}'$ for some convex EFF $\mathcal{E}'$. Since convexity is selfdual, $\mathcal{E}'^d$ is also convex, and hence it is also stable by Theorem 32. Because the inequality $\mathcal{E}'^d \geq \mathcal{E}^d$ holds, the stability of $\mathcal{E}^d$ is implied by Lemma 23, a contradiction with our assumptions.

Beside the duality (d) there is another important type of 'duality' for EFFs: Given an EFF $\mathcal{E} = (K, B; I, A)$ let us define EFF $\mathcal{E}^*$ by $\mathcal{E}^*(K, B) = 1 - \mathcal{E}(I - K, A - B)$ for all $K \subseteq I$, and $B \subseteq A$. Obviously, both (d) and (*) are involutive, i.e. $\mathcal{E}^{**} = \mathcal{E}$. Let us note that (d) and (*) are also commutative, i.e. $\mathcal{E}^{*d} = \mathcal{E}^{d*}$, that (d) is monotone, i.e. $\mathcal{E} \leq \mathcal{E}'$ iff $\mathcal{E}^{d*} \geq \mathcal{E}'^{d*}$.

Let us call an EFF $\mathcal{E}$ *-selfdual if $\mathcal{E} = (K, B; I, A)$ holds for all $K \subseteq I$, $B \subseteq A$, or in other words, if $\mathcal{E} = \mathcal{E}^*$. The stable *-selfdual EFFs can be characterized as follows.

**Proposition 33.** (Abdou, 1981) Stable *-selfdual EFFs are sub- and superadditive; furthermore *-selfdual, sub- and superadditive EFFs are stable.

It is also known that *-selfdual EFFs form a subclass of convex EFFs.

**Proposition 34.** (Gurvich, 1992) Stable *-selfdual EFFs are convex.
$\mathcal{E}^d$ is convex, too, consequently they are both stable by Theorem 32. Since we have $\mathcal{E}^d \succeq \mathcal{E}^d$, the stability of $\mathcal{E}^d$ is implied by Lemma 23, a contradiction again with our assumptions.

Furthermore, one could also conjecture that a stable EFF $\mathcal{E}$ is always majorized by a stable *-selfdual (or convex) one, in case both $\mathcal{E}$ and $\mathcal{E}^d$ are stable. This is, however, not true again, as the next example shows.

**Example 35.** (Gurvich, 1990) Let us consider the EFF $\mathcal{E} = (K, B; J, I, A)$ with three players $I = \{i_1, i_2, i_3\}$, six outcomes $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and for which $\mathcal{E}(K, B) = 1$ i f $|K| \geq 2$ and $B$ contains one (or more) of the subsets $\{a_1, a_3\}, \{a_2, a_4\}, \{a_5, a_6\}, \{a_4, a_5\}, \{a_5, a_1\},$ and $\{a_6, a_2\}$. One can show that the EFFs $\mathcal{E}$ and $\mathcal{E}^d$ are stable; however, there exist no convex (or *-selfdual) EFFs majorizing $\mathcal{E}$.

Thus, it seems to be difficult to characterize maximal stable EFFs.

On the other hand, minimal unstable EFFs are easier to describe because every unstable EFF has a CARS. Let us consider a graph $G = (V, E)$ and a suborientation $D$ of it, such that

(i) for every vertex $v \in V$ there exists an edge coming in and an edge going out;

(ii) orienting any of the non-directed edges of $D$ would create a new clique-cycle.

One can show that in fact every such suborientation $D$ is a CARS for some EFFs. Furthermore, among these EFFs there exists a unique minimum EFF $\mathcal{E}_p = (C, S; I, A; V)$, for which $\mathcal{E} = \{C, i \in I\}$ are all the maximal acyclic cliques of $D$, i.e. cliques of $G$ without clique-cycles in $D$, and $\mathcal{E} = \{S_a, a \in A\}$ are $n = |V|$ maximal subsets $S_a \subseteq V$ such that $S_a$ is rejected in $D$ by a vertex $v = v(a)$, and the correspondence $a \leftrightarrow v(a)$ is a bijection between $V$ and $A$, i.e. one can identify these sets and assume that $A = V$. It can be shown that in fact every minimal unstable effectivity function can be realized by some suborientation $D$ in the above way.

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**Appendix A. Perfect graphs**

Let $G = (V, E)$ be a graph, where $V$ denotes the set of vertices and $E$ denotes the set of edges. We denote the number of vertices and edges by $n = n(G)$ and $m = m(G)$, respectively, and call these numbers the *order* and the *size* of the graph $G$.

Given a graph $G = (V, E)$, we define the complementary graph as $G^c = (V, E^c)$, that is
the set of vertices is the same and two vertices are adjacent in $G^c$ iff they are not adjacent in $G$.

A subgraph $G' = (V', E')$ of a graph $G = (V, E)$ is defined by a subset of vertices $V' \subseteq V$ and a subset of edges $E' \subseteq E$. We say that $G'$ is an induced subgraph (or the subgraph of $G$ induced by the subset $V' \subseteq V$) if every edge of $E$ with both its vertices in $V'$ belongs to $E'$.

A graph is called complete if every two of its vertices are adjacent, i.e., connected by an edge. The complement to a complete graph is called an edge-free graph. A complete subgraph is called a clique, and an edge-free induced subgraph is called a stable set (or independent set). Each clique (stable set) which is not strictly contained in another clique (stable set) is called maximal, and each clique (stable set) of the maximal order is called maximum. The order of a maximum clique (stable set) is called the clique number ($\alpha$) of the graph $G$ and is denoted by $\omega(G)$ (respectively, by $\alpha = \alpha(G)$). The relations $\alpha(G) = \omega(G)$, $\omega(G') = \omega(G)$, $n(G^c) = n(G)$, and $m(G') + m(G) = n(G)(n(G) - 1)/2$ hold obviously.

The chromatic number $\chi = \chi(G)$ of a graph $G = (V, E)$ is the minimum possible number of stable sets the union of which is $V$. Obviously, $\chi \leq \omega$ holds for every graph.

A graph $G$ is called perfect if $\chi(G') = \omega(G')$ for every induced subgraph $G'$ of $G$ including $G$ itself. Perfect graphs were introduced by Berge (1961a) who suggested the following two conjectures.

Conjecture 36. (Weak Perfect Graph Conjecture (WPGC)) A graph $G$ is perfect iff the complementary graph $G^c$ is perfect.

Lovasz (1972b) proved that a graph $G$ is perfect iff $\alpha(G') \times \omega(G') \geq n(G')$ for every induced subgraph $G'$ of $G$ including $G$ itself. This result implies immediately the above conjecture.

A graph $G$ is called minimally imperfect if $G$ itself is not perfect but every induced subgraph $G'$ of $G$, different from $G$, is perfect.

Which graphs are minimally imperfect? Let us consider an odd hole $C_{2i+1}$, $i \geq 2$, i.e., an odd chordless cycle of length 5 or longer. It is easy to compute that $2 = \omega(C_{2i+1}) < \chi(C_{2i+1}) = 3$. Thus, odd holes are imperfect, and it is not difficult to check that they are minimal in the sense that by deleting any of its vertices, we obtain a perfect graph.

Are there other minimal imperfect graphs? Let us consider an odd antihole $C_{2i+1}^c$, $i \geq 2$, i.e., the complement to the odd hole $C_{2i+1}$. It is easy to compute that $i = \omega(C_{2i+1}^c) < \chi(C_{2i+1}^c) = i + 1$. Thus, odd holes are imperfect, and it is not difficult to check that they are minimal again with respect to taking induced subgraphs. Are there more minimal imperfect graphs? The second famous conjecture of Berge (1961b) claims that no other ones exist.

Conjecture 37. (Strong Perfect Graph Conjecture (SPGC)) The only minimally imperfect graphs are the odd holes and odd antiholes, or in other words, every imperfect graph contains an induced odd hole or an induced odd antihole.

It is immediate to see that SPGC is stronger than WPGC, and, as of today, SPGC is still open.
References