Nash equilibrium in a spatial model of coalition bargaining

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Abstract

In the model presented here, $n$ parties choose policy positions in a space $Z$ of dimension at least two. Each party is represented by a “principal” whose true policy preferences on $Z$ are unknown to other principals. In the first version of the model the party declarations determine the lottery outcome of coalition negotiation. The coalition risk functions are common knowledge to the parties. We assume these coalition probabilities are inversely proportional to the variance of the declarations of the parties in each coalition. It is shown that with this outcome function and with three parties there exists a stable, pure strategy Nash equilibrium, $z^*$ for certain classes of policy preferences. This Nash equilibrium represents the choice by each party principal of the position of the party leader and thus the policy platform to declare to the electorate. The equilibrium can be explicitly calculated in terms of the preferences of the parties and the scheme of private benefits from coalition membership. In particular, convergence in equilibrium party positions is shown to occur if the principals’ preferred policy points are close to colinear. Conversely, divergence in equilibrium party positions occurs if the bliss points are close to symmetric. If private benefits (the non-policy perquisites from coalition membership) are sufficiently large (that is, of the order of policy benefits), then the variance in equilibrium party positions is less than the variance in ideal points. The general model attempts to incorporate party beliefs concerning electoral responses to party declarations. Because of the continuity properties imposed on both the coalition and electoral risk functions, there will exist mixed strategy Nash equilibria. We suggest that the existence of stable, pure strategy Nash equilibria in general political games of this type accounts for the non-convergence of party platforms in multiparty electoral systems based on proportional representation.

Keywords: Nash equilibrium; Spatial models; Coalition bargaining

JEL classification: C72; C78; D72; C11

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1. Introduction

Spatial models of competitive party behavior in representative democracies often give “convergence” results that are at odds with the way political parties seem to behave. In two-party competition, for example, it is usually assumed that the only motivation of a party is to win as many seats as possible. If the electoral model is “deterministic”, then pure strategy Nash equilibria will generally not exist (Saari, 1996), but the support of the mixed strategy Nash equilibria will be within the so-called “uncovered set” (Banks et al., 1998; Cox, 1987; McKelvey, 1986). This set is centrally located with respect to the distribution of voter ideal points. If the electoral model is probabilistic, or “stochastic”, and parties choose positions to maximize expected vote, then the pure strategy Nash equilibrium will also be centrally located, at the mean of the voter distribution (Coughlin, 1992; Enelow and Hinich, 1984; Hinich, 1977).

A two-party model, due to Cox (1984), does not exhibit such “Downsian” (Downs, 1957) convergence, because it assumes that elections are inherently risky and that parties actually care about policy. Cox supposes that each party, i, has a “sincere” point, o, say, in the policy space, Z, and that the electoral outcome of a pair of party declarations, \( z = (z_1, z_2) \), is described by a “lottery” \( \{ (p_i(z), z_i), (p_j(z), z_j), (p_k(z), z_k) \} \). Here \( p_i(z) \) for \( i = 1, 2 \), is the probability that party i wins, and implements \( z_i \), while \( p_j(z) \) is the probability of a draw after which \( \{ 1, 2 \} \) form a government and implement a compromise position, \( z_o \), say. Given ideal points \( (o_1, o_2) \) for the two parties, and appropriate “spatially” defined utilities, Cox suggests there is a pure strategy Nash equilibrium \( (z_1^*, z_2^*) \in Z^2 \). One problem not fully addressed by Cox is why a winning party i say, would choose to implement \( z_i^* \) after the election, rather than its preferred position, \( o_i \). (See Banks, 1990, for a discussion.) One way to deal with this problem is to suppose that each party, i, is a heterogeneous collection of elite actors, described by \( \{ o_k \} \) who choose one of their members as the party principal. This principal then chooses a second elite member as leader, or agent for the party, whose ideal point is identical to the party declaration, \( z_i \). By this method, Nash equilibrium selection of leaders gives policy choices which are credible to the electorate. However, Cox’s model of competition does not readily generalize to the multiparty case (where the number of parties, n, is at least three). It is typical of such a situation that no party wins a majority, and that coalitions are necessary for government formation (Laver and Schofield, 1990).

Recent analyses of multiparty competition, based on stochastic models of electoral behavior and expected vote maximizing behavior, have extended the earlier two-party analysis of Downs (1957) and have shown that the unique Nash equilibria are “convergent” and at the mean of the electoral distribution (Lin et al., forthcoming). In contrast, if the voter model is deterministic, then typically there are no pure strategy Nash equilibria. Indeed, it is possible that discontinuity in the electoral response means that there are not even mixed strategy equilibria (Dasgupta and Maskin, 1986; Osborne, 1993).

However, the empirical evidence available (Budge, 1987; Laver and Schofield, 1990; Schofield, 1995a) suggests that the selection of party positions, under electoral systems based on proportional representation, is neither convergent (as suggested by the stochastic model), nor chaotic (as indicated by the deterministic model).
This paper is an attempt to account for the non-convergence of party positions in multiparty situations. We postulate that the choice of party leader by each party principal is made in the context of a "political game". This game is determined not just in terms of electoral results, but also through the relationship between party choices and post-election coalition negotiation.

Post-election negotiation was originally studied by Riker (1962) in situations where the seat strengths of the parties, and thus the coalition possibilities, were fixed. Riker's original work was presented in the context of (constant sum) cooperative game theory. More recent work (Banks and Duggan, 1998; Baron, 1989; Baron, 1991), in this "Rikerian" tradition, has supposed that the party leader positions (z) are common knowledge, and that party weights (and thus the coalition structure D, say) are predetermined. However, if the bargaining game, which we call $\tilde{g}(z)$, is determined by both $D$ and $z$, then "rational" party principals will choose party leaders, and thus party positions, in such a way as to affect $\tilde{g}(z)$ to their advantage, with respect to the preferred policy outcomes of the principals.

We are concerned here with the choice of party leaders before the election. To model such a choice by party principals it is necessary to model the effect of the choice of party leaders, z, not only on $\tilde{g}(z)$ but on $D$ itself. To our knowledge, the first attempt at such a game-theoretic analysis was by Austen-Smith and Banks (1988). To briefly describe their model, they assumed that the party space was 1-dimensional, that there were three parties, and that party principals were essentially only concerned with coalition perquisites. However, party leaders were concerned with policy, and voters strategically chose parties on the basis of their beliefs about post-election negotiation.

The framework presented in this paper can be seen as an attempt to generalize certain aspects of the Austen-Smith and Banks model. Most importantly, we do not suppose that Z is 1-dimensional. Because Z can be of dimension two or more, the outcome $\tilde{g}(z)$, at the coalition structure $D$ and vector of positions, $z$, need not be a singleton. We assume instead that it is a lottery, involving coalition positions and probabilities. We do not attempt to determine $\tilde{g}(z)$ from first principles, but instead assume it has a structural form that is "common knowledge" to the political elite. In a way to be described below, the heterogeneous preferences within each party elite define a sincere policy point, which identifies the party principal. The optimization problem for each party principal is to choose a party leader in such a way as to balance policy objectives and electoral constraints. We follow Cox (1984) in supposing that the election is risky. We contend that strategic calculations by the electorate in response to a vector $z = (z_1, \ldots, z_n)$ of party leader positions cannot be fully determined by the parties. Instead, electoral risk is described by a stochastic operator $\Psi$ which maps the vector, $z = (z_1, \ldots, z_n)$, of the declaration of the n parties, to a vector of random variables describing seat shares in the "parliament". As indicated above, we assume that the choice of declaration, $z_i$, of party $i$ is equivalent to choosing a leader of that party, in the sense that the party’s declaration coincides with the most preferred policy position, $z_i \in Z$, of the party leader. The resulting choice vector, $z$, is also assumed to be in Nash equilibrium with respect to a "game form", $\tilde{g}$, which we now briefly describe. We assume that a particular "state of the world", $t \in T$, brought into being by the election can be described by a specific set, $D_t$ (a family of winning coalitions). For example, in the two-party situation, considered
by Cox (1984), the class \( \mathcal{D}_2 \) of possibilities would be simply \( \mathcal{D}_1 \) (party 1 wins), \( \mathcal{D}_2 \) (party 2 wins) or \( \mathcal{D}_0 \) (both parties gain an equal number of seats). In the more general multiparty political world in which we are interested, \( \mathcal{D}_2 \) comprises the class of all conceivable “winning” coalition structures, and this class is of order 2\(^n\). Since the election is “stochastic”, the parties cannot determine prior to the election (even with knowledge of the vector \( z \)) precisely which state of the world, \( t \), will occur. However, we shall assume that the probability, \( p_z(z) \), that \( \mathcal{D}_2 \) occurs, given \( z \), is common knowledge, for all \( z \in Z^n \). Moreover, \( p_z: Z^n \rightarrow [0,1] \) is a continuous function for all \( t \in T \), where \( T \) indexes the various states of the political world. Since each \( p_z \) is a probability function, this defines a continuous “electoral risk” function \( p: Z^n \rightarrow \Delta \), where \( \Delta \) is the \( |T| - 1 \) dimensional simplex.

Once the coalition structure, \( \mathcal{D}_2 \), is determined by the election, then the party leaders negotiate among themselves to choose government coalitions, policy positions and distributions of portfolios. Following recent work by Baron (1989, 1991), and Banks and Duggan (1998), the outcome of this bargaining is not a single government coalition, but a lottery, \( \tilde{g}(z) \), that depends both on the vector, \( z \), of party leader positions, and on the coalition structure, \( \mathcal{D}_2 \). For each \( \mathcal{D}_2 \), we assume this lottery, \( \tilde{g}_i \), at the vector \( z \), can be represented as \( \{(\rho_M(z), a_M(z), \sigma_M(z)): M \in \mathcal{D}_2\} = \tilde{g}_i(z) \). Here, for each \( M \in \mathcal{D}_2 \), \( \rho_M(z) \) is the probability that coalition \( M \) occurs, \( a_M(z) \in Z \) is a unique policy compromise point for the coalition \( M \), and \( \sigma_M(z) \) is a division of the “perquisites” among the leaders of the parties in \( M \).

This recent work has utilized models of post-election bargaining to infer the nature of \( \tilde{g}_i(z) \), for given \( \mathcal{D}_2 \), and fixed \( z \in Z^n \). If we denote the image space of \( \tilde{g}_i \) as \( \tilde{W} \), then we may more generally conceive of \( \tilde{g}_i \) as a map \( \tilde{g}_i: Z^n \rightarrow \tilde{W} \). We assume that \( \tilde{W} \) is endowed with the weak topology (Parthasathy, 1967) and that for each \( t \), \( \tilde{g}_i: Z^n \rightarrow \tilde{W} \) is continuous, with respect to these topologies.

Although \( \tilde{g}_i \) describes the result of bargaining among party leaders, in state \( t \) (or \( \mathcal{D}_2 \)), we may view it as a social welfare function that encapsulates the nature of coalition risk in the post-election world described by \( \mathcal{D}_2 \) and any chosen vector of party leader positions. (As we have emphasized, the vector, \( z \), of party positions encodes the preferences of party leaders.) We assume that for each \( t \), the coalition map, \( \tilde{g}_i: Z^n \rightarrow \tilde{W} \), is common knowledge to the political elite. The game form \( \tilde{g} = \tilde{H}(p, g): Z^n \rightarrow \tilde{W} \) is given by \( \tilde{g}(z) = \{(p(z), \tilde{g}(z)), t \in T\} \). This map describes both electoral and coalition risk and is both common knowledge and continuous, under our assumptions.

Having set up the general model, we are in a position to describe what we mean by a Nash equilibrium to the game form \( \tilde{g} \).

We assume that each elite member \( k_i \) (in party \( i \)) has separable preferences derived from a utility which takes the form \( u_{k_i}(y, \sigma_i) = -\frac{1}{2}\|y - \alpha_i\|_2^2 + \sigma_i \), whenever a policy point, \( y \), is chosen and a portfolio perquisite, \( \sigma_i \), is allocated to the \( i \)th party. The point \( \alpha_i \) is usually referred to as \( k_i \)'s bliss point. Without great loss of generality, we may suppose that the pre-election decision maker, or principal, of party \( i \) is that party member, \( k_i \), whose bliss point \( \alpha_i \) is at the multi-dimensional median of the bliss points of the members of the same party. We denote the utility function of the principal of party \( i \) as \( u_i \) and let \( \alpha_i \) denote this principal’s bliss point. We also refer to \( \alpha_i \) as the “sincere”
policy point\(^1\) of party \(i\). In the obvious fashion this defines a function \(U_i = \tilde{u}_i \circ \tilde{g} : Z' \to \tilde{W} \to \mathfrak{P}\), where \(\tilde{u}_i\) is the von Neumann-Morgenstern extension of the function \(u_i\) to the lottery space \(\tilde{W}\). If we let \(\tilde{Z}\) and \(\tilde{Z}'\) be the lottery spaces over \(Z\) and \(Z''\) endowed with the weak topology, then we can\(^2\) define \(\tilde{U} : \tilde{Z}' \to \mathfrak{P}\). A Nash equilibrium of \(\tilde{U} : \tilde{Z}' \to \mathfrak{P}\) is a vector \(\tilde{z}^* \in \tilde{Z}'\) such that, for each \(i = 1, \ldots, n\), it is the case that \(\tilde{U}_i(\tilde{z}_i^*, \ldots, \tilde{z}_i, \ldots, \tilde{z}_n) > \tilde{U}_i(\tilde{z}_1^*, \ldots, \tilde{z}_i^*, \ldots, \tilde{z}_n)\) for no \(\tilde{z}_i \in \tilde{Z}\). A pure strategy Nash equilibrium (PSNE) is a Nash equilibrium, \(z^*\), under the restriction that \(z^* \in Z''\). A vector \(z^* \in Z''\) is a local pure strategy Nash equilibrium (LNE) iff there is a neighborhood \(V\) of \(z^*\) in \(Z''\) such that \(z\) is a PSNE where each \(U_i\) is restricted to the domain \(V\).

Recent work by Schofield and Sénéd (1998) has considered a model of the above form under the stronger assumption that the “game form” \(\tilde{g}\) is differentiable, and has argued that \(\tilde{g}\) will generically exhibit a LNE.

Since we have assumed here that \(\tilde{g}\) is continuous, it follows naturally from the usual application of Glicksberg’s Theorem (Glicksberg, 1952) that the game form \(\tilde{g}\) exhibits a (mixed strategy) Nash equilibrium. We contend that local pure strategy Nash equilibria can give insight into political choice, when party principals reason about the choice of party leaders in a context where both electoral and coalition risk are relevant. Thus, we assume that the electoral operator, \(\Psi\), is common knowledge, and continuous in \(z\). We envisage that the parties may sample electoral preferences, and have access to the same unbiased information. Moreover, once the election is over and a class, \(D_i\), of winning coalitions is electorally determined, all parties may compute the probability and outcome for each coalition, as determined by \(z\). A pure strategy Nash equilibrium is a vector \(z^* = (z_1^*, \ldots, z_n^*)\) of policy positions chosen prior to the election. At \(z^*\), each party principal has a party leader, with preferred policy point \(z_i^*\), to represent party \(i\) both to the electorate and in post-election coalition bargaining. The policy \(z_i^*\) we also refer to as the equilibrium strategic choice of party \(i\). There is obviously no reason why the party principal’s bliss point, \(o_i\), and the party leader’s bliss point, \(z_i^*\), should coincide. However, prior to the election the policy \(z_i^*\) is believed by the \(i\)th principal to be the best policy to offer to the electorate (given the other parties’ declarations). It should be evident that we implicitly assume that the party is committed, prior to the election, to the party leader’s position. This follows because the leader of party \(i\) chooses, or indeed dictates, the party manifesto, and necessarily attempts to implement

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\(^1\)Suppose that the delegates in each party vote “sincerely” for a policy to offer to the electorate. This “sincere” voting equilibrium or “core” for the party can be interpreted as that position which is unbeaten by majority rule within the party. Even if there is no core given the distribution of bliss points inside the party, the so-called “uncovered” set (Banks et al., 1998; Cox, 1987; McKelvey, 1986), within the party will be non-empty, small and located close to this “multi-dimensional median” (Schofield, 1999). Thus, for each party \(i\), the principal’s preferred position \(o_i\), can be interpreted as a proxy for the sincere policy choice of the party.

Section 3 gives more formal definitions of \(\tilde{W}, \tilde{Z}, \tilde{Z}'\). Briefly, \(W = Z \times \Delta\), where \(\Delta\) is a simplex in \(\mathfrak{P}\) representing all possible distributions of perquisites among the government members. If \(u_i(c, \sigma) = u_i(c, \sigma')\) for any \((c, \sigma) \in W\), then \(u_i'\) can be extended to \(u_i : \tilde{W} \to \mathfrak{P}\). Composition with the game form \(\tilde{g}\), and extension to \(\tilde{Z}'\), then gives (under certain conditions) a continuous utility function \(\tilde{U} : \tilde{Z}' \to \mathfrak{P}\) for each \(i \in N\). The function \(\tilde{U} : \tilde{Z}' \to \mathfrak{P}\) so obtained describes the political game that we consider.
the policy positions, $z^*$, in post-election bargaining. It would of course be possible to modify the model to allow for party leaders to be overthrown immediately after a low-probability election outcome. Such occurrences seem relatively rare in multiparty polities, and would make an already complex model even more cumbersome. From our point of view the equilibrium vector $z^* = (z_1^*, \ldots, z_n^*)$ is a message both to the electorate and to all parties. Although the vector of sincere policy positions $o = (o_1, \ldots, o_n)$ is, in principle, computable, it is immaterial to the calculations of the electorate and to the parties in post-election bargaining, once $z^*$ has been collectively chosen.

As we have emphasized, one purpose of the theoretical model proposed here is to attempt to account for the fact that parties, in typical multiparty situations, do not “converge” to an electoral center. Indeed Duverger (1954) noted this characteristic of certain polities, but gave no formal justification for his intuition. “Downsian” models of electoral competition typically assume that parties attempt to maximize “power”, whether measured in terms of the vote shares, seat shares, or “probability of winning”. Nearly all such models conclude that there is a “centripetal” tendency of convergence in Nash equilibrium to an electoral center. Even “Rikerian” models of post-election coalition bargaining have implicit within them a centripetal tendency. For example, Axelrod’s (1970) earlier work essentially posited that, within a one-dimensional policy space, the “median” party (whose leader’s policy point was at the median of the legislature’s policies) would for certain belong to the government coalition. It is intuitively obvious that if the party principals strongly desire to belong to government (in the sense that their preferences for perquisites dominates their policy preferences), then a pure strategy Nash equilibrium would coincide with all party principals choosing (if possible) an identical policy position at the median of the principals’ bliss points. A similar inference can be drawn from the two-dimensional multiparty model offered by Grofman (1982, 1996). This model suggests that the government coalition that exhibits the lowest variance in the bliss points of the party principals within the coalition is the one that is created. Again, if perquisites dominate over policy preferences, then the “Nash equilibrium” in policy choices would result in minimal (zero) variance in the declared policies.

Since electoral risk can be seen to generate a “centripetal” tendency, (Cox, 1990), we seek a counter-centripetal or centrifugal tendency resulting from coalition risk. To study the simplest case, we consider a three-party situation in two dimensions, assuming first of all that the electoral risk can be ignored. Suppose that, for every vector $z$ of declarations, it is the case that no one party may gain a majority of the seats. In this case there is a fixed coalition structure, which we denote by $\mathcal{D}_1$. Without loss of generality we assume that $\mathcal{D}_1$ is the coalition structure comprising each of the three different two-party coalitions, say $\{1,2\}$, $\{2,3\}$, $\{1,3\}$.

To analyze this situation in detail we make specific assumptions about the relationship between the lottery $\tilde{g}(z)$ and $z$. Because post-election bargaining is carried out by the party leaders, and because their bliss points $\{z_1, z_2, z_3\}$ coincide with the party declarations, we assume all coalition policies lie inside the convex hull of $\{z_1, z_2, z_3\}$. For simplicity we further assume that coalition $M = \{i, j\}$, if it occurs, chooses a compromise policy point $z_{ij} = \frac{1}{2}(z_i + z_j)$. An alternative, of course, would be that such a coalition chooses a weighted sum of the leaders’ positions (where the weight is
functions (and thus their bliss points). We say a PSNE of party leader, with preferred point \( z \)
showed that parties in most of the European polities could increase vote and seat shares.

The probability functions \( p \) and \( h \) of \( z \) gives the general result. (Theorem 3 below gives a more general version of this result.) On the other hand, if the principals’ bliss points are colinear, and ranked \( o_1 < o_2 < o_3 \), say, then there is a PSNE \( z^* \) with \( \{z_1^*, z_2^*, z_3^*\} \) colinear and all “close” to \( o_2 \). (See Theorem 2 for a more general result.) Again, in the symmetric case, if \( \|o_i - o_j\| = r \) for each pair \( i, j \), and any party in a government coalition can be certain of obtaining a perquisite of value \( \sigma \) greater than \( \frac{1}{5} r^2 \), then there is a “convergent” PSNE such that \( \|z_i^* - z_j^*\| < \|o_i - o_j\| \) for each pair, \( i, j \). (Theorem 4 gives the general result.)

Because the coalition probabilities are given in terms of the distances, \( \|z_i - z_j\| \), it is necessary to deal with the question of potential singularities or degeneracies as \( z_i \to z_j \), say. For example, suppose that two principals \( \{i, j\} \) were to choose party positions \( \{z_i, z_j\} \) with \( z_i = z_j = \frac{1}{2} (o_i + o_j) \). Then under our assumptions \( a_{ij} = a_{ji} = z_i = z_j \), and \( \rho_{ij} = 1 \) irrespective of \( z_i \). Although this is a PSNE, it need not be stable in the following sense.

We envisage that the process by which the PSNE is attained occurs via the reiteration of the “best response” correspondence \( h: Z^3 \to Z^3 \), induced by the three principals’ utility functions (and thus their bliss points). We say a PSNE \( z^* \) is stable if, for any neighborhood \( V \) of \( z^* \) in \( Z^3 \), there is a proper subneighborhood \( V' \subset V \) of \( z^* \) such that if \( z \in V \), then \( h(z) \in V' \). Certain degenerate PSNE of the kind just considered can be shown not to be stable. Such unstable PSNE are eliminated from the analysis.

The paper is structured in the following way. The effect of “convergence” in Downsian models of electoral competition is discussed in Section 2. Recent work in this tradition has used Markov Chain Monte Carlo techniques to estimate a multinomial probit model (Quinn et al., forthcoming; Schofield et al., 1998a, 1999). This empirical work is presented to justify the assumption, introduced above, that the electoral probability functions \( p_i: Z^2 \to [0, 1] \) are continuous. Moreover, these analyses have shown that parties in most of the European polities could increase vote and seat shares...
by moving to the electoral center. Counter to this observation, however, a detailed analysis of the Netherlands in 1979 suggests, first of all, that no major party chose a leader near the electoral center. Secondly there is some evidence from the Dutch elections of 1977 and 1981 that at least one party adopted a strategic “radical” position. A very simple bargaining model is offered in Section 2 to illustrate how a centrifugal tendency may balance the centripetal tendency induced by vote maximization. Section 3 sets out the general model of choice of party leader and position in a political game with both electoral and coalition risk. Section 4 presents the detailed assumptions of the three-party model, without electoral risk. As just indicated, the main theme of the results obtained is that if perquisites are of sufficiently high value for each party, then there will exist a “convergent” stable pure strategy Nash equilibrium. Conversely, with perquisites of low value, and certain configurations of principals’ bliss points, there exist divergent stable PSNE. Various theorems are presented which give the conditions for divergence and convergence in stable Nash equilibrium strategies. This section also discusses results obtained through computation of the model. Section 5 draws out implications of the model to argue for existence of local pure strategy Nash equilibria in the general model with electoral risk. Section 6 concludes. All formal proofs are presented in Appendix A.

2. Convergence or divergence in spatial models

The electoral models based on the early work of Hotelling (1929) and Downs (1957) essentially suppose that the motivation of parties is to win a majority of the votes or seats. While this is a reasonable assumption in two-party situations, where there usually will be a winner, it is not so clear that it matches the incentives of parties in multiparty situations where coalitions are typically necessary to construct governments.

To introduce the complexity of party decision-making in a spatial policy context, consider Fig. 1.

This figure is adapted from earlier work (Quinn et al., forthcoming; Schofield et al., 1998a) on the Netherlands, using survey (Rabier and Inglehart, 1982) and elite (ISEIUM, 1983) data. Factor analysis was used on a survey of approximately 1000 voters in 1979 to generate a two dimensional space, \( Z \), representing voter beliefs on economic and social issues (the background to this figure illustrates an estimated probability density function of voter “preferred” policy points). “Elite” data for the delegates of the four parties were also interpreted in identical fashion to obtain a distribution of delegate positions. The two dimensional median, for each of these parties, was then calculated, to give the position of the “party principal”, as described in the previous section. The “positions” of these four parties are given in Fig. 1 (CDA is the Christian Democratic Appeal, VVD is the Liberal Party, D66 are the Democrats ’66, and PVDA is the Labor Party). The positions of voters and parties are normalized about (0, 0), the mean of the voter distribution. The “economic dimension” is constructed from factor loadings on a number of questions involving income distribution, control of public enterprises, etc. As expected, the median PVDA delegate position on this scale is to the left of the median D66 position, etc. The second, “social”, dimension may be thought of in terms of a Libertarian/scope of government scale. The key question that characterized the “social” scale concerned the freedom of women to decide on abortion. As may be
expected, the median CDA delegate position was very different from the other party
delegate medians. A typical CDA delegate disagreed very strongly with the proposition
that women should be free to decide on abortion. The left–right ranking of the parties
agrees with most scholarly analyses (Daalder, 1987; Laver and Schofield, 1990). The
existence of a second, social dimension adds a complexity that makes the usual
Downsian model difficult to apply.

We can however use the sample of voter preferred points to perform a thought
experiment to examine party behavior under the usual vote maximizing assumptions.

As usual in spatial models, we suppose the “utility” of voter $v$ for party $i$ given the
positions of the four parties is $u_{vi} = -\beta d_{vi}^2$, where $d_{vi}$ is the distance between
the position of voter $v$ and the position of party $i$, as believed by voter $v$. Thus voter $v$
chooses party $i$ iff $u_{vi} > u_{vj}$ (for all $j \neq i$). Under such a “deterministic” voting model,
we can estimate how election results change as parties modify their positions. As Table
1 illustrates, the PVDA gained 38% of the popular vote (of the four major parties) in the
1977 election, taking 53 seats out of the 138 controlled by these four parties. Had it
located itself at the center (0, 0) of the electoral distribution, we estimate (by a method
to be described below) that it would have gained approximately 45% of the vote (or 62
seats). Assuming the other three parties kept their positions, and ignoring the behavior
of six small parties (controlling 11 seats), the PVDA would not have controlled a majority
(requiring 75 out of 149).

If this move by the PVDA to the center had been associated with an appropriate
choice of party leader, so as to commit the party to this centrist position, then it is likely
Table 1

Vote shares in the Netherlands’ elections, 1977 to 1981

<table>
<thead>
<tr>
<th>Party</th>
<th>Share of national vote, %</th>
<th>Share of national vote, %</th>
<th>Sample share in 1979</th>
<th>Estimated share</th>
<th>95% Confidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>D66</td>
<td>6.1</td>
<td>12.6</td>
<td>10.4</td>
<td>10.6</td>
<td>(3.8, 18.2)</td>
</tr>
<tr>
<td>PVDA</td>
<td>38.0</td>
<td>32.4</td>
<td>36.9</td>
<td>35.3</td>
<td>(30.9, 39.7)</td>
</tr>
<tr>
<td>CDA</td>
<td>35.9</td>
<td>35.2</td>
<td>33.8</td>
<td>29.9</td>
<td>(25.5, 34.2)</td>
</tr>
<tr>
<td>VVD</td>
<td>20.0</td>
<td>19.8</td>
<td>18.9</td>
<td>24.2</td>
<td>(20.8, 28.0)</td>
</tr>
</tbody>
</table>

* Vote shares are very close to seat shares, because of proportional representation. Shares are calculated on the basis of total vote to the four large parties, using Keesing’s Contemporary Archives.

* Based on Euro-barometer sample with sample size 1000.

* Estimated share is the expectation of the MNP probit vote share variables.

...that the party would have been able to ensure itself membership in one of the government coalitions (probably with the CDA). Then, however, the government policy would be one that was quite close to the position of the CDA party principal’s position. Whether such a move would be rational for the PVDA delegates would depend both on their valuations of policy choice and government perquisites, and on calculations about changing coalition probabilities.

If we continue with the thought experiment, vote maximization obviously drives all four parties to adopt centrist positions near the mean voter position, (0, 0). At this point, under the deterministic voter rule, all four parties gain the same popular vote. The deterministic voter rule has the disadvantage of discontinuity: if the four parties are located at (0, 0), then each has an incentive to move away from the center, so as to increase its vote (Osborne, 1993). However, it is not clear that such a move by the PVDA, say, implemented so as to increase its vote share from 25% to 26%, would increase its power to affect coalition policy to its advantage.

In two-party deterministic voting models, it is well-known that pure-strategy Nash equilibria (under the vote maximizing hypothesis) will “generically” not exist in “high” dimensions (McKelvey and Schofield, 1986; Saari, 1996). However, mixed strategy Nash equilibria may exist under certain conditions (Kramer, 1978), and their support will generally lie inside a subset of the support of the electoral distribution, known as the “uncovered set” (McKelvey, 1986; Cox, 1987; Banks et al., 1998).

Computation of MSNE (under vote maximization in the deterministic model) is a difficult task because of discontinuity in the electoral response function (see Dasgupta and Maskin (1986) for theoretical results on a class of models of this kind).

Existence results are easier to obtain in the probabilistic voting model. Here the realized utility of voter $v$, for party $j$, given the party position, $z_j$, is $u_{vj} = u_{uv} + \varepsilon_j$, where $\varepsilon_j$ is a perceptual error term. The standard two-party probabilistic models (Hinich, 1977; Enelow and Hinich, 1984; Coughlin, 1992), assume the error terms $\{\varepsilon_j\}$ are iid, in that they are sampled from independent, identical normal distributions with specified variance. In such a model the probability $\chi_{vi}$ that voter $v$ chooses party $i$ is

$$
\chi_{vi} = \text{Prob}(u_{vi} > u_{vj}) = \text{Prob}((\varepsilon_i - \varepsilon_j) > \beta(d_{vi}^2 - d_{vj}^2)), \text{ for all } j \neq i.
$$

(1)
Such a model can be readily extended to the case of three or more parties. Since the voter probabilities are derived from the distance matrix \((d_{ij})\) and these are determined by the vector, \(z\), of party positions, it is possible to compute the expected vote-share vector \(E(\Psi(z))\). Here the expected vote share for party \(i\) is simply \(1/|K| \sum_{v \in K} \chi_{vi}\), where \(|K|\) is the size of the electorate.

Lin et al. (forthcoming) have recently shown that, under the two assumptions of iid errors and expected vote maximization, there is a PSNE, \(z^*\). This result clearly depends on the continuity of the expected vote function, and for this it is necessary that the error variance be sufficiently large. For vanishingly small error variance, the probabilistic vote model will lose continuity and resemble the deterministic vote model, in the sense that PSNE may not exist.

In empirical applications, estimation of the error distribution and of the \(\beta\) constant, requires knowledge of actual voter choices. Recent empirical work (Quinn et al., forthcoming; Schofield et al., 1998a), based on the party positions given in Fig. 1, and using survey responses over voter intended choice, estimated a multinomial probit (MNP) electoral model, which we shall denote by \(\Psi\). Instead of assuming iid errors, the covariance structure, \(\Sigma\), of the errors was estimated using a Bayesian iteration technique to construct the maximum likelihood estimator. It is important to note that this MNP does not assume that all voters vote sincerely for the nearest party. As we observe below, the covariance structure \(\Sigma\), on the errors, captures the possible complicated strategic calculations of the electorate. This “pure” spatial model just described was compared with a “sociological” model of voting (based on individual properties such as religion, income, demography) and a “retrospective” model (based on satisfaction with government choice). Analysis of the Bayes factors of these various models made clear the superiority of the probabilistic spatial voting model.

Table 1 reports the estimated expected vote shares \(E(\Psi(z))\) among the four parties, using the vector \(z\) of party positions as given in Fig. 1. The estimation, of course, depends on the sample shares. Table 1 indicates that the estimated shares are close to sample shares and national vote shares (in the two elections of 1977 and 1981).

Notice one anomaly: the VVD vote share is estimated at 24% (or 36 seats in the 1977 election). In fact the VVD gained only 28 seats in 1977. One way to address this misestimation is to modify the utility model for the voters by assuming that \(\bar{u}_{vi} = \beta - \beta d_{i}\), where \(\beta\) is a party specific term. The procedure allowed an estimation of these party specific constants, with the constants for the VVD and D66 found to be lower than for the CDA and PVDA. The consequence was to change the VVD expected vote share to 18.9% (with confidence interval ranging from 14.4% to 23.8%). Although this modification of the spatial model is reasonable for reasons of prediction, it has no obvious theoretical justification. However, a plausible justification for the modification could be in terms of strategic choice by voters (preferring “large” parties over “small”, for reasons other than policy) or because of voter belief in various levels of competency of the parties.

A second point to note is that the small D66 party has (by our estimation) a sincere policy point close to the center, \((0, 0)\), of the electoral distribution. Under the usual stochastic voter model assumptions with iid errors, the D66 would gain the greatest share of the popular vote. Table 1 clearly indicates that this was not so. The MNP model
does however allow for differing variance in the error terms. This reflects, in our view, the possibility that uncertainty in the electorate over party capabilities can outweigh the proximity to the electoral center.

The more general point to note is that the usual Downsian stochastic voter models do not allow for strategic behavior on the part of members of the electorate. In contrast, the MNP model is compatible with strategic voter behavior, and gives insight into its effects.

The MNP model can also be used to examine the possibility of strategic behavior by parties. As discussed in the previous section, the principal of party $i$, with preferred position $o_i$, may strategically choose a party leader, with position $z_i$, different from $o_i$, because of the principal’s understanding of the political game. In contrast, the MNP analysis that we have just outlined used the vector $o = (o_1, o_2, o_3, o_4)$ of sincere party positions to compute the distance matrix $(d_{ij})$. If we were to suppose instead that the VVD declared position was at $(1.2, -1)$ instead of $(1.2, -0.2)$ in Fig. 1, then we would obtain a precise match between sample vote shares and estimated vote shares, without the need to introduce party specific constant terms in the voter model. To see why such a strategic position by a party may be rational, we now explore the coalition bargaining possibilities inherent in Fig. 1. To ascertain the possible coalition structures, let us identify $\{PVDA, CDA, VVD, D66\}=\{1, 2, 3, 4\}$. To simplify the analysis, let us ignore the impact of the various small parties in the parliament, and concentrate on the two most obvious coalition structures. The first, $D_1 = \{1, 2, 1, 3, 2, 3\}$, is the most likely given our estimates of the variance of the electoral operator $\Psi(o)$, at the vector of party positions, $o$, represented in Fig. 1. Note that $D_1$ did indeed occur at the 1977 election. The second coalition structure $D_2 = \{1, 3, 4, 1, 2, 2, 3, 4\}$ is somewhat less likely, given our estimation for 1979. In the 1981 election, the coalition $\{VVD, CDA\}=\{3, 2\}$ only gained 72 seats out of a possible 150. To obtain a majority, this two party coalition required the D66. In fact after the 1977 election, $\{VVD, CDA\}$ first formed.

One advantage of the estimation procedure is that it allows us to determine the relationship between any vector of declared party positions, $z$, and the “stochastic” vote share vector described by $\Psi(z)$. As in the case of deterministic voting, all parties could increase expected vote shares by moving to the electoral center. Had the three other parties’ positions remained fixed at the positions given in Fig. 1, the PVDA could have increased its expected vote share to 45% of the four party vote, or 60\pm4 seats, in 1981, by moving from its sincere position $o_1 = (-1, 0)$ to the mean position $(0, 0)$. Similarly, all four parties could have increased vote share by moving to the electoral center.

Unlike the deterministic voting model, the expected vote-share operator

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Notice that the MNP electoral operator $\Psi$ is stochastic, so that at any vector $z$ of party positions, there is significant variance in the predicted vote shares. The empirical probabilities $p_1$ and $p_2$ of the two coalition structures $D_1, D_2$ can be computed directly from the information encoded in $\Psi$. Of course, our estimation gave vote shares, not seat shares. However, the Dutch electoral system is highly “proportional”, so no significant error is induced from estimating $p_1$ and $p_2$ directly from $\Psi$. 
mapping to the 3-dimensional simplex of vote shares, is continuous in the vector of party positions. Indeed, if we identify \( E(\Psi) \) as the payoff function, \( u = (u_1, u_2, u_3, u_4) \), then each \( u_i \) is strictly pseudo-concave, in the appropriate component. Clearly the declaration \( z^* \), where each \( z_i = (0, 0) \) is a stable PSNE, under vote maximization in the context of the MNP electoral model.

However, there is no evidence that any of the three major parties in the Netherlands adopted a position near the center of the electoral distribution (see also Budge et al., 1987; Laver and Schofield, 1990). Other work (Schofield et al., 1998b, 1999; Sened, 1996) has made the same inference based on MNP electoral models of various European polities as well as Israel. These empirical analyses lead to the conclusion that expected vote maximization is not an adequate model for explaining party choice in these varied polities. The analyses have also indicated that in most cases it is only necessary to base the analysis on a small number \([T]\) of possible coalition structures.

The MNP electoral model has one feature which suggests that it can be used to construct an equilibrium model of party positioning. In the context of the MNP electoral operator, the probability function \( p: Z^4 \rightarrow \Delta \) is continuous (indeed smooth). Here \( p = (p_1, \ldots, p_4, \ldots) \) has image in the \( [T]-1 \) dimensional simplex, and \( T \) indexes the states of the political world.

This allows us in principle to separate the game form \( \tilde{g} \), described in the previous section, into its various components \( \tilde{g}_j \), and to determine whether Nash equilibria exist that exhibit a centrifugal tendency.

To explore such a possibility, consider the fixed post-election coalition structure \( D_2 = \{\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}\} \) that in fact resulted from the 1981 election. As we have indicated, had the PVDA principal chosen a party leader at the mean electoral position \((0, 0)\), we estimate it would have gained approximately 60 seats, leaving a majority (78 seats) to the three other parties. Under the assumption of Euclidean preferences, the “legislative core” of this spatial game would be empty. That is the coalition \{CDA, VVD, D66\} would be able to find a policy point that their leaders preferred to \((0, 0)\). However, were the PVDA to appoint a centrist party delegate as leader, with preferred policy \( z_1 \), say, lying inside the convex hull of the positions \{CDA, VVD, D66\} as given in Fig. 1, then this point would be a legislative core.\(^4\) Both empirical (Schofield, 1995a; Schofield et al., 1998a,b; Sened, 1996) and theoretical work on coalition bargaining (Banks and Duggan, 1998) with a fixed coalition structure suggests that this point, \( z_1 \), would be the policy outcome. Moreover Laver and Schofield (1990) suggest that the PVDA would, in such a situation, be able to effect a minority government, and control (under plausible conditions) all government perquisites. Whether such a choice is rational

---

\(^4\)As observed in the previous footnote, the MNP electoral operator gives vote shares, not seat shares. Electoral systems based on majoritarian principles, rather than proportionality, may indeed introduce discontinuities in the probability function. As a first order approximation, however, the probability function \( p \) in a proportional electoral system will be smooth.

\(^5\)That is, this anti-PVDA coalition \{CDA, VVD, D66\} would be able to find no policy point that the three party leaders preferred to \( z_1 \).
for the party would depend, of course, on the balance within the party over preferences for policy and perquisites. However, if a centrist position was deemed rational by the PVDA, then it would also be rational for the three other parties to choose leaders with such centrist policy preferences.

Thus empirical and theoretical work based on a “bargaining model” of negotiation in the context of a fixed coalition structure $\mathcal{D}_2$ provides no obvious explanation as to why parties do not converge to an electoral center.

To provide an intuitive explanation of why a party such as the VVD could rationally choose a position more “radical” than the sincere position of its principal, consider Fig. 2. For simplicity let us ignore the role of the D66, and consider the fixed coalition structure $\mathcal{D}_1$. Let $\{\text{PVDA, CDA, VVD}\}$ represent the sincere policy positions of these three parties. It is natural to assume that policy outcomes will occur within the triangle or convex hull of the positions, $\{(\text{VVD, PVDA, CDA})\}$. Whatever the specific lottery, $\tilde{g}_1$, of outcomes, let $U(\{\text{VVD, PVDA, CDA}\})$ denote the value of the von Neumann-Morgenstern utility for the VVD principal resulting from this lottery. Now consider a situation where the VVD principal chooses a party leader at the position VVD'. Denote by $U(\{\text{VVD', PVDA, CDA}\})$ the von Neumann Morgenstern utility for the VVD
principal resulting from this choice. It is natural to expect that symmetry properties of the lottery give the equality

\[
U(\{PVDA, CDA, VVD'\}) = \frac{1}{3}U(\{VVD', VVD, PVDA\}) + \frac{1}{3}U(\{VVD', VVD, CDA\}) + \frac{1}{3}U(\{VVD, PVDA, CDA\}).
\]

Because of the symmetry of the figure, and the assumption of Euclidean preference we obtain

\[
U(\{VVD', PVDA, CDA\}) = U(\{VVD, PVDA, CDA\}).
\]

If we further assume that \(U\) is continuous, then there will be some position, VVD" say, on the arc [VVD, VVD'] which maximizes \(U\). This position is the strategic best response to \{CDA, PVDA\} by the party principal of VVD. If we now consider other delegates for this party with preferred positions symmetrically distributed about VVD, then their best responses will also be symmetrically distributed about VVD". Thus the majority rule choice for best response by the party will be to choose a leader with preferred policy position at VVD". Note that the motivation to choose VVD" satisfies the centrifugal tendency proposed earlier.

To actually compute VVD" it is obviously necessary to make some further assumptions on the nature of the lottery, \(g\). In the section that follows we make the following extremely simple assumption. Under the coalition structure \(\mathcal{D}_1\), we suppose that each of the two-party “bounding” coalitions forms with probability inversely proportional to the square of the distance between the declared points of the two parties. Further, we suppose that each such coalition adopts a compromise policy at the mean of the declared positions of the two parties forming the coalition.

Thus, given a declaration at the position VVD, the coalition \{CDA, PVDA\} forms with probability \(\frac{1}{3}\) and adopts a policy a unit distance from the ideal point VVD, while \{PVDA, VVD\} and \{CDA, VVD\} each form with probability \(\frac{2}{3}\) and each adopt policies a unit distance from VVD.

At the declaration VVD', all three coalitions are equiprobable. Each coalition adopts a policy a unit distance from VVD. Thus utility from declaring VVD is identical to the utility for declaring VVD'.

The optimal position VVD" is such that the point \(C\), the midpoint of the arc \{PVDA, VVD', VVD\}, minimizes the distance to VVD. At such a position, the probability that coalition \{VVD, PVDA\} forms is \(\frac{12}{31}\), while \{CDA, PVDA\} forms with probability \(\frac{17}{31}\). Notice that this argument on the optimality of VVD" only depends on the assumption that policy preference is Euclidean. For example if we assume that the policy utility for VVD is of the form \(u(z) = -\|z - VVD\|\), then the von Neumann Morgenstern utility at the positions VVD or VVD' is \(-1\), while at VVD" it is \(-0.99\). With a policy utility of the form \(u(z) = -\|z - VVD\|^2\), we obtain utility at VVD of \(-1\) and at VVD" of \(-0.8\).

Clearly the idea here is that, under certain conditions, it is rational for a party to adopt a more radical position, even if it lowers the chance of coalition membership, as long as
some of the eventual policy outcomes, associated with the lottery \( \tilde{g}_i \), are changed to the advantage of the party principal.

Note however, in the example, that the probability that VVD will be a member of a majority coalition drops from \( \frac{5}{7} \) to \( \frac{24}{31} \) as the party moves to the more extreme position. If coalition members obtain perquisites from being in government, then for sufficiently high value perquisites, the expected utility of the VVD principal at VVD' would be lower than at VVD. Thus coalition perquisites may transform the centrifugal tendencies generated by policy negotiations to centripetal tendencies, similar to those generated by electoral concerns.

The above discussion concentrated on the best response of a single party, namely the VVD, to the location of the other two parties. To deal formally with the strategic choices of parties in the bargaining game just described we now consider the nature of the Nash equilibria in such a game.

### 3. Formal preliminaries of the political game

In the previous section we supposed that the set of parties \( N = \{1, \ldots, i, \ldots, n\} \) was exogenously given, as was the set \( K = \{1, \ldots, j, \ldots, k\} \) of voters. In fact a deeper model would be to suppose that there is a subset \( K' \) of \( K \) of elite actors who form coalitions called parties, thus generating \( N \). Some of the results presented below suggest how groups of elite actors might cohere to form parties.

The political game takes place in a policy space, \( Z \). For ease of analysis we shall assume that \( Z \) is two-dimensional, but there is no impediment to assuming \( Z \) is of higher dimension. Each voter, \( v \), has a quasi-concave utility function \( u_v: Z \to \mathbb{R} \), which we can (without great loss of generality) assume is Euclidean, and takes the form \( u_v(y) = -a\|y - x_v\|^2 \), where \( x_v \) is the voter’s ideal point and \( a \) is a positive constant. Each party, \( i \), makes a declaration \( z_i \), so \( z = (z_1, \ldots, z_n) \in Z^n \) is the policy or “manifesto” profile. Let \( \Delta_N \) be the \((n-1)\) dimensional unit simplex. A vector \( v = (v_1, \ldots, v_n) \in \Delta_N \) represents the shares of the vote for the parties, and a vector \( e(v) = (e_1, \ldots, e_n) \in \Delta_N \) represents the share of seats of each of the parties.

The vector \( e(v) \) is, of course, determined by the nature of the electoral system in use. The two vectors \( v \) and \( e(v) \) represent the post-election realizations of the behavior of the electorate. Prior to the election, all agents hold common knowledge beliefs about the stochastic relationship between the vector \( z \in Z^n \) of party declarations and the electoral response. In particular we shall assume that the response by voter \( v \) is described by a continuous probability function

\[
\chi_v: Z^n \to \Delta_N.
\]

Thus \( \chi_v(z) = (\ldots, \chi_{v_i}(z), \ldots) \), where \( \chi_{v_i}(z) \) is the probability that voter \( v \) picks party \( i \) at the manifesto profile \( z \). The MNP model (Quinn et al., forthcoming; Schofield et al., 1998a, 1999) mentioned previously does not assume that voter behavior is pairwise independent; nor does it assume away the possibility of strategic voting in the electorate. However MNP does permit an estimate of the stochastic vote function \( \Psi: Z^n \to \Delta_N \). Here
\(\Psi(z)\) is a continuous random variable, whose components are the random variables characterizing the vote shares of the various parties. Obviously \(\Psi\) is determined fundamentally by the probability functions \(\{\chi_c\}\), and the nature of the covariance structure on the error function.

In particular, there is an expectation operator \(E(\Psi): Z^n \rightarrow \Delta_n\), where \(E(\Psi(z)) = E(\Psi(z))\) is the expectation of the vote shares at the profile \(z\). In the same way \(e(\Psi(z))\) is the random variable describing the seat shares of parties at \(z\), and \(E(e(\Psi(z)))\) is the expectation of this vector. Given any realized vector, \(e\), of seat shares, we can compute the class of winning coalitions, \(\mathcal{D}\) say, where coalition \(M\) is a member of \(\mathcal{D}\) iff \(\sum_{i \in M} e_i > \frac{1}{2}\). Let \(2^N\) be the family of all subsets of \(N\). Clearly a family of winning coalitions, \(\mathcal{D}\) say, is a subset of \(2^N\). Thus the information encoded in \(e(\Psi(z))\) can be interpreted as a probability distribution over subsets of \(2^N\). In other words the choice of the policy vector \(z\) determines a finite lottery \(\{(p_t(z), \mathcal{D}_i): t = 1, \ldots, n^*\}\). Here \(\mathcal{D}_1, \ldots, \mathcal{D}_{n^*}\) are all the coalition structures that are believed possible, and \(p_t(z)\) is the probability associated with \(\mathcal{D}_t\) at \(z\). A particular election result is a realization, say \(\mathcal{D}_t\), from this set of possible coalition structures. As noted above, one property that the MNP voting model possesses in the context of a proportional electoral system is that these probabilities, \(p_t\), are smooth functions of \(z\). We emphasize this by declaring it as an assumption.

**Assumption 1.** The electoral probability function \(p: Z^n \rightarrow \Delta\) is a smooth function from \(Z^n\) to the simplex \(\Delta\) (of dimension \(n^* - 1\)).

Note that in general \(n^*\) is of order \(2^n - 1\). As we observed in the Introduction, in the case \(n = 2\) there are three possibilities, either party 1 wins, or party 2 wins or 1 and 2 gain exactly the same number of seats. The situation with \(n = 3\) is obviously more complicated, since there are three cases where a single party wins, and three cases where one party gains no seats and the other two have the same number of seats. Finally there is the non-degenerate case where each two-party coalition wins, so \(\mathcal{D} = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}\). For convenience we may ignore the grand coalition \(\{1, 2, 3\}\). Clearly the formulation presented here is a natural generalization of the two party case.

It is important to note that the usual probabilistic models of electoral behavior (Coughlin, 1992) may not satisfy Assumption 1. However Assumption 1 is satisfied by the electoral probability function induced by the MNP model, constructed in Schofield et al. (1998a) and Quinn et al. (forthcoming), for any “strongly proportional” electoral system. By this term we mean an electoral system where vote shares and seat shares are almost identical. Of course, the coalition structure is determined by numbers of seats, and this may introduce some small discontinuities in empirical analysis. For theoretical work, the covariance structure on the errors and the resulting variance in vote shares can be interpreted to mean that the electoral probability function is indeed smooth. More importantly, Assumption 1 emphasizes that elections necessarily involve risk. We may suppose that the electoral information encoded in \(p\) is consistent with an MNP model of the form so indicated, and accessible to the parties through electoral sampling.

Suppose then that the policy vector \(z\) is declared. Prior to the election the beliefs of the parties are described by the lottery \(\{(p_t(z), \mathcal{D}_t)\}\) in the manner described. After the
election a particular winning or decisive coalition structure, say $D$, is realized. For convenience we shall let $D(z)$ denote this realized coalition structure. For the moment, let us ignore the portfolio payoff to coalition members. Given this electoral realization, we shall assume that all government policy outcomes which result will belong to a subset of $Z$ termed the "heart", $H_D(z)$. The "heart" is a concept from spatial "committee" voting theory (Austen-Smith, 1996; Schofield, 1995b, 1999, forthcoming) obtained by localizing the "uncovered set" (Banks et al., 1998; Cox, 1987; McKelvey, 1986). For a fixed coalition structure, $D$, it is known (Schofield, 1999), that the heart $H_D$ is a lower hemi-continuous (lhc) correspondence $H_D:Z^n \rightarrow Z$, and also admits a continuous selection say $g_D:Z^n \rightarrow Z$. In the discussion of the previous section based on Fig. 1 the heart, under $D$, can generally be identified with the convex hull of the declared position of the three large parties, {PVDA, CDA, VVD}.

As in the previous section, we suppose that in choosing the policy declaration $z$, party $i$ chooses at the same time a leader or representative of the party called $s_i$, say, who has a preferred point at $z_i$. Without loss of generality, we may suppose that this player, $s_i$, has Euclidean preferences derived from the utility function $u_i(y) = -\frac{1}{2} \|y - z_i\|^2$. To determine the heart at the decisive structure $D$ it is sufficient to compute the "median" hyperplanes through the set of points $z_1, \ldots, z_N$.

Thus for example, if $N = \{1, 2, 3\}$ and $D$ is the family, $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, then $H_D(z)$ will be the convex hull of the three points $\{z_1, z_2, z_3\}$ whenever these are not colinear. For preference profiles that are convex (representable by quasi-concave utilities) the heart lies within the Pareto set of the "committee" of party leaders. Moreover if $z^*$ is a profile of positions such that the parliamentary game described by $D$ has a legislative core\(^6\), then $H_D(z^*)$ and the core coincide. In a situation characterized by $D$, the policy $z_1$ can be at a legislative core only if $z_1$ lies within the arc $[z_2, z_3]$. With four parties, more complicated situations can arise. For example, under the coalition structure $D$, in the Dutch example, the PVDA position, $z_1$, will be a legislative core point if it lies within the convex hull of the positions {CDA, VVD, D66}. The concept of the heart is utilized because of the theoretical and empirical inference that bargaining between parties must result in a "core" policy, when it happens that the legislative core is non-empty (Banks and Duggan, 1998; Schofield, 1995a).

We view $H_D$ as a natural "cooperative" theory that, in a sense, constrains the possibilities of coalition bargaining over policy in the post-election situation. Current work (Winter and Schofield, 1999) suggests that the outcome of such bargaining will be a lottery $g_D(x)$, say, from the set of alternatives in $H_D(z)$.

For a fixed coalition structure, $D$, let $g_D:Z^n \rightarrow \tilde{Z}$ be the correspondence that assigns to each policy profile $z \in Z^n$ the space of Borel probability measures on the set $H_D(z)$. With respect to the topology of weak convergence (Fudenberg and Tirole, 1991; Parthasathy, 1967) on $\tilde{Z}$ (the set of Borel probability measures in $Z$), this correspondence is also known to be lhc (Schofield, forthcoming), and to admit a continuous

---

\(^6\)As indicated in the Dutch example in Section 2, if there is a legislative core at $z$, then there will be some party $i$ whose position $z_i$ cannot be beaten by a winning coalition $M$ in $D$, offering a policy $x$, say, in $Z$ which is preferred by all the leaders of the parties in $M$. 

selection (Michael, 1956), namely a continuous function \( \tilde{g}_\phi: Z^n \rightarrow \tilde{Z} \) such that \( \tilde{g}_\phi(z) \in \mathcal{H}_\phi(z) \), for all \( z \in Z^n \). Assumption 2' requires that \( \tilde{g}_\phi(z) \) be a lottery chosen from \( \mathcal{H}_\phi(z) \), for each \( z \in Z \), which is continuous on the space \( Z^n \). The weaker Assumption 2 merely requires that \( \tilde{g}_\phi(z) \) be chosen from the Pareto sets associated with the coalitions in \( \mathcal{D} \).

**Assumption 2.** The policy outcome \( \tilde{g}_\phi(z) \in \tilde{Z} \) resulting from the realized fixed coalition structure \( \mathcal{D} \) at a policy profile \( z \) is a finite lottery \( \tilde{g}_\phi(z) = \{ (\rho_M(z), a_M(z)) : M \in \mathcal{D} \} \), chosen in such a way that each \( a_M(z) \) is a “compromise point” for coalition \( M \), in the convex hull \( \{ z_i : i \in M \} \), while \( \rho_M(z) \) is the probability that coalition \( M \) forms.

Moreover, the function \( \tilde{g}_\phi: Z^n \rightarrow \tilde{Z} \) is continuous with respect to the specified topologies.

**Assumption 2'.** In addition, the function \( \tilde{g}_\phi: Z^n \rightarrow \tilde{Z} \) is a continuous selection of the heart correspondence \( \mathcal{H}_\phi: Z^n \rightarrow \tilde{Z} \).

It is important to note that we assume that the lottery \( \tilde{g}_\phi(z) \in \tilde{Z} \) does not depend directly on the seat or vote shares of the parties, but rather on the coalition structure \( \mathcal{D} \). That is, we view the negotiation game between the parties to be based on coalition possibilities. If we assumed instead that a party could gain power by increasing its vote or seat share, even though \( \mathcal{D} \) remained unchanged, then parties would have a strong incentive to seek the electoral center. No centrifugal tendency would be possible.

As we have emphasized, there may be a legislative core to the bargaining game given by the Euclidean preferences based on the vector of leader positions, \( z \), and coalition structure, \( \mathcal{D} \). In particular, if the legislative core is non-empty, then it will typically be at the declared point of one of the parties, \( z_j \), say. Under Assumption 2', the lottery \( \tilde{g}_\phi(z) \) will be a singleton,

$$ z_j \text{ occurring with probability 1.} $$

To examine existence of equilibria in this model we must introduce the preferences of the parties, and the private perquisites that result from coalition membership. Assuming therefore that the coalition structure \( \mathcal{D} \) is realized, then coalition \( M \in \mathcal{D} \) forms with probability \( \rho_M(z) \), chooses a point \( a_M(z) \in Z \), and allocates a vector \( \{ \sigma^M_i : i \in M \} \) of shares of portfolios or perquisites to the members of the coalition \( M \). In the most general case it would be appropriate to make \( \{ \sigma_i \} \) dependent on the realized seat shares (Browne and Franklin, 1973) within coalition \( M \). In Model 1 below we shall just assume that the portfolio shares are prespecified.\(^8\) Of course the behavior of the model will depend on the balance between policy preferences and the utility gained from government perquisites. For convenience below, we sometimes specify \( \{ \sigma^M_i \} \) in terms of a scale factor, which we shall denote by “\( r \)

\(^7\)Notice that each \( a_M(z) \) will lie in the convex hull of \( \{ z_i : i \in M \} \). If \( z \rightarrow z' \) for some profile \( z' \) with a legislative core \( z'_j \), then Assumption 2' requires that \( a_M(z) \rightarrow a'_M \) for any \( M \) containing \( i \), and \( \rho_M(z) \rightarrow 0 \) for any \( M \) that does not include \( i \). We shall call this the “core convergence” property for \( \tilde{g}_\phi \).

\(^8\)However, the expected portfolio share of a party, in the coalition situation \( \mathcal{D} \), will depend on \( \mathcal{D} \). In early empirical work attempting to explain the distribution of portfolios, it was found that a theory based on a transferable value bargaining set predicted portfolio payoffs (Schotfield, 1982; Laver and Schofield, 1990). This predictor was based on \( \mathcal{D} \), and not on actual seat shares.
**Assumption 3.** At the policy profile \( z \in \mathbb{Z}^n \) the “pre-election” von Neumann-Morgenstern utility for the principal of party \( i \) is

\[
U_i(z) = \sum_i p_i(z) \left( \sum_{M \in \mathcal{A}_i} p_M(z) \left[ u_i(a_{im}(z)) + \sigma_i^M \right] \right).
\]

Moreover each \( U_i: \mathbb{Z}^n \to \mathbb{R} \) is a continuous function in the argument, and there exists a continuous extension \( \hat{U}_i: \hat{Z}^n \to \mathbb{R} \) to the space \( \hat{Z}^n \) of Borel probability measures on \( Z^n \). (Obviously, to extend \( U_i \) in this fashion, it is necessary to extend the electoral probability function \( p \) to \( \hat{p}: \hat{Z}^n \to \Delta \).)

To complete the model we need to be precise about the nature of the “policy utility”, \( u_i \), of party \( i \). Consistent with the earlier discussion we assume that each elite member, \( k_i \), of party \( i \) has Euclidean policy preferences generated by a “sincere” policy utility of the form

\[
u_{k_i}(y) = -\frac{1}{2} \| y - o_{k_i} \|^2\]

where \( o_{k_i} \) is \( k_i \)'s ideal point. As above, the ideal point \( o_i \) for party \( i \) is chosen to be that point which is at the multidimensional median of \( \{ o_{k_i} : k_i \text{ belongs to party } i \} \). This point is a proxy for the sincere “internal” voting “equilibrium” within the party, using majority rule. Let \( o = (o_1, \ldots, o_n) \in \mathbb{Z}^n \) be the vector of party ideal points.

**Assumption 4.** For each party \( i \in N \), the “policy utility”, \( u_i \), is given by

\[
u_i(y) = -\frac{1}{2} \| y - o_i \|^2.
\]

Assumptions 2′ and 3 characterize a model in which coalition bargaining gives outcomes within the “policy heart” \( \mathcal{H}_{\mathcal{D}}(z) \). However this model can in principle be extended to a situation where bargaining is over both policy and perquisites (Schofield and Sened, 1998). To see this, let \( W = \mathbb{Z} \times \Delta \) be the product space of policy and distributions of perquisites (\( \Delta \) is the \((n-1)\)-dimensional simplex). In the obvious fashion the policy utility \( u_i: \mathbb{Z} \to \mathbb{R} \) of \( i \) can be extended to \( u'_i: \mathbb{Z} \to \mathbb{R} \) by defining

\[
u'_i(y, \sigma) = u_i(y) + \pi_i(\sigma)
\]

where \( \pi_i: \Delta \to \mathbb{R} \) is the projection onto the \( i^{th} \) component. Let \( \tilde{W} \) be the space of all probability measures on \( W \), endowed with the weak topology (Parthasathy, 1967). We assume \( u'_i: \tilde{W} \to \mathbb{R} \) is measurable with respect to the Borel sigma-algebra on \( W \).

If we assume the vector \( z \in \mathbb{Z}^n \) encodes information about the chosen party leader’s policy preferences and preferences for perquisites, then for each coalition structure \( \mathcal{D} \), the more general heart correspondence \( \mathcal{H}_{\mathcal{D}}: \mathbb{Z}^n \to \mathbb{W} \) can be constructed (Schofield, 1999). \( \mathcal{H}_{\mathcal{D}}(z) \subseteq W \) is conceived of as the general constraint on bargaining over policy and perquisites, in the context of the coalition structure \( \mathcal{D} \) and leader characteristics, \( z \). As in the simpler case, \( \mathcal{H}_{\mathcal{D}}: \mathbb{Z}^n \to \tilde{W} \) is a lhc correspondence (Schofield, forthcoming) and admits a continuous selection \( \tilde{g}_{\mathcal{D}}: \mathbb{Z}^n \to \tilde{W} \). Combining \( \{ \tilde{g}_{\mathcal{D}}: t \in T \} \) with the risk function of Assumption 1, gives the game form
\[ \tilde{g} = \Pi((p_1, \tilde{g}_i)): Z^n \rightarrow \tilde{W}. \]

Composition of \( \tilde{u}_i \) with \( \tilde{g} \), and extension to \( \tilde{Z}^n \) gives the utility function
\[ \tilde{U}_i: \tilde{Z}^n \rightarrow \mathbb{R}. \]

We present this framework as our most general Assumption 5.

**Assumption 5.** The political game \( \tilde{U}: \tilde{Z}^n \rightarrow \mathbb{R}^n \) is induced from a continuous game form \( \tilde{g} = H((p_1, \tilde{g}_i)): Z^n \rightarrow \tilde{W} \) where each \( \tilde{g}_i: Z^n \rightarrow \tilde{W} \) is a continuous selection of the general heart correspondence \( \tilde{H}_{\alpha_i}: Z^n \rightarrow \tilde{W} \) induced by the post-election coalition structure \( \alpha_i \).

This framework allows us to introduce the definition of Nash equilibrium. In the analyses that follow, the Nash equilibrium notion can be interpreted in the appropriate fashion for the simpler games that are studied.

**Definition 1.** Consider the game \((\tilde{U}, \tilde{Z}^n)\) where \( \tilde{U}: \tilde{Z}^n \rightarrow \mathbb{R}^n \) is the utility “profile” for the committee of party principals. The induced preference correspondence for \( \alpha_i \), \( f_i: \tilde{Z} \rightarrow \tilde{Z} \) is given by
\[
f_i(z_1, \ldots, z_i, \ldots, z_n) = \{z_i^* \in \tilde{Z} : U_i(z_1, \ldots, z_i^*, \ldots, z_n) > U_i(z_1, \ldots, z_i, \ldots, z_n) \forall z_i \in \tilde{Z} \},
\]
and the best response correspondence \( h_i: \tilde{Z}^n \rightarrow \tilde{Z} \) for \( i \) is given by \( h_i(z_1, \ldots, z_i, \ldots, z_n) = \{z_i^* \in \tilde{Z} : f_i(z_1, \ldots, z_i^*, \ldots, z_n) = \emptyset \} \). Since \( h_i(z_1, \ldots, z_i, \ldots, z_n) = h_i(z_1, \ldots, z_i', \ldots, z_n) \) for any \( z_i, z_i' \), we can combine \( \{h_i\} \) to form the joint best response correspondence \( h: \tilde{Z}^n \rightarrow \tilde{Z}^n \) in the obvious way.

A mixed strategy Nash equilibrium (MSNE) \( z^* \in \tilde{Z}^n \) is a fixed point of \( h \), such that \( z^* \in h(z^*) \). Since \( f_i, h_i, h \) can all be restricted to \( Z^n \), we may define a pure strategy Nash equilibrium (PSNE) as a fixed point, \( z^* \in Z^n \), of \( h \) (if one exists) when \( h \) is restricted to have domain \( Z^n \).

The models described by these Assumptions give a fully specified continuous game \((\tilde{U}, \tilde{Z}^n)\). Using standard arguments (Bergstrom, 1975, 1992; Glicksberg, 1952) there will, in general, exist a MSNE (Fudenberg and Tirole, 1991) when \( \tilde{Z} \) is compact, convex, and \( h \) is continuous on \( \tilde{Z}^n \). However PSNE may fail to exist when \( h \) is not continuous on \( Z^n \). As we now observe, there may exist equilibria that are “unstable” in a certain sense, and we wish to exclude these.

**Definition 2.** A MSNE \( z^* \) for \((\tilde{U}, \tilde{Z}^n)\) is stable iff for any neighborhood \( V \) of \( z^* \) in \( \tilde{Z}^n \), there is a proper subneighborhood \( V' \) of \( z^* \) in \( V \) such that for any \( z \in V \), then \( h(z) \in V' \). Similarly, a PSNE \( z^* \) is stable if \( h(z) \in V' \), where \( h \) is restricted to the domain \( Z^n \).

Clearly if \( z^* \) (in \( V \)) is not stable (i.e., unstable) then “best response” from a perturbation \( z \) (in \( V \)) of \( z^* \) may lead to a point outside \( V \).

In the model just described we have left unspecified the electoral probability functions \( \{p_i\} \) and the coalition selections \( \{\tilde{g}_i\} \), where \( \tilde{g}_i = g_{\alpha_i} \). We shall denote these by \( p \) and \( g \).
respectively, and leave the assumptions on these terms until the next section. For the model based on Assumption 2, let $I' = \{a_i^M\}$ denote the scheme of private benefits or perquisites that describe the private benefit to player $i$ in coalition $M$.

The remaining parameters describe the distribution of ideal points of the elite members of each of the parties. By Assumption 4, we may restrict attention to the vector, $o$, of party ideal points.

**Definition 3.** The Nash correspondence, $\mathcal{E}_f(p, \tilde{g})$, maps the vector $o \in Z^n$ of party principals’ ideal points to the stable MSNE, $z^*$. Thus $\mathcal{E}_f(p, \tilde{g}) : Z^n \rightarrow Z^n$.

It is our contention that, under certain constraints on $p$, $\tilde{g}$, the Nash correspondence gives PSNE. In this case, and when $p$, $\tilde{g}$ are specified we shall write $\mathcal{E}_f : Z^n \rightarrow Z^n$ or simply $\mathcal{E} : Z^n \rightarrow Z^n$.

**Assumption 6.** In a game described by the parameters $(p, \tilde{g}, I')$, if there exists a unique PSNE $z^* = \mathcal{E}(o)$ at $o \in Z^n$, then each party, $i$, declares the policy $z_i^*$ and chooses as a leader of the party that member of the party elite whose ideal point coincides with $z_i^*$.

We have assumed that the principal of each party determines the pure strategy Nash equilibrium policy to declare, and chooses as a leader that party colleague whose “sincere” or ideal policy coincides with the party’s Nash equilibrium choice.

The case with a mixed strategy Nash equilibrium can be less easily interpreted. However it is our contention that unique PSNE are obtained in simple versions of this model.

In the general model, a result on generic existence of local PSNE can be obtained (Schofield and Sened, 1998). Of course there may well be multiple equilibria of this kind, so the selection problem must be solved. However, the results of the simple model are offered in the following section, because they suggest that non-convergent Nash equilibria are possible.

### 4. A model of three-party bargaining

The general model just described is intended to extend a two-party model presented earlier by Cox (1984). In the two party case described by Cox there are only three possible states of the post-election world: $D_1 = \{1\}$ so 1 wins, $D_2 = \{2\}$ so 2 wins, or $D_0 = \{1, 2\}$ where 1 and 2 have exactly the same number of seats. In the case 1 wins, then $\sigma_1^1 = 0$ (so 2 receives nothing). When neither wins, it is natural to assume that $\sigma_1^{12} = \sigma_2^{12} = \frac{1}{2} \sigma_1^1$.

Thus the utility function for 1, say, is

$$U_1(z_1, z_2) = p_1(z)(u_1(z_1) + \sigma_1^1) + p_2(z)(u_1(z_2))$$

$$+ (1 - p_1(z) - p_2(z))\left(u_1\left(\frac{z_1 + z_2}{2}\right) + \frac{1}{2} \sigma_1^1\right).$$
In the case $D_0 = \{1, 2\}$, party 1 and 2 compromise over policy and adopt the midpoint $z_M = \frac{1}{2}(z_1 + z_2)$. Cox argued that natural assumptions on the nature of the electoral probabilities would lead to a PSNE, $z^\ast$. Clearly Cox’s model concentrated on “electoral risk”, induced by the uncertainty associated with the electoral probabilities, $p_1(z)$ and $p_2(z)$.

In contrast, recent models (Baron, 1989, 1991; Banks and Duggan, 1998) have concentrated on “coalitional risk” associated with the probabilities $\{\mu_M\}$, introduced in Assumption 2. In general, the idea behind models of coalition bargaining is that the true ideal points $\{o_i\}$ of the parties are declared, and the coalition probabilities and compromise points $\{a_{ij}(o)\}$ are deduced from a bargaining game. Although existence of these bargaining solutions is known, it is often unclear whether the solution is continuous in the ideal points $\{o_i\}$. A model currently being analyzed (Winter and Schofield, 1999) indicates that, in the three party case, the compromise point for coalition $\{i, j\}$ will be linear in $\{o_i, o_j\}$ and the coalition probabilities will be approximately inversely proportional to $\|o_i - o_j\|^2$. As indicated above, we suppose that the parties declare, not the vector of ideal points $(o_1, o_2, o_3)$ but a vector $(z_1, z_2, z_3)$ of leader positions, chosen with the knowledge of the bargaining that will then occur.

Since we are interested in the effect of coalitional incentives on electoral motivations, it seems appropriate to examine the three party case without electoral risk. This suggests that we impose a fixed coalition structure $D$. For symmetry, we suppose $D = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. We suppose further that the “costs of bargaining” $c_{ij}$ for coalition $\{i, j\}$ are proportional to $\|z_i - z_j\|^2$ and that the probability that coalition $\{i, j\}$ forms is proportional to $c_{ij}^{-1}$. When this coalition $M = \{i, j\}$ forms it adopts the policy point $a_{ij}(z) = z_M = (z_i + z_j)/2$, and assigns a perquisite $\sigma_{ij} = s_{ij}$ to party $i$.

Model 1 which we examine, satisfies Assumptions 1–4 above, but not Assumption 2’, since the “core convergence” property is not assumed. That is, in Model 1 we do not suppose that when $z = \{z_1, z_2, z_3\}$ are colinear, then $g_M(z)$ is the singleton $\{z\}$.

Model 1. Suppose parties declare $z \in Z^3$ and have ideal points given by $o \in Z^3$, where $Z$ is a disc $D$ in $\mathbb{R}^2$ of sufficiently large radius. We assume:

1. $p_t(z) = 1$ and $p_t(z) = 0$ for all $t \neq 1$, where $t = 1$ corresponds to the state where each two-party coalition wins.
2. the Pareto set of the party leaders is the convex hull of $\{z_1, z_2, z_3\}$.
3. the selection $g_M = \{\mu_M, a_M\}$ is a lottery selected from the Pareto set. If $M = \{i, j\}$, then $\mu_M(z) = c_{ij}^{-1}(c_{ij}^{-1} + c_{ik}^{-1} + c_{jk}^{-1})^{-1}$, and $a_M(z) = (z_i + z_j)/2 = z_{ij}$. The perquisite to $i$ in coalition $\{ij\}$ is $\sigma_{ij} \in \mathbb{R}$.

\*An important first step in formulating such a game can be found in Austen-Smith (1986) in the one-dimensional case. Just as in the model here, Austen-Smith assumes that each minimal winning coalition, $M$, forms with probability $\mu_M$, inversely proportional to its variances and adopts the mean policy point $z_M$. An obvious motivation in Austen-Smith’s paper was to model the creation of parties (that is coalitions of heterogeneous candidates). The convergence phenomena found in the model presented here may provide a theoretical framework for the coalescence of elites into parties.
Thus for principal of party 1, the von Neumann-Morgenstern utility at \( z = (z_1, z_2, z_3) \) is \( U_i(z) = \rho_{i1}(z)(u_1(z_{12}) + \sigma_{12}) + \rho_{i2}(z)(u_2(z_{13}) + \sigma_{13}) + \rho_{i3}(z)u_3(z_{23}) \).

Since the utility functions are smooth we show in Appendix A how to compute their critical points, and thus to determine the best response correspondences.

The following four theorems (whose proofs are presented in Appendix A) characterize Model 1.

**Theorem 1.** If \( Z \) is a compact, convex subset of \( \mathbb{R}^2 \), then for each vector of ideal points \( o = \{ o_i, o_2, o_3 \} \in Z^3 \), and scheme \( \{ I = (\sigma_i): i, j \in \{1, 2, 3\}, i \neq j \} \) of perquisites, there exists a mixed strategy Nash equilibrium. For each \( o \in Z^3 \), there exists \( \sigma^* > 0 \), dependent on \( o \), such that whenever \( \sigma_{ij} > \sigma^* \) for each \( \sigma_{ij} \) in \( I \), then there exists a stable PSNE.

We show in Appendix A that the utility functions \( U_i \) need not be quasi-concave in the strategy variable, \( z_i \). Nonetheless the \( U_i \) are continuous (indeed differentiable) in the strategy variables and thus MSNE exist. We also show in Appendix A, when the perquisites are sufficiently large, that, for each profile \( o \in Z^3 \), the joint best response correspondence, \( h(o) \), is single-valued and continuous in the strategies of players other than \( i \). This implies that \( h(o) \) has a fixed point, which corresponds to a PSNE. Moreover specific configurations of bliss points give rise to unique, stable PSNE.

Two different cases giving unique PSNE are considered in detail.

(A) Consider first the symmetric case in which \( \| o_i - o_j \| = r \) constant for all pairs \( \{ i, j \} \). We suppose \( Z \) is a disc \( D \) centered at the barycenter of \( \{ o_1, o_2, o_3 \} \) with radius \( r \gg r \). If perquisites are zero, then the best response correspondence of each player is single-valued. The fixed point \( z^* \) of the joint best response function satisfies \( \| z_i^* - z_j^* \| = \| o_i - o_j \| \) for each pair \( \{ i, j \} \). Since \( \| z_i^* - z_j^* \| > \| o_i - o_j \| \) for each pair, say divergence occurs. (See Fig. 3). Note in particular, in this case, that no joint strategy, where all principals choose the same policy, can be stable.

(B) In the second case, suppose the bliss points are colinear. Write the bliss points as \( o_i = (0, y_i) \) and suppose \( y_i > y_j \). Assuming zero perquisites, the best response by 3 is to choose \( y_3^* \) closer to the mid-point \( \frac{1}{2}(y_1 + y_2) \). On the other hand the best response by both 1 and 2 is to move closer to \( y_3^* \). In this second, degenerate, case we find, as we might expect, that convergence occurs. By “convergence”, we mean that \( \| z_i^* - z_j^* \| < \| o_i - o_j \| \) for each pair \( i, j \).

In this colinear case, there is an attractor \( (z_1, z_2, z_3) \) of the best response function, where each player chooses the same policy. By definition this is a stable PSNE. Note that this is a form of Downsian convergence.

To solve the general case, we consider the problem of the best response by party 3 to \( z_1 = (0, r_1/2), z_2 = (0, -r_2/2) \). The best response in \( x \) is essentially a function of \( (r_1 + r_2) \), while the best response in \( y \) is a function of \( (r_1 - r_2) \). Either the response equation in \( x \) or the equation in \( y \) will dominate, giving either divergence or convergence. Full mathematical analysis of the general equations has not been possible.

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10 Although the analysis is performed in \( \mathbb{R}^2 \), it is evident that the analysis is valid if \( Z \subset \mathbb{R}^q \) for any \( q \gg 2 \). In this case all party behavior will lie in the two-dimensional affine subspace of \( Z \) generated by \( \{ o_1, o_2, o_3 \} \).
but computer simulation indicates that for almost any assignment of bliss points, a stable
pure strategy Nash equilibrium occurs in the interior of the space.

Even in the symmetric case (A), if there are positive perquisites from coalition
membership, and if these perquisites are sufficiently large, then the Nash equilibrium $z^*$
is convergent.

The results obtained in the Appendix on the relationship between bliss points,
perquisites and Nash equilibria are described in Theorems 2, 3, and 4. We first need
generalizations of the notions “colinear” and “symmetric”.

**Definition 4.** Say three points $\{z_i, z_j, z_k\}$ are $\varepsilon$-bounded in linearity if
\[
\min_{i,j,k \in \mathbb{N}} \left\{ \| z_i - \lambda_i z_j - \lambda_k z_k \| \right\} \leq \varepsilon.
\]

Say three points $\{z_i, z_j, z_k\}$ are $\varepsilon$-bounded in symmetry if
\[
\max_{i,j,k} \| z_i - z_j \| - \| z_i - z_k \| \leq \varepsilon,
\]
where $\max_{i,j,k}$ means across all permutations of $i, j, k$.

In the case $\varepsilon = 0$, say simply that the points are symmetric.
Note of course that if three points are $\epsilon$-bounded in linearity, for $\epsilon = 0$, then the degree of symmetry they exhibit will be low. These two definitions attempt to capture the difference between the extreme cases $A$ and $B$.

**Theorem 2.** Suppose that perquisites are zero. There exists $\epsilon^* > 0$ such that, if the ideal points are $\epsilon$-bounded in linearity, for any $\epsilon < \epsilon^*$, then there exists a unique, stable, pure strategy Nash equilibrium which is convergent. That is $\{z_1^*, z_2^*, z_3^*\}$ all lie within the convex hull of $\{o_1, o_2, o_3\}$. Moreover, the Nash equilibrium strategies are also $\epsilon$-bounded in linearity.

**Theorem 3.** Suppose that perquisites are zero and $Z$ is the disc, $D$, as above. Then there exists $\epsilon^* > 0$ such that, if the ideal points are $\epsilon$-bounded in symmetry, for $\epsilon < \epsilon^*$, then there exists a unique stable pure strategy Nash equilibrium $z^*$ in the interior of $Z$ which satisfies

$$\frac{\|z_i^* - z_j^*\|}{\|o_i - o_j\|} = b_{ij}(\epsilon), \text{ for each pair } (i, j).$$

Here, $b_{ij}$ is a continuous real-valued function of $\epsilon$, with $b_{ij}(\epsilon) > 1$ and $b_{ij}(0) = 2$.

Theorem 3 asserts that the Nash equilibrium $z^*$ is divergent in the sense that $\|z_i^* - z_j^*\| > \|o_i - o_j\|$ for each pair. Moreover, if the ideal points are symmetrically located ($\epsilon = 0$), then $z^*$ is symmetric, i.e.,

$$\|z_1^* - z_2^*\| = \|z_1^* - z_3^*\| = \|z_2^* - z_3^*\|.$$

Note that the Nash equilibrium positions do not lie in the Pareto set of the party principals, namely the convex hull of $\{o_1, o_2, o_3\}$.

The effect of perquisites is captured by the following result.

**Theorem 4.** If perquisites are non-zero and constant ($\sigma_{ij} = \sigma$ for all $i, j$), then for each $\{o_1, o_2, o_3\}$, which is $\epsilon$-bounded in symmetry, there exists a unique, stable, pure strategy Nash equilibrium $z^*$ which satisfies

$$\|z_i^* - z_j^*\| = d_{ij}(\epsilon, \sigma)\|o_i - o_j\|, \text{ for each pair } (i, j).$$

Here $d_{ij}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a positive-valued continuous function, for each $i, j$, which satisfies the following properties:

1. $d_{ij}(\epsilon, \sigma)$ decreases as $\sigma$ increases (for each $\epsilon$);
2. if the ideal points are symmetric ($\epsilon = 0$), then $d_{ij}(0, \sigma) = d(0, \sigma)$ for each pair, so that the Nash equilibrium will be symmetric;
3. in this case, there is a bound $\sigma^* < \frac{1}{2\sigma} \|o_i - o_j\|^2$ such that $d(0, \sigma) < 1$ for all $\sigma > \sigma^*$.

See Fig. 4 for an illustration of the convergence result. Theorem 4 shows that if the ideal points are symmetric, and the perquisites are
sufficiently high, then there exists a stable symmetric, convergent Nash equilibrium which is uniquely determined by the parameters \( \{o_1, o_2, o_3, I\} \). Clearly \( \{z^*_1, z^*_2, z^*_3\} \) all lie in the convex hull of the principals’ bliss points.

Observe that, in the symmetric case, the expectation of the coalition outcomes is the center of the distribution of ideal points. Since the probability associated with each coalition is \( \frac{1}{3} \), the expectation of private benefits or portfolios of each party is \( \frac{2}{3} \sigma \).

These four theorems all use the properties of the reaction functions, together with Fort’s Theorem (Fort, 1950), to assert that the Nash equilibrium mapping

\[ \mathcal{E}_I: Z^n \rightarrow \hat{Z}^n \]

defined by the scheme \( I \) of private benefits, with \( |N| = 3 \), and which maps the ideal points to the Nash equilibria, is a continuous function on specific open domains in \( Z^n \).

In order to understand the foundations of the convergence and divergence results, let us consider an example which is not covered by these theorems but which does illustrate the nature of the equilibria.

Consider again Fig. 2, and let us relabel the CDA and PVDA as \( \{1, 2\} \) and the VVD as
{3}. Suppose that 1 and 2 adopt positions \((0, \sqrt{3})\) and \((0, -\sqrt{3})\) a distance \(r = 2\sqrt{3}\) apart. To generalize Fig. 2, suppose the ideal point of 3 is on the x-axis at \((L, 0)\), say, and consider the best response of 3. By symmetry, it must be on the x-axis at \((x, 0)\), say. We can normalize by letting \(x = \alpha r\) and \(L = \beta r\) for some parameters \(\alpha, \beta > 0\). Eq. (4) of the Appendix shows, for zero perquisites \(\sigma = 0\), that \(\alpha\) and \(\beta\) are related by the quadratic expression \(\alpha^2 + (\frac{9}{\beta} - \frac{y}{4}) = 0\) with solution

\[
\alpha = \frac{1}{2} \left( -\frac{1}{\beta} \pm \sqrt{\frac{9}{\beta} + \frac{1}{\beta^2}} \right). \tag{1}
\]

This is obtained by setting \(dU_3 = 0\). In fact the term involving the negative sign corresponds to a minimum, as we need only consider the positive term. An easy calculation shows that \(\alpha = \beta\) iff \(\beta = \frac{1}{\sqrt{3}}\). This defines a fixed point of the best response function, \(h_3\). In particular if \(L \in (0, \frac{\sqrt{5}}{\beta})\), then \(\alpha > \beta\) but \(x \in (0, \frac{\sqrt{5}}{\beta})\). In other words (3) diverges from \(z_1, z_2\) but \(z_3\) is still bounded by the limit \(\frac{1}{\sqrt{3}}\). On the other hand if \(L \in (\frac{\sqrt{5}}{\beta}, \infty)\) then \(\alpha < \beta\) and \(x \in (\frac{\sqrt{5}}{\beta}, \frac{\sqrt{5}}{\beta})\). Of particular interest is the solution \(\beta = \frac{1}{\sqrt{3}}\) and \(\alpha = \frac{3}{\sqrt{3}}\). As Fig. 3 illustrates, if the bliss points are symmetrically distributed, so \(\|o_i - o_j\| = r\) for each pair \(\{i, j\}\), and we choose coordinates so that for player 3, \(o_3 = (L, 0) = (\frac{\sqrt{5}}{\beta}, 0)\) then 3's best response to \((z_1^*, z_2^*)\) is at \((z, 0) = (2r\frac{\sqrt{5}}{\beta}, 0)\). By symmetry, the three best responses in fact are symmetrically located, so \(\|z_i^* - z_j^*\| = 2\|o_i - o_j\|\) for each pair. This gives a symmetric, unique and divergent PSNE, from which Theorem 3 can be derived.

To see how the Nash equilibrium is changed as \(L\) is decreased, let us modify Fig. 3 by supposing \(o_3 = (0, \frac{\sqrt{5}}{\beta})\), \(o_2 = (0, -\frac{\sqrt{5}}{\beta})\) and \(o_3 = (L, 0)\). If we choose \(o_3\) to be a distance \(\frac{\sqrt{5}}{\beta}\) from the y axis, then the equilibrium is \((z_1^*, z_2^*, z_3^*)\) with \(z_3^*\) approximately \((0.93r\frac{\sqrt{5}}{\beta}, 0)\) while \(z_1^*\) is close to the y-axis but with its x-coordinate at approximately \((0.64)\frac{\sqrt{5}}{\beta}\). In other words, 3 still “diverges”, but 1 and 2 converge. On the other hand if \(L = \frac{\sqrt{5}}{8}\), then 3 initially diverges, but as 1 and 3 continuously converge in best response, the scale parameter, \(\|z_1 - z_2\|\) drops until 3 goes through its “best response fixed point”. After this stage, 3 also converges towards the positions of 1 and 2. Computation of the round of best responses shows all three players converging towards the bliss point of party 3. If we regard \(r = \|o_1 - o_3\|\) as an indication of the initial scale parameter, then by round 33 of best responses, all three players are within \((0.001)\) of this point of convergence. Thus the degree of colinearity required for Theorem 2 is not high.

To determine the effect of perquisites in determining best response, we may suppose that the perquisite \(\sigma = tr\), where \(t\) is a coefficient and \(r\) is the scale parameter.

Instead of (1) above we now obtain

\[
\alpha = \frac{1}{2} \left( -\frac{1 + 2t}{\beta} \pm \sqrt{\left(\frac{1 + 2t}{\beta}\right)^2 + \frac{9}{\beta^2}} \right). \tag{2}
\]

For example, if \(t = 1\) and \(\beta = \sqrt{3}\), then \(\alpha = \frac{3}{\sqrt{3}}\). This immediately implies that the parties converge. Indeed it is easy to see that if \(t > \frac{3}{\sqrt{3}}\), then there is no solution to the fixed point requirement, and so \(\beta > \alpha\). In the symmetric case, presented in Fig. 4, if
\[ \sigma = r^2 = \frac{2}{3} \| o_i - o_j \|^2, \] then there is a convergent Nash equilibrium with \( \| z_i - z_j \| = \frac{2}{3} \| o_i - o_j \| \). Theorem 4 follows by continuity.

On the other hand, in the skew-symmetric case if

\[ \| o_3 - o_1 \| = \| o_3 - o_2 \| > \| o_2 - o_1 \|, \]

then in the Nash equilibrium 3 diverges from 1 and 2, but 1 and 2 may converge to one another. Sufficiently high perquisites will force convergence of all parties. Computation of this three party game always resulted in stable PSNE.

In computing Nash equilibria the procedure we adopted was, for each profile \( o \in Z^3 \) of bliss points, to start the process of best response from random initial points and to terminate once the process of best response had “stabilized”. This was interpreted to mean that no movement was observed greater than \((0.001)\) times the scale parameter, \( r \). In all cases we found that the computed PSNE was unique, and a global attractor. That is to say the process of best response led into a neighborhood of the PSNE, \( z^* \), from all initial points. Our analysis suggests that the best response correspondence is a contraction mapping.

**Computation Fact 1.** The Nash equilibrium map of Model 1 with \( |N| = 3 \), given by \( \mathcal{E}_r : Z^3 \to \hat{Z}^3 \) is a function, with image in \( Z^3 \). Moreover, for each \( o \in Z^3 \), the PSNE \( \mathcal{E}_r(o) \) is a global attractor of the joint best response function.

**Computation Fact 2.** Suppose all perquisites are zero in Model 1. Consider the skew symmetric case with \( \| o_3 - o_1 \| = \| o_3 - o_2 \| = e \| o_2 - o_1 \| \).

If \( e > 0.616 \), then all parties diverge in Nash equilibrium, so \( \| z^*_i - z^*_j \| > \| o_i - o_j \| , \forall \ i, j \).

If \( 0.54 < e < 0.616 \), then 1 and 2 “weakly” converge, but 3 diverges, so \( \| z^*_i - z^*_j \| < \| o_i - o_j \| \) but \( \| z^*_i - z^*_j \| > \| o_i - o_j \| \) for \( i = 1, 2 \).

If \( e < 0.54 \) then all three parties converge to a neighborhood of \( o_3 \).}

While the specific details on the equilibria in Model 1 depend on our assumptions on \( \rho \) and \( \breve{g} \), we regard the above results as being qualitatively valid for any similar model. That is, the qualitative aspects of convergence or divergence in Nash equilibrium appear to be unchanged for certain small perturbations of \( \rho \) and \( \breve{g} \), with \( |N| = 3 \).

**Model 2.** To illustrate this observation, consider the following perturbation of the lottery \( \breve{g}_\sigma \) given in Model 1. For any \( z \in Z^3 \) with \( \{ z_1, z_2, z_3 \} \) colinear and \( z_i \) the median, let \( \breve{g}_\sigma(z) = \{ z_i \} \) be a singleton. For any \( e > 0 \), let \( \mathcal{V}_e \) be an open set in \( Z^3 \) such that, for any \( z = \{ z_1, z_2, z_3 \} \) with \( \{ z_1, z_2, z_3 \} \) colinear, there is an open \( e \)-ball \( B_e \) containing \( z \), and contained in \( \mathcal{V}_e \). Then we can construct a smooth selection \( \breve{g}_\sigma : Z^3 \to \hat{Z} \) of \( \mathcal{F}_\sigma : Z^3 \to \hat{Z} \) such that \( \breve{g}_\sigma(z) = \breve{g}_\sigma(z) \) whenever \( \{ z_1, z_2, z_3 \} \) are colinear, and \( \breve{g}_\sigma(z) = \breve{g}_\sigma(z) \) for all \( z \in Z^3 \setminus \mathcal{V}_e \). Here \( \breve{g}_\sigma \) is the lottery outcome function of Model 1. Call \( \breve{g}_\sigma \) an \( e \)-perturbation of \( \breve{g} \). The proof procedure of Schofield and Sened (1998) can then be adapted to show that the Nash equilibrium mappings for \( \breve{g}_\sigma \) and \( \breve{g} \) will be “close”. Thus the qualitative results obtained for \( \breve{g}_\sigma \) in Model 1 will also hold for some smooth
perturbation $\tilde{g}_\varepsilon$ for $\varepsilon$ sufficiently small, where $\tilde{g}_\varepsilon$ satisfies the core convergence property of Assumption 2.

**Theorem 5.** For sufficiently small $\varepsilon > 0$, there exists an $\varepsilon$-perturbation $\tilde{g}_\varepsilon$ of $\tilde{g}_\varepsilon$ in Model 1 such that $\tilde{g}_\varepsilon$ satisfies Assumption 2'. Moreover, Theorems 1–4 are valid for the political game induced by $\tilde{g}_\varepsilon$.

Note however that not every $\varepsilon$-perturbation of $\tilde{g}_\varepsilon$ will necessarily satisfy the convergence result of Theorem 2. Whether convergence in the degenerate colinear case does occur will depend on the precise details of the assumptions made on $\tilde{g}_\varepsilon$ in the neighborhood $V$. The following section mentions an example from Germany to illustrate this point.

**Model 3.** In Models 1 and 2, it has been assumed that the vector of perquisites obtained by the members of coalition $M$ (if it forms) are prespecified in some fashion. Assumption 5 is designed to deal with situations where bargaining between party leaders generates the payoffs of perquisites (see also Austen-Smith and Banks, 1988). If we use $I'$ to denote the total value of perquisites available to coalition members, and $I'$ is suitably constrained, the Nash equilibria policy outcomes under this more general game form $\tilde{g}_\varepsilon: Z'' \to W$ will be close to those generated by the game form $\tilde{g}_\varepsilon: Z'' \to \tilde{Z}$ of Model 2. We therefore expect that when the coalition structure $\tilde{D}$ is held constant, as in Models 1 and 2, and $\tilde{g}_\varepsilon: Z'' \to W$ is a continuous selection of $\tilde{\mathcal{R}}: Z'' \to \tilde{W}$, then the Nash equilibria of the induced political game will have qualitatively similar properties to those presented in Theorems 1, 3 and 4.

5. **Extension of the model to include electoral risk**

To introduce an extension of Model 1, let us consider again the four party case discussed in Section 2. As before, identify $\{\text{PVDA}, \text{CDA}, \text{VVD}, \text{D66}\}$ with $\{1, 2, 3, 4\}$.

Let $\{o_1, o_2, o_3, o_4\}$ represent the bliss points, and suppose the electoral probability function, $p$, is compatible with the MNP estimation of Section 2. To set up the model we need to introduce all possible winning coalition structures. For convenience we shall only consider the party declarations that are within restricted open neighborhoods of the party principals’ bliss points. With this assumption, our MNP estimation gave no single party a non-zero probability of winning a majority. As the discussion of Section 2 suggested, we need only consider two coalition structures, namely:

$\mathcal{D}_1 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$

$\mathcal{D}_2 = \{\{1, 3, 4\}, \{1, 2\}, \{2, 3, 4\}\}$.

In fact the coalition structure occurring after the 1977 election in the Netherlands was $\mathcal{D}_1$, while $\mathcal{D}_2$ occurred after the 1981 election. The structure $\mathcal{D}_1$ is the 3-party case analyzed in the previous section. As we noted in Section 2, the coalition structure $\mathcal{D}_2$ is
interesting since it permits a non-empty legislative core. To be specific, if party 1 (the
PVDA) adopts a policy position within the convex hull of \( \{z_2, z_3, z_4\} \) then it is at the
legislative core, and one would expect coalition bargaining to result in precisely that
outcome (Banks and Duggan, 1998; Laver and Schofield, 1990). We have incorporated
this feature in Assumption 2’, because \( \hat{g}_d(z) \) will then be a singleton. Clearly, in
computing best response under the structure \( \mathcal{D}_2 \), party 1 will have a motivation to choose
\( z_1 \) within the convex hull of \( \{z_2, z_3, z_4\} \). On the other hand, if the probability \( p_1(z) \) that
coalition structure \( \mathcal{D}_1 \) forms is high, then the best response by 1 may well be to diverge
(for the reasons given in Section 4). The best response by 1 will therefore be to choose
\( z_1 \) so that the “marginal utility” resulting from these two events is zero.

To be more specific, we may extend the analysis of Model 2 and write

\[
U_i(z) = p_1(z)U^1_i(z) + p_2(z)U^2_i(z)
\]

where \( \{U_i^t(z)\} \) represents the von Neumann Morgenstern utility obtained from policy
and perquisites in the two different post-election states of the world given by \( t = 1 \) or \( 2 \). To
first order approximation, \( p_1 \), and \( p_2 \) may be regarded as constant, so the first order
marginal utility condition for best response becomes

\[
dU_i = p_1 dU^1_i + p_2 dU^2_i = 0.
\]

As suggested in the analysis of Model 1, choosing \( dU^1_i = 0 \), in the case where policy
is “more important” than perquisites, leads to divergent Nash equilibria. The discussion
in Section 2 suggests that choosing \( dU^2_i = 0 \) leads to a centrist best response, \( z^*_1 \).
Whether the “divergent” or “convergent” tendency dominates depends obviously on
the relative balance of \( p_1 \) and \( p_2 \).

In the most general model, there will certainly exist MSNE. If the equilibria that exist
are not PSNE, then it is difficult to interpret the choice of party leaders. It seems to us
that the natural route to avoid this difficulty is by considering LNE. We believe this
makes sense, because it seems to us plausible that party principals will engage in local
search for the “optimal” leaders of the parties. A related paper (Schofield and Sened,
1998) argues that local Nash equilibria (LNE) will almost always exist, when the
parameters \((p, g)\) are smooth.

We offer a last example from Germany to illustrate some of the properties of local
pure strategy Nash equilibria. Data for 1979 on Germany suggest that the positions of
the principals of the three major parties were almost colinear. Fig. 5 (taken from
Schofield et al., 1998a) gives our estimated positions of the principals of the SPD (Social
Democrats), FDP (Free Democrats), and CDU (Christian Democrats).

Until the recent election of 1998, the coalition structure was \( \mathcal{D}_1 \) (so each two-party
coalition was winning). If we were to assume that positions of principals and leaders
were identical, then the FDP would be close to a legislative core position. However,
because the FDP is the smallest of the three parties, the qualitative analysis given in
Laver and Schofield (1990) and Schofield (1995a), suggests that it could not form a
minority government. It is the case, however, because the FDP “pivots” (between the
SPD and CDU), that we expect it to belong to each coalition government. We suggest
that our Model 1 does not fully capture the extent of the rational calculations of the party
 principals. Both the SPD and CDU party principals should reason with respect to electoral possibilities that could give them majorities in the Bundestag. Thus the underlying electoral model resembles that of Cox (1984). Consequently we would not expect the convergence result, as presented in Theorem 2, to be valid. However, we suggest that compromise within each party would result in a local Nash equilibrium in some neighborhood of the principal’s position. The results of the very simple model presented in the body of the paper suggest that more radical members of each party would prefer an even more radical party leader. Thus, greater heterogeneity within a party may diminish any convergence effect. In fact, ideal points of the SPD delegates do tend to be quite heterogeneous. A second feature that may explain the distance between the SPD position and the electoral center in Fig. 5 is the plausible belief by the SPD delegates that future elections could change the coalition structure.

In fact, in the September 1998 election, the coalition structure was changed, as a
“new” small Green party was able to gain sufficient votes and seats, so as to form a majority coalition with the SPD. As Fig. 5 illustrates, there was ample opportunity for the entrance of such a new party in the policy space in the German polity.

These two examples from the Netherlands and Germany are offered to give some insight into the complicated calculations that party delegates should rationally make in choosing party leaders prior to elections. It is difficult to come to definite conclusions about the equilibrium characteristics of such political games. However, we feel that it is reasonable to make the following plausible generalizations about the relationship between convergence or divergence in party platforms, and the degree of heterogeneity among the parties’ delegates:

1. the more important are perquisites over policy objectives, then the more convergent will be the LNE.
2. the more likely a party believes it is able to occupy a legislative core position, then the more likely it is that its chosen position in LNE will be near the center of the electoral distribution. However, greater heterogeneity in the distribution of ideal points of the party delegates will weaken this tendency.
3. If delegates of a party believe that they are unlikely to obtain sufficient votes near the electoral center to be able to occupy a legislative core position, then they will tend to choose a radical leader, far from the electoral center. Greater heterogeneity in delegates’ ideal points will increase this degree of divergence.

Although the framework presented here may appear unduly complex, it does give some theoretical reasons why there is such great variety in political configurations in the multiparty systems in Europe (Laver and Schofield, 1990; Schofield et al., 1999).

### 6. Conclusion

The model analyzed here has taken a specific, and what appears to us to be a fairly natural structural form for the coalition probabilities and outcomes. It is crucial to our general model that, for each fixed coalition structure, $\mathcal{D}$, the lottery function $g_{\mathcal{D}}: Z \rightarrow W$ is continuous in the space $Z$ of political choices. In fact, $g_{\mathcal{D}}$ is intended to represent the common beliefs by the political elite of the bargaining process under $\mathcal{D}$, and it seems necessary to us to assume these beliefs are continuous. The three-party case, examined as Model 1, exhibits a number of intuitively expected properties, such as convergence to a median in the degenerate one-dimensional or colinear case. As we have indicated, even this convergence property is not robust, as perturbations of the assumptions can result in non-convergent positions. If two dimensions are relevant, then divergence of party declarations occurs when there is a degree of two-dimensional symmetry in the ideal points of party principals. If perquisites, of the same order of magnitude as policy payoffs, are introduced into the model, then a weak form of convergence of party declarations is observed. Only if these private benefits or perquisites dominate policy benefits does Downsian convergence to a single policy point occur. We suggest that the
balance between convergence and divergence may depend on quite subtle properties of the game forms \(g_{ij}\) describing coalition bargaining in different situations.

The model offered here attempts to generalize, in a sense, both the two-party model under electoral risk (Cox, 1984) and the recent multiparty bargaining models (Austen-Smith and Banks, 1988; Banks and Duggan, 1998; Baron, 1989, 1991). With further refinement, we hope to be able to provide a theoretical basis for comparison of two party political systems and multiparty, coalition systems. The eventual goal is to better understand the wide variation that can be observed in political systems, whether based on proportional representation or some form of majoritarian electoral rule.

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Appendix A. Analysis of Model 1 and proof of the Theorems

To solve the response problem for party 3 in Model 1, choose coordinates such that 
\[ z_1 = (0, \frac{r_1}{2}), \quad z_2 = (0, -\frac{r_2}{2}), \quad z_3 = (x, y), \]
and let party 3 have a Euclidean utility function 
\[ u_3(x, y) = -\frac{1}{2}[(x - L)^2 + y^2]. \]
That is party 3 has a bliss point, \(z_3\), at \((L, 0)\). Now 
\[ \|z_1 - z_3\| = \frac{1}{2}(r_1 + r_2). \]
Define 
\[ s_i = \|z_i - z_3\| \]
where \(z_i \in Z\) is the strategy of party 3. For \(i = 1, 2\), let \(\rho_i = \rho_{i3}(z)\) be the probability that coalition \(\{i3\}\) forms and define 
\[ \rho_3 = 1 - \rho_1 - \rho_2. \]
We also assume that if coalition \(\{i3\}\) forms, then party 3 receives a perquisite of \(\sigma_{i3} \geq 0\). Thus given \((z_1, z_2, z_3)\), the outcome for 3 is a lottery:

1. policy \(\frac{z_1 + z_3}{2}\) and bonus \(\sigma_{31}\), with probability \(\rho_1\)
2. policy \(\frac{z_2 + z_3}{2}\) and bonus \(\sigma_{32}\), with probability \(\rho_2\), and
3. policy \(0, \frac{z_1 - z_2}{2}\) and no bonus with probability \(1 - \rho_1 - \rho_2\).

See Fig. 6.

Note we do not consider the formation of the grand coalition \(\{1, 2, 3\}\). Write \(U(x, y)\) for \(U_3(z_1, z_2, z_3)\) and \(u\) for \(u_3\). Then the response problem for 3 is to maximize 
\[ U(x, y) = \rho_1 \left\{ u\left(\frac{z_1 + z_3}{2}\right) + \sigma_{31}\right\} + \rho_2 \left\{ u\left(\frac{z_2 + z_3}{2}\right) + \sigma_{32}\right\} + (1 - \rho_1 - \rho_2) \left\{ u\left(\frac{z_1 + z_2}{2}\right)\right\}. \]
Note of course that $\rho_i$ are both functions of $s_i$, $s_j$ and $\|z_1 - z_2\|$. By assumption

$$\rho_i = \left( \frac{1}{s_i^2} + \frac{1}{s_j^2} + \left( \frac{2}{r_1 + r_2} \right)^2 \right)^{-\frac{1}{2}}$$

Note that $\frac{\partial \rho_i}{\partial s_i} = -\frac{1}{2} (\rho_i - \rho_j^2)$ and $\frac{\partial \rho_i}{\partial s_j} = \frac{2 \rho_i \rho_j}{s_i}$.

Since $s_1^2 = x^2 + \left( \frac{x}{2} - y \right)^2$ and $s_2^2 = x^2 + \left( \frac{y}{2} + y \right)^2$ we find that $\frac{\partial s_i}{\partial x} = \frac{s_i}{s_i}, \frac{\partial s_i}{\partial y} = (y + (-1)(\frac{y}{2})) \frac{1}{s_i}$. Then

$$\begin{pmatrix} \frac{\partial s_1}{\partial x} & \frac{\partial s_1}{\partial y} \\ \frac{\partial s_2}{\partial x} & \frac{\partial s_2}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial \rho_1}{\partial s_1} & \frac{\partial \rho_1}{\partial s_2} \\ \frac{\partial \rho_2}{\partial s_1} & \frac{\partial \rho_2}{\partial s_2} \end{pmatrix} = 2 \begin{pmatrix} x & x \\ y - \frac{r_1}{2} & y + \frac{r_2}{2} \end{pmatrix} \begin{pmatrix} -\frac{\rho_1(1 - \rho_1)}{s_1^2} & 0 \\ 0 & -\frac{\rho_2(1 - \rho_2)}{s_2^2} \end{pmatrix} \begin{pmatrix} \frac{\rho_1 \rho_2}{s_1^2} \\ \frac{\rho_1 \rho_2}{s_2^2} \end{pmatrix} = \begin{pmatrix} \frac{\rho_1 \rho_2}{s_1^2} \\ \frac{\rho_1 \rho_2}{s_2^2} \end{pmatrix}.$$

(1)
Differentiating $U(x, y)$ gives:

\[
\left( \begin{array}{c}
\frac{\partial U}{\partial x} \\
\frac{\partial U}{\partial y}
\end{array} \right) = \left( \begin{array}{c}
\frac{1}{2}(\rho_1 + \rho_2)(L - \frac{x}{2}) \\
- \frac{\rho_1 y + r_1}{2} + \frac{\rho_2 y}{2} - \frac{r_2}{4}
\end{array} \right) + (d\rho) \left( \begin{array}{c}
\delta_1 \\
\delta_2
\end{array} \right).
\]

(2)

Here

\[
\delta_1 = \sigma_{31} + u\left( \frac{x}{2} + \frac{r_1}{4} \right) - u\left( 0, \frac{r_1 - r_2}{4} \right)
\]

\[
\delta_2 = \sigma_{32} + u\left( \frac{x}{2} - \frac{r_2}{4} \right) - u\left( 0, \frac{r_1 - r_2}{4} \right).
\]

The first term on the right hand side of Eq. (2) involves the differential

\[
\left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = \frac{1}{2}\left( L - \frac{x}{2}, -\left( \frac{y}{2} + \frac{r_1}{4} \right) \right),
\]

etc.

It is relatively easy to solve Eq. (2) in the symmetric case when $\sigma = \sigma_{31} = \sigma_{32}$ and $r_1 = r_2$ (so $s_1 = s_2$). To consider the symmetric case (A), where the equation in $x$ dominates, assume that $L \gg 0$ and $x = L$. Then the equation $\partial U/\partial y = 0$ immediately gives $y^* = 0$.

By symmetry, $\rho_1 = \rho_2$ and so $\partial U/\partial y = 0$ implies that the solution in $y$ is $y^* = 0$.

Since $y = 0$, and $\sigma_{31} = \sigma_{32} = \sigma$, then $\delta_1 = \delta_2 = \delta$, and computation shows $\delta = (\sigma + Lx - x^2/4 - r^2/16)$.

Moreover,

\[
\rho_1 = \rho_2 = \rho = \frac{s^{-2}}{2s^{-2} + r^{-2}}
\]

and so $\rho = \frac{r^2}{2s^{-2} + r^{-2}}$. Then

\[
1 - 2\rho = \frac{r^2}{2s^{-2} + r^{-2}} = \frac{s^2}{r^2}, \rho.
\]

Substituting in the equation $\partial U/\partial x = 0$ then gives $\rho(L - x/2) - [2x\rho(1 - 2\rho)d]/s^2 = 0$, which can be rewritten as

\[
\rho \left\{ \left( L - \frac{x}{2} \right) - \frac{2x\rho}{r^2} \left( \sigma + Lx - \frac{x^2}{4} - \frac{r^2}{16} \right) \right\} = 0.
\]

(3)

Assume, until specified below, that $\sigma = 0$.

To solve this equation, consider the case $L = \beta r$, $x = \alpha r$. We obtain

\[
\rho \left\{ \left( \beta - \frac{\alpha}{2} \right) - 2\alpha \rho \left( \alpha \beta - \frac{\alpha^2}{4} - \frac{1}{16} \right) \right\} = 0.
\]

(4)

Now $\rho = \frac{r^2}{s^2 + x^2} = \frac{1}{2} + \alpha^2$ since $s^2 = \left( \frac{r}{2} \right)^2 + x^2$. Assuming $\rho \neq 0$, gives a quadratic expression in $\alpha$, $\beta$ with solution

\[
\alpha = \frac{1}{2} \left( -1 \beta \pm \sqrt{1 - 9} \beta^2 \pm 9 \right).
\]

(5)
Thus if \( L \gg r \), so that \( \beta \gg 1 \), then \( x^* = 3r/2 \). Notice that there is only one positive solution, so that \( U(x, 0) \) is concave on the positive \( x \)-axis.

Examination of \( \partial U/\partial x, \partial U/\partial y \) in a neighborhood \((x^*, 0)\) shows that \((x^*, 0)\) maximizes \( U \). (This is because the determinant of \( dU \) is a function of \( \rho_1, \rho_2(1 - \rho_1 - \rho_2) > 0 \) and has negative diagonal terms.) The negative solution for \( x \) in Eq. (5) corresponds to a minimum. Note also that as \( x \to \pm \infty \), then \( \rho \to 0 \) and the outcome approaches \((0, 0)\).

Clearly, if \( r_1 = r_2 \neq 0 \), the (maximum) solution to Eq. (2) is unique and gives a global maximum for \( U \), for each fixed \( z_1, z_2 \). Moreover, the solution to Eq. (2) is continuous in \((z_1, z_2)\). If the bliss points satisfy the symmetry condition \( \|o_1 - o_2\| = \|o_2 - o_3\| = \|o_1 - o_3\| \), then the conditions of the Glicksburg theorem are satisfied and a stable, pure-strategy Nash equilibrium exists.

Note however that if \( x = 3r/2 \), then \( s^2 = (r/4)^2 + x^2 > r^2 \). Thus best response produces a divergence of the position of party 3 from the positions of parties 1 and 2. To examine the symmetric case further, suppose that \( \beta = 1 \). It is easy to show that if \( \beta = \sqrt{5}/2 \), then \( \alpha = \beta \). In particular, if \( \beta \in (\sqrt{5}/2, \infty) \), then \( \alpha \in (\sqrt{5}/2, 3/2) \) with \( \alpha < \beta \), while if \( \beta \in (0, \sqrt{5}/2) \) then \( \alpha > \beta \), but \( \alpha \in (0, \sqrt{5}/2) \). If \( \beta = 1/\sqrt{3} \), then \( \alpha = \sqrt{3}/2 \), and \( s^2 = r^2 \).

Thus a symmetric Nash equilibrium \( \|z_i^* - z_j^*\| = \|o_i^* - o_j^*\| = \|o_i^* - z_i^*\| \) can occur. The equilibrium is related to the ideal points \( \{o_1, o_2, o_3\} \) by the condition

\[
\frac{\|z_i^* - z_j^*\|}{\|o_i - o_j\|} = 2
\]

for each pair \( i, j \).

In the non-symmetric case with \( r_1 \neq r_2 \), consider first a perturbation of the above situation with \( r_1 = r_2 \), and \( \rho_1 = \rho_2 \). In Eq. (2), note that \( \delta_1 = \delta_2 = \delta \).

Then Eq. (3) becomes

\[
\frac{\rho_1 + \rho_2}{2}(L - x) - 2x \left( \frac{\rho_1^2 + \rho_2^2}{r^2} \right) \left( L_x - \frac{x^2}{4} - \xi \right) = 0
\]

where \( r = \frac{r_1 + r_2}{2} \) and \( \xi = \frac{\rho^2}{2r^2} \).

Equation \( \partial U/\partial y = 0 \) gives

\[
y(\rho_1 + \rho_2) = \frac{1}{2}(\rho_1 r_1 - \rho_2 r_2) - 8y \left( \frac{\rho_1^2}{r_1^2} + \frac{\rho_2^2}{r_2^2} \right) \delta + 4 \left( \frac{r_1 \rho_1^2}{r_1^2} - \frac{r_2 \rho_2^2}{r_2^2} \right) \delta.
\]

Clearly, for \( r_1 > r_2 \), the optimal choice \( y^* < 0 \), since \( \rho_1, \rho_2 \) are dominated by the choice in \( x \). Thus \( y^* \) is essentially a function of \( r_1 - r_2 \). Hence for \( r_1 = r_2 \), \( y^* = 0 \). For this case let us say that the \( x \)-solution dominates the \( y \)-solution.

We have shown above that if \( r_1 = r_2 \) then Eq. (3) is a quadratic expression in \( x \), with one positive root. For general \( r_1, r_2 \), Eq. (6) involves higher powers of \( x \). Nonetheless the quadratic terms dominate and the critical point in the domain \( x > 0 \) always corresponds to a global maximum of \( U \).

To deal with the situation where the bliss points are non-symmetric, consider the colinear case \((B)\) when the third party has bliss point \((0, 0)\) and the two other parties are
positioned at \( z_1 = (0, r_1/2) \), \( z_2 = (0, -r_2/2) \) as before. Let us first consider the response by \( z \) on the \( y \)-axis.

As a first order approximation suppose \( r_1 \approx r_2 \), with \( r_1 > r_2 \). Then from Eq. (7), at \((x, y) = (0, 0)\) we obtain

\[
\frac{dU}{dy} \approx \frac{1}{8}(\rho_1 r_1 - \rho_2 r_2) \approx -\frac{1}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) > 0.
\]

Note here that \( \rho_1 = 4/r_1^2 \).

Thus the best response is at a point \( y^* > 0 \). In fact it can be shown that \( y^* \approx (r_1 - r_2)/12 \), so the optimal response is to partially equalize the distances to \( z_1 \) and \( z_2 \). We can also determine the optimal response by \( z \) when both \( z_1 \) and \( z_2 \) lie on the same side of the origin. To illustrate, suppose \( r_1 = 1, 0; r_2 = -0.8 \). Then it is easy to show that \((\rho_1, \rho_2, \rho_3) = (0.04, 0.06, 0.9)\).

For \( y = 0 \), Eq. (7) can be rewritten as:

\[
\frac{\partial U}{\partial y} = -\frac{1}{8}(\rho_1 r_1 - \rho_2 r_2) + \frac{\rho_1 r_1 (1 - \rho_1) \delta_1 - \rho_2 \delta_2}{s_2} + \frac{\rho_2 r_2 (\rho_1 \delta_1 - \delta_2 (1 - \rho_2))}{s_2}.
\]

In this case \( \delta_1 \) can be regarded as the utility gain from having an outcome at \( \frac{1}{2}(z_1 + z_2) \) rather than \( \frac{1}{2}(z_1 + z_2) \). Easy computation shows that \( \delta_1 = 0.07, \delta_2 = 0.08 \).

Thus at \( y = 0 \),

\[
\frac{dU}{dy} = -\frac{1}{8}(0.04 + (0.06)(0.8)) + 0.027 = 0.016 > 0,
\]

Thus the optimal response is to move to a position \( y^* \) closer to both \( z_1 \) and \( z_2 \).

**Preliminaries to a proof of the Theorems.**

Proof of existence of a PSNE in any spatial model must deal with problems of failure of quasi-concavity and degenerate cases (Dasgupta and Maskin, 1986). As we have noted in the previous examples, the induced utility functions are not quasi-concave. However the utility function \( U \) considered above has a unique maximum. To see this, note that the optimality equations \( \partial U/\partial \delta = dU/\partial y = 0 \) are fundamentally quadratic equations, although they involve higher order terms in \( \rho_{13} \) and \( \rho_{23}^2 \). If the scheme \( I' = \{a_i\} \) involves benefits that are sufficiently large, then the coefficients of the higher order terms will be small, and each optimality equation will have only two roots, corresponding to a maximum and minimum. We may parameterize the joint best reaction correspondence, \( h \), by \( o = (o_1, o_2, o_3) \in Z^3 \). It was shown that when the scheme, \( I' \), of private benefits is sufficiently large, then

\[
h(o):Z^3 \to Z^3
\]

is a continuous function, which will exhibit a fixed point.

“Degeneracy”, when \( z_1 = z_2 \), say, so that \( \rho_{12} = 1 \), was analyzed by computation to show that such a Nash equilibrium is unstable. In particular, in cases where \( \{o_1, o_2, o_3\} \) were not colinear, then the best reaction correspondence \( h(o):Z^3 \to Z^3 \) led away from \( z \) (z, z, z). That is, for any neighborhood \( V \) of \( z \), it was possible to find \( z^* \in V \), such that \( h(o)(z^*) \cap V = \phi \).
Proof of Theorem 1. For a general scheme $I$ it is possible that some best reaction correspondences will not be single-valued. Convexification of the best reaction correspondence, and application of the Glicksburg Theorem gives existence of an MSNE.

However the previous argument implies that there exists $\sigma^*$ such that when $\sigma^*_i > \sigma^*$, then each individual best response will be single-valued. Examination of $h(o):Z^3 \to Z^3$ shows that it is a continuous function. Thus there exists a fixed point, giving a PSNE. Moreover any such PSNE, $z^*$, simultaneously satisfies the optimality equations for each party, and is therefore stable. \hfill \Box

Proof of Theorem 2. Above we considered the extreme case of $I$ identically zero with colinear ideal points, and showed that best response by each player required convergence to a point on the ideal point axis. The resulting fixed point, $\mathcal{E}(o)$, of $h(o):Z^3 \to Z^3$ is unique and corresponds to a stable Nash equilibrium. Now $h(o)$ is a contraction on $Z^3$, in the sense that $\|h(o)(z_1) - h(o)(z_2)\| < \|z_1 - z_2\|$ for any $z_1, z_2 \in Z^3$. Moreover there is a neighborhood $V$ of $o$ in $Z^3$ such that $h(o')$ is single-valued, as well as continuous for all $o'$ on $V$. Thus there exists a neighborhood $V$ of $o$ in $Z^3$ such that for all $o' \in V$, $\mathcal{E}(o')$ is a unique stable Nash equilibrium. By Fort’s Theorem (Fort, 1950) there exists a neighborhood $V_1$ of $o$ in $Z^3$, with $V_1 \subset V$, and a neighborhood $V_2$ of $\mathcal{E}(o)$ in $Z^3$ such that $\mathcal{E}(o') \in V_2$ for all $o' \in V_1$.

Since “$o$” comprises colinear points, there exists $\varepsilon^* > 0$ such that $o' \in V_1$ implies $o'$ is $\varepsilon$-bounded in linearity for some $\varepsilon < \varepsilon^*$. Moreover since $\mathcal{E}(o)$ is convergent, $\mathcal{E}(o') \in V_2$ implies that $\mathcal{E}(o')$ is also $\varepsilon$-bounded in linearity. \hfill \Box

Proof of Theorem 3. Suppose now that the profile of bliss points, $o$, is symmetric. Setting $\sigma = 0$ in Eq. (3) shows that for the best response there is a unique scale relationship between the bliss points and the Nash equilibrium. As we have shown $\|z_i^* - z_j^*\| = 2\|o_i - o_j\|$, for each pair $i, j$. Just as in the proof of Theorem 2, there exist neighborhoods $V_1$ of $o$ and $V_2$ of $\mathcal{E}(o)$ such that the Nash equilibrium correspondence is a function

$$\mathcal{E}: V_1 \to V_2.$$ 

Again there exists $\varepsilon^* > 0$ such that any profile which is $\varepsilon$-bounded in symmetry, for $\varepsilon < \varepsilon^*$, lies in $V_1$. Thus $\mathcal{E}(o') = (z_1^*, z_2^*, z_3^*)$ will satisfy

$$\|z_i^* - z_j^*\| = b_{ij}(\varepsilon)\|o_i' - o_j'\|$$

for $b_{ij}(\varepsilon)$ close to 2. \hfill \Box

Proof of Theorem 4. Consider Eq. (3) in the case $r_1 = r_2$, and $\sigma \neq 0$. If $\sigma = r^2$, instead of Eq. (4) we obtain

$$\left( \beta - \frac{\sigma}{2} \right) - 2\alpha\rho \left( 1 + \alpha\beta - \frac{\alpha^2}{4} - \frac{1}{16} \right) = 0,$$

with solution
\[
\alpha = \frac{1}{2} \left( \frac{-3}{\beta} \pm \frac{1}{\beta^2 + 1} \right).
\]

Again, there is only one positive root, so that if \( \beta = \sqrt{3} \), then \( \alpha = \sqrt{3}/2 \). This implies that a symmetric Nash equilibrium exists, but satisfying

\[
\frac{\|z_i - z_j\|}{\|o_i - o_j\|} < 1
\]

for each pair \( i, j \). Thus the equilibrium is convergent.

The computation implicit in Fig. 4 shows that the equilibrium satisfies the equation

\[
\|z_i^* - z_j^*\|^2 = \frac{3}{23} \|o_i - o_j\|^2.
\]

It is evident that a unique symmetric equilibrium occurs if \( \sigma = r^* = \frac{1}{23} \|o_i - o_j\|^2 \). Thus there is some bound, \( \sigma^* < \frac{1}{23} \|o_i - o_j\|^2 \), such that there exists a unique symmetric convergent equilibrium, \( z^* \), satisfying \( \|z_i^* - z_j^*\| < \|o_i - o_j\| \), whenever \( \sigma > \sigma^* \). A continuity argument, together with Fort's Theorem completes the proof. \( \square \)

References

Axelrod, R., 1970. Conflict of Interest, Markham, Chicago.


