Existence of stable outcomes and the lattice property for a unified matching market

Marilda Sotomayor*

FEA-USP-Department of Economics, Universidade de São Paulo, 908 Butantá SP, 05508-900 CEP Sao Paulo, Brazil

Received 1 December 1997; received in revised form 1 June 1998; accepted 1 February 1999

Abstract

This paper establishes the existence of stable matchings, the lattice property of the core and the existence of optimal stable payoffs for each side of the market. We do this in a matching market consisting of a 'mixed economy' in which some firms compete by means of salary and others have no flexibility over terms of appointment. Common proofs are used, which unify the traditional discrete and continuous cases, namely the marriage and assignment models. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Matching; Stable matching; Core

JEL classification: C78; D78

1. Introduction

There is now a large literature on matching, both theoretical and empirical, concerned with stable matchings and mechanisms which achieve them. An unusual feature of the theoretical literature is that quite similar results have been established for both discrete and continuous models, initially with fundamentally dissimilar proofs, and it has turned out to be difficult to unify these models. This divide goes back to Gale and Shapley (1962) and Shapley and Shubik (1972) which show the non-emptiness of the set of stable matchings for discrete and continuous models, respectively, using combinatorial arguments in one case and linear programming arguments in the other. These same families of arguments can then be extended to show that the sets of stable matchings in
each kind of model share many common properties such as a lattice structure in terms of the common preferences of agents on one side of the market, with corresponding optimal outcomes for each side of the market at which agents on that side will have no incentive to misrepresent their preferences etc. (see Roth and Sotomayor, 1990, for a review of the literature).

The empirical importance of both kinds of models comes from the variety of labor markets. For example, new law school graduates may enter the market for associate positions in private law firms, which compete with each other in terms of salary, or they may seek employment as law clerks to federal circuit court judges, which are civil service positions with predetermined fixed salaries. Traditionally the former kind of market has been modeled as an Assignment Game, in which salary may be negotiated and may vary continuously on the set of real numbers, while the latter has been modeled as a Marriage Market. In this market salary is modeled as part of the job description and it is one of the factors that determine the preferences that workers have over firms.

One line of work which went some way towards unifying these disparate models involved creating linear programming formulations of stable matchings in discrete markets (for example, Vande Vate, 1989; Rothblum, 1992; Roth et al., 1993). Those papers revealed a good deal of surprising algebraic structure in the set of stable matchings of discrete markets, but the large differences between the linear programming formulations for the discrete case and the continuous case only emphasized that similar results were being obtained for very different reasons in the two cases.

Another approach has been to look at models which generalize both the Marriage Model and the Assignment Model, and try to obtain the common results for the two in the more general model. This was the approach taken in Roth and Sotomayor (1996). In contrast with previous treatments, this paper derives the parallel conclusions for the two sets of models in the same way from the same assumptions. It shows the lattice property of the core, the existence of optimal stable outcomes and some comparative statics results.

However, that paper does not provide any existence theorem of stable outcomes. Furthermore, in the model of Roth and Sotomayor (1996) a given agent either is in the discrete market or is in the continuous one. This depends on the set of allowable salaries. For empirical purposes, the problem with such a model is of course that all individuals have their choices restricted to only one of the markets, although some of them may wish to enter both kinds of markets simultaneously. In the real world these markets are really parts of a single market.

Kaneko (1982) provides a quite general and complex model that generalizes the Assignment Game of Shapley and Shubik. He observes that the Marriage model is a special case of this model. He proves the non-emptiness of the core (but not the lattice result or related structural properties).

Eriksson and Karlander (1997) also provides a single market in which the two kinds of agents can trade. However, they prove the existence of stable outcomes and the lattice property of the core for a particular case of this model, which does not include the continuous market as it was proposed by Shapley and Shubik.

In the present paper, following the idea of Eriksson and Karlander (1997), we unify the two kinds of markets by presenting a single market that includes the discrete and
continuous markets. The interest of this model is that it mimics real life to the effect that
an agent may trade in both markets simultaneously.

For this market we propose to fill some existing gaps. We first prove the non-
emptiness of the set of stable outcomes. Gale and Shapley (1962) uses an algorithm to
prove the existence of stable outcomes for the marriage market; Shapley and Shubik
(1972) uses the Duality Theorem to prove the same result for the continuous Assignment
Game; Demange et al. (1986) proves the existence of stable outcomes for the
Assignment Game with an integer payoff matrix by using a mechanism where objects
are auctioned simultaneously.

An old challenge in the literature is to find a simple proof of the existence result for
both models, using similar arguments. The proof of Kaneko (1982), which uses the
technique of balanced games, is very complicated because of the complexity of its
model. The recent proof of Eriksson and Karlander (1997) does not hold for all possible
assignment games. It uses a version of the proof of Demange et al. (1986) which only
applies when the payoff matrices have integer entries.

In this paper we use combinatorial arguments. The main feature of our proof is that it
is simpler and shorter than the proofs of the other authors. Moreover, it holds for both
models without any restriction. When restricted to the Marriage Model it coincides with
the existence proof of Sotomayor (1996). When restricted to the Assignment Game it is
also very short and it can be used as an alternative proof of the non-emptiness of the
core of this model.

Next we prove that the set of stable payoffs is endowed with the structure of a
complete lattice. Roughly speaking, if A and B are two stable outcomes then it might be
that some workers (resp. firms) will get more income under A than under B and others
will get more under B than under A. The lattice theorem implies that there is a stable
outcome C, which gives each worker (resp. firm) the larger of the two amounts and also
one, D, which gives each of them the smaller amount. This is of interest because, together with the compactness of the set of stable payoffs, it shows that among all stable
payoffs there is one (and only one) which is worker optimal (resp. firm optimal),
meaning that all workers (resp. firms) get at much income under it as under any other
stable payoff.

Eriksson and Karlander (1997) proves the lattice result under a very restrictive
assumption. It assumes strict preferences for the Marriage Market and unicity of the
optimal matching for the Assignment Game. John Conway, in Knuth (1976), requires
strictness of the preferences to prove the lattice property for the Marriage Market. Our
assumption is that the strong core and the weak core coincide. In particular our result
holds for the Marriage Market when preferences are strict. It always holds for the
Assignment Market independently of the number of optimal matchings. In the marriage
model, the assumption of strict preferences causes those two sets to coincide, while in
the continuous models the two sets coincide because agents have continuous preferences
and prices can be adjusted continuously.

Finally, as a corollary of the lattice property and its completeness, we show that, when
the strong core and the weak core coincide, the optimal stable payoff for each side
exists.

This paper is organized as follows: Section 2 describes the generalized model. Section
3 proves the existence of stable outcomes. Section 4 shows that the core is a complete lattice and that it has a unique optimal stable payoff for each side of the market. Two illustrative examples are presented at the end of this section.

2. The mathematical model

There are two finite and disjoint sets of players $P = \{1, \ldots, i, \ldots, m\}$ and $Q = \{1, \ldots, j, \ldots, n\}$. The players in $P$ and $Q$ will be called firms and workers, respectively. There are two classes of players: $F$ and $R$ with $F \cup R = P \cup Q$. Set $F^* = \{(i, j) \in P \times Q; i \in F$ and $j \in F\}$ and $R^* = \{(i, j) \in P \times Q; i \in R$ or $j \in R\}$. Therefore, $P \times Q = R^* \cup F^*$. For each pair $(i, j) \in P \times Q$ there is a pair of numbers $(a_{ij}, b_{ij})$, where $a_{ij} + b_{ij}$ can be interpreted as being the productivity of the pair $(i, j)$. If the partnership $(i, j)$ is formed then the productivity $a_{ij} + b_{ij}$ can be divided between $i$ and $j$ so that $i$ receives the payoff $u_i$, $j$ receives the payoff $v_j$ and $u_i + v_j = a_{ij} + b_{ij}$. If the partnership $(i, j) \in R^*$ then $u_i = a_{ij}$ and $v_j = b_{ij}$.

**Definition 1.** A **matching** is a matrix $x = (x_{ij})$ of zeros and ones that satisfies $\Sigma_{i \in P} x_{ij} \leq 1$ and $\Sigma_{j \in Q} x_{ij} \leq 1$. If $x_{ij} = 1$ (resp. $x_{ij} = 0$) we say that $i$ and $j$ are **matched** (resp. **unmatched**) at $x$. If $\Sigma_{i \in P} x_{ij} = 0$ (resp. $\Sigma_{j \in Q} x_{ij} = 0$) we say that $j$ (resp. $i$) is unmatched at $x$. If $x_{ij} = 1$ we can write $x(i) = j$ or $x(j) = i$. Given $S \subseteq P$ we write $x(S) = \{j \in Q; x_{ij} = 1$ for some $i \in S\}$. Analogously we define $x(S)$ for $S \subseteq Q$.

**Definition 2.** An **outcome** is a matching $x$ and a pair of vectors $(u, v)$ called payoff, with $u \in R^m$ and $v \in R^n$. An outcome will be denoted by $(u, v; x)$.

**Definition 3.** An outcome $(u, v; x)$ is **feasible** if

(i) $u_i \geq 0$, $v_j \geq 0$, for all $(i, j) \in P \times Q$ and $u_i = 0$ (resp. $v_j = 0$) if $i$ (resp. $j$) is unmatched.
(ii) $u_i + v_j = a_{ij} + b_{ij}$ if $x_{ij} = 1$, and
(iii) $u_i = a_{ij}$ and $v_j = b_{ij}$ if $x_{ij} = 1$ and $(i, j) \in R^*$.

A feasible payoff is a pair of vectors $(u, v)$ with $u \in R^m$ and $v \in R^n$ such that for some matching $x$, $(u, v; x)$ is a feasible outcome; in this case we say that $x$ is **compatible** with $(u, v)$.

Given some matching $x$, we say that $i$ is **acceptable** to $j$ if $x_{ij} = 1$ and $v_j \geq 0$. Similarly $j$ is acceptable to $i$ if $x_{ij} = 1$ and $u_i \geq 0$.

**Definition 4.** An outcome $(u, v; x)$ is **stable** if it is feasible and for all $(i, j) \in P \times Q$ we have that:

(i) $u_i + v_j \geq a_{ij} + b_{ij}$ if $(i, j) \in F^*$, and
(ii) $u_i \geq a_{ij}$ or $v_j \geq b_{ij}$ if $(i, j) \in R^*$.
A payoff vector \((u, v)\) is stable if \((u, v; x)\) is stable for some matching \(x\).

A blocking pair of a feasible outcome \((u, v; x)\) (resp. a feasible payoff \((u, v)\)) is a pair \((i, j)\) such that either \((i, j) \in P \times Q\) and \(u_i + v_j \leq a_{ij} + b_{ij}\), or \((i, j) \in R^*\) and \(u_i < a_{ij}\) and \(v_j < b_{ij}\); in any case we also say that \(i\) and \(j\) are blocking partners. Therefore, a feasible outcome (resp. payoff) is stable if and only if it does not have any blocking pair.

Observe that the existence of a blocking pair is equivalent to the existence of a pair \((i, j)\) and a feasible outcome \((u', v'; x')\), with \(x_{ij}' = 1\), such that \(u_i' > u_i\) and \(v_j' > v_j\). This is to say that \((u, v)\) is stable if it is not dominated by any feasible payoff via some pair \((i, j)\) and matching \(x'\). Since the pairs \((i, j)\) are the only essential coalitions, a payoff is stable if and only if it is in the core, which will be denoted by \(C\).

**Definition 5.** Let \(A_i\) be the set of all \(i\)'s blocking partners for the feasible outcome \((u, v; x)\). For each \(j \in A_i\) define a feasible outcome \((u', v'; x')\) such that:

(i) \(x_{ij}' = 1\),
(ii) \(u_i' = a_{ij}\) if \((i, j) \in R^*\), and
(iii) \(u_i' = a_{ij} + b_{ij} - v_j\) if \((i, j) \in F^*\).

\((u_i'\) means the highest payoff that \(j\) would pay to \(i\) if they break their current partnership and work together.

We say that \(j\) is an \(i\)'s favorite blocking partner if \(u_i' \geq u_i^k\) for all \(k \in A_i\).

Definition 5 implies that if \(j\) is an \(i\)'s favorite blocking partner for \((u, v; x)\) then \(i\) cannot form a blocking pair for \((u', v'; x')\).

**Definition 6.** An outcome \((u, v; x)\) is strongly stable if it is stable and there is no pair \((i, j) \in P \times Q\) with \(x_{ij} = 0\) and a feasible outcome \((u', v'; x')\) such that \(x_{ij}' = 1\), and

(i) \(u_i' > u_i\) and \(v_j' = v_j\), or
(ii) \(u_i' = u_i\) and \(v_j' > v_j\).

Similarly we define a strongly stable payoff.

Definition 6 says that \((u, v)\) is strongly stable if it is stable and it is not weakly dominated by any feasible payoff via some coalition \((i, j)\). The pair \((i, j)\) is called a weak blocking pair. Thus, if \((i, j) \in R^*\) and \((i, j)\) weakly blocks \((u, v; x)\) then \(x_{ij} = 0\) and:

(i) \(a_{ij} > u_i\) and \(b_{ij} = v_j\); or
(ii) \(a_{ij} = u_i\) and \(b_{ij} > v_j\).

---

1 An outcome \((u, v; x)\) is in the core if it is not dominated by any other outcome via some coalition.
It follows from Definitions 4 and 6 that if \((i, j) \in F^*\) then \((i, j)\) weakly blocks \((u, v; x)\) if and only if it blocks \((u, v; x)\). Thus, if all players are in \(F\) then \((u, v; x)\) is strongly stable if and only if it is stable.

Since the pairs \((i, j)\) are the only essential coalitions, the set of strongly stable payoffs coincides with the core defined by weak domination, which is a subset of the core and will be denoted by \(C^*\).

The following models are well known special cases of the model we are treating here.

2.1. Marriage model

Here \(F = \emptyset\). Player \(i\) is acceptable to player \(j\) if \(b_{ij} \geq 0\). Analogously, player \(j\) is acceptable to player \(i\) if \(a_{ij} \geq 0\). If \(a_{ij} \geq 0\) and \(b_{ij} \geq 0\) we say that \(i\) and \(j\) are mutually acceptable. The players can list their potential partners, in order, in a finite list of preferences. Let \(L(i)\) denote the list of preferences of player \(i\). Thus, \(L(i) = h, [j, k], \ldots, i, p\) means that \(a_{ih} > a_{ij} = a_{ik} > 0 > a_{ip}\). That is, \(i\) prefers \(h\) to \(j\) and is indifferent between \(j\) and \(k\). Agent \(p\) is not acceptable to \(i\), so \(i\) prefers to be unmatched to be matched to \(p\).

In this model, any outcome \((u, v; x)\) is completely specified by the matching \(x\). Such a matching is called feasible (resp. stable) if the corresponding outcome is feasible (resp. stable). Therefore, a matching is feasible if every matched pair is mutually acceptable. (Here feasibility means individual rationality.) Hence, \(x\) is a stable matching if it is individually rational and there is no pair \((i, j) \in P \times Q\) such that \(u_i < a_{ij}\) and \(v_j < b_{ij}\). If we consider that an unmatched agent is self-matched then:

The matching \(x\) is stable if it is individually rational and there is no pair \((i, j) \in P \times Q\) such that \(i\) and \(j\) prefer each other to their respective mates.

The matching \(x\) is strongly stable if there is no pair \((i, j) \in P \times Q\) such that

(a) \(i\) prefers \(j\) to \(x(i)\) and \(j\) is indifferent between \(i\) and \(x(j)\), or
(b) \(i\) is indifferent between \(j\) and \(x(i)\) and \(j\) prefers \(i\) to \(x(j)\).

When preferences are strict, it follows from the definitions above that the set of stable matchings coincides with the set of strongly stable matchings. In this case the resulting model is the well known Marriage model, introduced in Gale and Shapley (1962).

2.2. The assignment game

In this case \(R = \emptyset\). If we define \(a_{ij} = \max\{a_{ij} + b_{ij}, 0\}\) then:

A feasible outcome \((u, v; x)\) is stable if and only if \(u_i + v_j \geq a_{ij}\) for all \((i, j) \in P \times Q\).

In this model the set of stable outcomes coincides with the set of strongly stable outcomes. In fact, if \((i, j)\) weakly blocks \((u, v; x)\) then \(i\) can transfer some payoff to \(j\) or \(j\) can transfer some payoff to \(i\), so that \((i, j)\) blocks \((u, v; x)\).

This model is the assignment game introduced in Shapley and Shubik (1972).
3. Existence of stable outcomes

Let \((u, v; x)\) be a feasible outcome. Define:
\[
D_i(u, v, x) = \{ j \in Q; x_j = 0 \text{ and } (u = a_j + b_{ij} - v_j \text{ if } (i, j) \in F^*) \text{ or } (u = a_j \text{ and } v_j < b_{ij} \text{ if } (i, j) \in R^*) \} \cup \{i's \text{ favorite blocking partners}\}.
\]

It follows from the definition above that, if \(j \in D_i(u, v, x)\) and \((i, j)\) is not a blocking pair, then we can construct a feasible outcome where \(i\) and \(j\) are matched to each other and \(i\) is indifferent between both outcomes. Moreover, \(j\) is indifferent between both outcomes when \((i, j) \in F^*\) and is better off when \((i, j) \in R^*\).

Set \(S = \{(u, v; x) \text{ feasible}; \text{ if } (i, j) \text{ blocks } (u, v; x) \text{ then } i \text{ must be unmatched}\}.

Thus, if \((u, v; x)\) is in \(S\) then no matched firm forms a blocking pair. The set \(S\) is non-empty, for the outcome where every player is unmatched is in \(S\). In fact, if every player is unmatched then no matched firm forms a blocking pair.

It follows from the construction of \(S\) that the set of payoffs \(v\) such that \((u, v; x) \in S\), for some \(u\) and \(x\), is a compact set of \(R^*\). Then there is some outcome \((u^*, v^*; x^*) \in S\) such that
\[
\sum_{j \in Q} v^*_j \geq \sum_{j \in Q} v_j \quad \text{for all } (u, v; x) \in S.
\]
and \(x^*\) has the minimum number of blocking pairs among all \(x\) which are compatible with \(v^*\).

We are going to show that \((u^*, v^*; x^*)\) is stable. Before we need the following result:

**Lemma 1.** Construct a graph whose vertices are \(P \cup Q\). There are two kinds of arcs. If \(x^*_ij = 1\) there is an arc from \(j\) to \(i\); if \(j \in D_i(u^*, v^*, x^*)\) there is an arc from \(i\) to \(j\). Suppose there exists some unmatched firm \(i^* \in P\) with \(D_i(u^*, v^*, x^*) \neq \emptyset\). Let \(j_1 \in D_i(u^*, v^*, x^*)\). Then there is an oriented path which starts from \(i^*\), has \(j_1\) as its second node and reaches a player of \(R\), or an unmatched worker, or a firm with payoff zero.

**Proof.** Suppose there is no such path, and denote by \(T\) and \(W\) the sets of firms and workers, respectively, that can be reached from \(j_1\). Then \(i^* \notin T\), there is no player of \(R\) in \(T \cup W\), all of \(W\) are matched and \(u^*_i > 0\) for all \(i \in T\), so all of \(T\) are also matched. Furthermore, \(W\) is non-empty since \(j_1 \in W\) and \(T\) is non-empty because all of \(W\) are matched to those of \(T\) at \(x^*\). If \(j \in W\), then there is no \(i \in T\) such that \(x^*_ij = 1\) or \(j \in D_i(u^*, v^*, x^*)\). (If \(x^*_ij = 1\) there is an arc from \(j\) to \(i\) and if \(j \in D_i(u^*, v^*, x^*)\) then there is an arc from \(i\) to \(j\). In both cases if \(j\) is not in \(W\), \(i\) cannot be in \(T\).) Then for all \(i \in T\) and all \(j \in W\), we must have that \(x^*_ij = 0\) and \(j \in D_i(u^*, v^*, x^*)\). Furthermore, \((i, j)\) is not a blocking pair, for all \(i \in T\) and all \(j \in W\), because \(i\) is matched and \((u^*, v^*; x^*) \in S\). Using the definition of \(D_i(u^*, v^*, x^*)\) and recalling that all players are in \(F\), it follows that \(u^*_i > a_{ij} + b_{ij} - v^*_j\) for all \(i \in T\) and \(j \in W\). Then there is some \(\lambda > 0\), sufficiently small, so that \(u^*_i - \lambda > 0\) and \(u^*_i - \lambda > a_{ij} + b_{ij} - v^*_j\) for all \(i \in T\) and all \(j \in W\). Hence, if we decrease \(u^*_i\) and increase \(v^*_j\) by \(\lambda > 0\), for all \((i, k) \in T \times W\), the resulting outcome is still in \(S\). (It is enough to see that no pair \((i, j) \in T \times (Q - W)\) blocks the new
outcome. However, all of \( W \) are better off, while the players in \( Q - W \) are indifferent between both outcomes, which contradicts inequality (1).

**Theorem 1.** The outcome \((u^*, v^*; x^*)\) is stable.

**Proof.** The plan of the proof is very simple. If \((u^*, v^*; x^*)\) is not stable then we can choose a blocking pair \((i^*, j_1)\), where \(j_1\) is some \( i^* \)'s favorite blocking partner, so \(j_1 \in D_{i^*}(u^*, v^*, x^*)\). The fact that \((u^*, v^*; x^*)\) \(\in S\) implies that \(i^*\) is unmatched. Lemma 1 guarantees the existence of an oriented path \(c = (i^* = i_1, j_1, i_2, j_2, \ldots)\) which starts from \(i^*\), has \(j_1\) as its second node, reaches a player \(y \in R\) or an unmatched worker \(j_s\), or a firm \(p\) with payoff zero. The basic idea is to construct a new outcome in \( S \) by rematching the nodes of the path. Then we show that the existence of this outcome contradicts the definitions of \( v^* \) or of \( x^* \). We consider three cases.

1st case: No oriented path which starts from \(i^*\) and has \(j_1\) as its second node contains an element of \( R \). Then there exists an oriented path, \( c \), which starts from \(i^*\), has \(j_1\) as its second node and reaches an unmatched worker \(j_s\) or a firm \(p\) with payoff zero. That is: \(c = (i^* = i_1, j_1, i_2, j_2, \ldots, i_s, j_s)\) or \(c = (i^* = i_1, \ldots, i_s, j_s, p, \ldots)\). Without loss of generality we can suppose that all elements in the path, until \(p\), are distinct. Construct the matching \(x'\) that matches \(i^*\) to \(j_1\), \(i_2\) to \(j_2\), \ldots, \(i_s\) to \(j_s\) and leaves \(p\) unmatched if \(p\) is in the path, and that otherwise agrees with \(x^*\) on every player who is not in the path. Now give the payoff \(a_{i_1j_1} + b_{i_1j_1} - v^*_{j_1}\) to \(i^*\) and keep unchanged the payoffs of the rest of the players. We claim that the resulting outcome is in \( S \). That it is feasible is immediate from the following facts: (a) \(i^*\) is matched to \(j_1\) with payoff \(a_{i_1j_1} + b_{i_1j_1} - v^*_{j_1}\); and (b) for all \(t = 2, 3, \ldots, s\), \(i_t\) is matched to \(j_t\), \(j_t \in D_{i_t}(u^*, v^*, x^*)\) and \((i_t, j_t) \in F^*\), so \(u^*_{i_t} = a_{i_tj_t} + b_{i_tj_t} - v^*_{j_t}\). That there is no new blocking pair follows from the fact that \(i^*\) is matched to a favorite blocking partner and the payoffs of the players other than \(i^*\) did not change. Now use the fact that \((u^*, v^*; x^*)\) is in \( S \). Therefore, we have decreased by one the number of blocking pairs, which contradicts the definition of \(x^*\).

2nd case: There is an oriented path which starts from \(i^*\), has \(j_1\) as its second node and reaches some player in \( R \).

Let \(y\) be the first player in \(c\) who belongs to \( R \) and let \(s\) be the first time \(y\) appears. Without loss of generality we can suppose that all elements in the path, until \(y\), are distinct. There are three subcases:

(A): \(c = (i^* = i_1, j_1, \ldots, i_s, j_s)\), where \(y = j_s\) or \(y = i_s\) and \(j_k \neq j_k\) for all \(k < s\).

Then match \(i_s\) to \(j_s\) for all \(t = 1, \ldots, s\), leave the partner of \(j_t\) unmatched, if it exists, and keep unchanged the rest of the matching \(x^*\). Give to \(j_t\) the payoff \(b_{i_tj_t} - v^*_{j_t}\) to \(i_t\), the payoff \(a_{i_tj_t} + b_{i_tj_t} - v^*_{j_t}\) and to the partner of \(j_t\), the payoff zero, if \(j_t\)'s partner exists. If \(i^* \neq i_s\), give the payoff \(a_{i_sj_s} + b_{i_sj_s} - v^*_{j_s}\) to \(i^*\). Keep unchanged the payoffs of the rest of the players. Now observe that if \(i^* \neq i_s\), then \(i_s\) is matched at \(x^*\). In this case, since \((u^*, v^*; x^*)\) \(\in S\) it follows that \(i_s\) does not form any blocking pair at \(x^*\), so, by recalling that \(j_t \in D_{i_t}(u^*, v^*, x^*)\), we have that \(i_s\) is indifferent between \(x^*(i_s)\) and \(j_s\). Thus, we must have that \(u^*_{i_s} = a_{i_sj_s}\). This means that \(i_s\) did not change its payoff, so it does not form any blocking partner under the new outcome. Hence, the resulting outcome is feasible. (The argument is analogous to the one used in the 1st case.) Since \(u^*_{i_s} = a_{i_sj_s}\), if some \((p,
$q \in P \times Q$ is a new blocking pair then $p$ is $j_s$’s partner, which is unmatched. Then the resulting outcome is in $S$. However, $j_s$ is better off while all the $Q$-players keep their old payoffs, contradicting inequality (1).

(B): $c = (i^* = i_1, j_1, \ldots, i_s, j_s, \ldots)$, where $y = i_s$ and $j_s = j_k$ for some $k < s$.

Then match $i_t$ to $j_t$ for all $t = k + 1, \ldots, s$. Now proceed as in the previous case. The only difference is that the partner of $j_s$ under $x^*$ is $i_{k+1}$, who will be matched under the new matching to $j_{k+1}$ instead of being unmatched. As before the resulting outcome is in $S$ but $j_s$ is better off while all the $Q$-players keep their old payoffs, contradicting inequality (1).

(C): $c = (i^* = i_1, j_1, \ldots, i_s)$, where $y = i_s$ and $D_i(u^*, v^*, x^*) = \phi$.

In this case $i_s \neq i^*$. Then match $i_t$ to $j_t$ for all $t = 1, \ldots, s - 1$, leave $y = i_s$ unmatched and keep unchanged the rest of the matching. Give the payoff $a_{i_sj_s} + b_{i_sj_s} - v^*_j$ to $i_s$ and keep unchanged the payoffs of the rest of the players. The resulting outcome is clearly in $S$. If $i_s$ does not form any blocking pair under the new outcome, we have reduced the number of blocking pairs, contradicting the definition of $x^*$. Otherwise let $(u', v'^*; x')$ be the new outcome. Let $k$ be some $i_s$’s favorite blocking partner. Then match $i_s$ to $k$, leave the partner of $k$ unmatched, if it exists, and keep unchanged the rest of the matching $x'$. Now give to $k$ the payoff $b_{i_sk} > v^*_k$ and to $i_s$ the payoff $a_{i_sk}$. Keep unchanged the payoffs of the rest of the players. Thus, the resulting outcome is feasible and it is clearly in $S$. But $k$ is better off, contradicting inequality (1).

Hence, we have proved that $(u^*, v^*; x^*)$ does not have any blocking pair, so it is stable. □

The proof of Theorem 1 restricted to the Marriage Model consists of the proof of the 2nd case (A) and coincides with the existence proof of Sotomayor (1996). When restricted to the Assignment Game it corresponds to the proof of the 1st case.

4. The lattice property of the core

The existence of optimal stable payoffs for each side of the market in the discrete and continuous cases is known to be related to the structure of the entire set of stable payoffs. [For example the set of stable payoffs forms a lattice both in the marriage model with strict preferences (Knuth, 1976) and in the assignment game (Shapley and Shubik, 1972)]. The compactness of the core in both markets implies that the lattice is complete, so it has one and only one maximal and one and only one minimal element.

In this section we show that the underlying assumption sufficient to produce these structural results in both classes of models is that the core coincides with the core defined by weak domination. At the end of this section we provide an example which shows that no optimal stable payoff may exist and the core may not be a lattice if $C \neq C^*$. This example also illustrates that strict preferences for the agents in $R$ are no longer enough to preserve these results, when both kinds of agents can trade in the same market.

Each of the results presented below have been separately established for the Marriage
market with strict preferences and for the Assignment game of Shapley and Shubik (1972), using incompatible assumptions.

The key result is Lemma 2. Lemma 3 is very close in spirit to the analogous result of Demange and Gale (1985).

**Lemma 2.** Let \((u^1, v^1; x^1)\) and \((u^2, v^2; x^2)\) be stable outcomes. Let \(P^1 = \{i \in P; u^1_i > u^1_j\}\), \(P^2 = \{i \in P; u^2_i > u^1_i\}\) and \(P^2 = \{i \in P; u^1_i = u^2_i\}\). Analogously define \(Q^1\), \(Q^2\) and \(Q^0\). If \(C = C^*\) then \(P^1 \neq \phi\) if and only if \(Q^0 = \phi\) if and only if \(Q^1 \neq \phi\). Furthermore, \(x^1(P^1) = x^2(P^2) = Q^2\), \(x^1(P^0) = x^2(P^0) = Q^2\) and \(x^1(P^0 \cup Q^0) = x^2(P^0 \cup Q^0) = Q^0\).

**Proof.** Suppose \(P^1 \neq \phi\) and let \(i \in P^1\). Then \(u^1_i > u^2_i \geq 0\), so \(x^1_i = 1\) for some \(j \in Q\). We claim that \(v^1_j = 0\). In fact, if \(x^1_j = 1\) this fact is obvious by the feasibility of the outcomes. If \(x^2_j = 0\) and \(v^1_j > v^2_j\) then \((u^1, v^1)\) dominates \((u^2, v^1)\) via \((i, j)\), which contradicts the assumption that \((u^2, v^1) \in C\) (resp. \(C^*\)). So \(j \in Q^1\). \(Q^2 \neq \phi\) and all \(P^1\) are matched to \(Q^2\) at \(x^1\). Then \(x^1(P^1) \subseteq Q^2\).

Now suppose that \(Q^2 \neq \phi\). Let \(j \in Q^2\). Then \(v^1_j > v^2_j \geq 0\), so \(x^1_j = 1\) for some \(i \in P\). We claim that \(u^2_i < u^1_i\). In fact, if \(x^1_j = 1\) it is immediate from the feasibility of the outcomes. If \(x^1_j = 0\) and \(u^2_i > u^1_i\) then \((u^2, v^1)\) dominates \((u^1, v^1)\) via \((i, j)\), which contradicts the assumption that \((u^1, v^1) \in C\) (resp. \(C^*\)). So \(i \in P^1\), \(P^1 \neq \phi\) and all \(Q^2\) are matched to \(P^1\) at \(x^2\). Then \(x^2(Q^2) \subseteq P^1\).

Now, \(|P^1| = |x^1(P^1)| \leq |Q^2| = |x^2(Q^2)| \leq |P^1|\) implies that \(x^1(P^1) = Q^2\). It also implies that \(x^2(Q^2) = P^1\), and so \(x^2(P^2) = Q^1\).

An analogous argument shows that \(x^1(P^2) = x^2(P^2) = Q^1\). The last assertion follows from the fact that \(P^0 = P - (P^1 \cup P^2)\) and \(Q^0 = Q - (Q^1 \cup Q^2)\) and the first two assertions. \(\square\)

The following result states that if \(i\) is unmatched at some stable outcome then it receives the payoff 0 at any stable outcome. When \(F = \phi\) the result says that if \(i\) is unmatched at some stable outcome then it is unmatched at every stable outcome.

**Corollary.** Let \((u^1, v^1; x^1)\) and \((u^2, v^2; x^2)\) be stable outcomes. Suppose \(C = C^*\). If \(i \in P\) (resp. \(j \in Q\)) is unmatched under \(x^1\) then \(u^1_i = 0\) (resp. \(v^1_j = 0\)).

**Proof.** Suppose that \(i \in P\) (resp. \(j \in Q\)) is unmatched at \(x^1\) and \(u^1_i > 0\) (resp. \(v^1_j > 0\)). Then \(i \in P^2\) (resp. \(j \in Q^2\)). But Lemma 2 implies that all of \(P^2\) (resp. \(Q^2\)) are matched under \(x^1\), a contradiction. \(\square\)

For any vectors \(Y^1\) and \(Y^2\) of the same dimension, define \((Y^1 \vee Y^2) = Y^+\) to be the pointwise maximum of \(Y^1\) and \(Y^2\), and \((Y^1 \wedge Y^2) = Y^-\) to be their pointwise minimum.

**Lemma 3.** Let \((u^1, v^1)\) and \((u^2, v^2)\) be in \(C\). If \(C = C^*\) then

(a) \((u^1 \vee u^2, v^1 \wedge v^2) = (u^+, v^-) \in C\).
(b) \((u^1 \land u^2, v^1 \lor v^2) = (u^-, v^+) \in C\).

**Proof.** Define \(P^2\) and \(Q^1\) as in Lemma 2 and let \(x^1\) and \(x^2\) be compatible matchings with \((u^1, v^1)\) and \((u^2, v^2)\), respectively. If \(i \in P^2\) and \(j \in Q^1\) we have that \(u^i_j > u^j_i \geq 0\) and \(v^i_j > v^j_i \geq 0\). Then all of \(P^2\) are matched at \(x^2\) and all of \(Q^1\) are matched at \(x^1\). Lemma 2 implies that \(x^1\) and \(x^2\) match \(P^1\) with \(Q^1\). To prove (a) define the matching \(x^*\) by: \(P^2\) and \(Q^1\) are matched by \(x^2\); \(P - P^2\) and \(Q - Q^1\) are matched by \(x^1\).

The conditions (ii) and (iii) of Definition 3 hold for \((u^1, v^1; x^1)\) and \((u^2, v^2; x^2)\), so they follow for \((u^+, v^-; x^*).\) If \(i\) is unmatched at \(x^*\) then \(i \in P - P^2\) and is unmatched by \(x^1\) so \(u^i = u^i_1 = 0.\) If \(j\) is unmatched by \(x^*\) then \(j \in Q - Q^1\) and \(v^j = v^j_1 = 0.\) Hence, in \((u^+, v^-; x^*)\) is feasible. The conditions (i) and (ii) of Definition 4 are also immediate if \(u^i = u^i_1\) and \(v^j = v^j_1\) then use the stability of \((u^*, v^*; x^*)\) to see that:

(i) \(u^i + v^j = u^i_1 + v^j_1 = u^i_2 + v^j_2 = a_{ij} + b_{ij}\) if \((i, j) \in F^*,\) and

(ii) \(u^i = u^i_1 \geq a_{ij}\) or \(v^j = v^j_1 \geq b_{ij}\).

Thus, \((u^+, v^-; x^*)\) is stable, so \((u^+, v^-) \in C).\)

To prove (b) define \(x^\#\) by

\[P^2\] and \(Q^1\) are matched by \(x^1;\)

\[P - P^2\] and \(Q - Q^1\) are matched by \(x^2.\)

The arguments are similar to those used to prove part (a). \(\square\)

From Lemma 3 it follows that if \((u^1, v^1)\) and \((u^2, v^2)\) are in \(C\) and \(C = C^*\) then \(u^i \succeq u^i_j\) for all \(i \in P\) if and only if \(v^i_j \succeq v^i_j\) for all \(j \in Q.\) Therefore, we can define two binary relations on \(C\) (one is the dual of the other). (i) \((u^1, v^1) \succeq (u^2, v^2)\) if and only if \(u^i \geq u^i_j\) and \(v^i \geq v^i_j\) for all \(i \in P\) and all \(j \in Q.\) (ii) \((u^1, v^1) \succeq (u^2, v^2)\) if and only if \(u^i \leq u^i_j\) and \(v^i \geq v^i_j\) for all \(i \in P\) and all \(j \in Q.\) It is clear that \(\succeq\) and \(\succeq\) are partial orders on the set of stable payoffs.

**Theorem 2.** If \(C = C^*\) then \(C\) is a lattice under \(\succeq\) and \(\succeq\).

**Proof.** Immediate from Lemma 3 and the definition of a lattice. \(\square\)

When \(F = \phi\), Theorem 2 establishes that the core is a lattice. Recall that we have defined the core as a set of payoffs, not a set of matchings. The analogous binary relations defined on the set of stable matchings \(x \succeq\) and \(x \succeq\) may fail the anti-symmetric property when preferences are not strict, even when \(C = C^*\). Hence, in this case these relations are not, in general, partial orders and the conclusion of Theorem 2 does not necessarily hold for the set of stable matchings. That is, in the case \(F = \phi\), the set of stable payoffs is a lattice when \(C = C^*\), although the set of stable matchings may not be. When preferences are strict it can be easily seen that there is a one-to-one correspondence between the set of stable payoffs and the set of stable matchings which preserves these binary relations. That is, if \((u, v; x)\) and \((u', v'; x')\) are stable outcomes
then \((u, v) >_p (u', v')\) [resp. \((u, v) >_q (u', v')\)] if and only if \(x >_p x'\) (resp. \(x >_q x'\)). So these binary relations on the set of stable matchings are partial orders and Theorem 2 implies that the set of stable matchings is a complete lattice under both partial orders.

**Definition 7.** We say that the stable payoff \((u^*, v^*_u)\) is \(P\)-optimal if \((u^*, v^*_u) \succeq_p (u, v)\) for all \((u, v) \in C\); similarly we say that the stable payoff \((u^*_q, v^*)\) is \(Q\)-optimal if \((u^*_q, v^*) \succeq_q (u, v)\) for all \((u, v) \in C\).

**Theorem 3.** If \(C = C^*\) then there exists a unique \(P\)-optimal stable payoff and a unique \(Q\)-optimal stable payoff.

**Proof.** By Theorem 2, \(C\) is a lattice under both partial orders: \(\succeq_p\) and \(\succeq_q\). \(C\) is obviously closed, since it is given by a set of inequalities; it is bounded since the set of feasible payoffs is feasible. Now, for each one of the partial orders, use the fact that a compact lattice has a unique maximal element. □

The existence of \(P\)-optimal and \(Q\)-optimal stable payoffs for the Marriage market with strict preferences and for the Assignment game is guaranteed by the fact that the core of both models is a compact lattice. The lattice property follows from the fact that \(C = C^*\). The core and the core defined by weak domination coincide if and only if no weak blocking pair exist. In the discrete model with strict preferences, it can never happen that one agent can benefit by being matched to another agent who remains indifferent, since no agent is indifferent between different mates, and so no weak blocking pairs exist. In the continuous case no weak blocking pairs exist because whenever there exists a pair of agents \((i, j)\) who can be matched in a way that benefits one of them without hurting the other, it is also possible to match them in a way that benefits both of them (since the price that one pays to the other can be continuously adjusted, and both agents have continuous utility functions). So the core coincides with the core defined by weak domination.

To see the role played by the assumption that \(C = C^*\) consider the following two examples. In the first example \(C \neq C^*\) and there is no \(P\)-optimal stable outcome. The core of this example is not a lattice. In the second example \(C = C^*\).

**Example 1.** Consider \(P = \{p_1, p_2, p_3\}\) and \(Q = \{q_1, q_2\}\), where \(R = \{p_1, p_2\}\) and \(F = \{p_3, q_1, q_2\}\). The numbers \(a_{i,j}\) and \(b_{i,j}\) are given in Table 1 for \(i, j = 1, 2\). The numbers \(a + b\) and \(a + b\) are 2 and 6 respectively. There are exactly three stable

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>((p_1))</td>
<td>(4, 8)</td>
</tr>
<tr>
<td>((p_2))</td>
<td>(3, 2)</td>
</tr>
<tr>
<td>((p_3))</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1
Values of \(a_{i,j}\), \(b_{i,j}\) for Example 1
Table 2
Values of $a_{ij}, b_{ij}$ for Example 2

<table>
<thead>
<tr>
<th></th>
<th>$a_{11}$</th>
<th>$a_{12}$</th>
<th>$a_{21}$</th>
<th>$a_{22}$</th>
<th>$a_{31}$</th>
<th>$a_{32}$</th>
<th>$a_{41}$</th>
<th>$a_{42}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>(2,2)</td>
<td>(1,3)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_2$</td>
<td>(1,3)</td>
<td>(2,2)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_3$</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(2,2)</td>
<td>(1,3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_4$</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,3)</td>
<td>(2,2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 2. Let $P = \{p_1, p_2, p_3, p_4\}$ and $Q = \{q_1, q_2, q_3, q_4\}$, where $R = \{p_1, p_2, q_3, q_4\}$ and $F = \{p_3, p_4, q_1, q_2\}$. The numbers $a_{ij}$ and $b_{ij}$ are given in Table 2. There are four stable outcomes in this market $(u^1, v^1; x^1), (u^2, v^2; x^2), (u^3, v^3; x^3), (u^4, v^4; x^4)$ (see Fig. 1).

To see that these are the only stable outcomes is enough to observe that any outcome at which $p_1$ or $p_2$ is matched to $q_3$ or $q_4$ is blocked either by $(p_1,q_1)$ or by $(p_2,q_2)$. It is a matter of verification that $C^* = C$.

To see that $C$ is a lattice under both partial orders: $\succeq_p$ and $\succeq_q$ compute:

- $x^1 = (u^1, v^1; x^1)$ and $v^1 = (2,2,3,3)$
- $u^1 = (2,2,1,1)$ and $v^1 = (2,2,3,3)$
- $u^2 = (1,1,2,2)$ and $v^2 = (3,3,2,2)$
- $u^3 = (2,2,2,2)$ and $v^3 = (2,2,2,2)$
- $u^4 = (1,1,1,1)$ and $v^4 = (3,3,3,3)$

Fig. 1. Stable outcomes in Example 2.
\[(u^1 \lor u^2, v^1 \land v^3) = (u^1 \lor u^3, v^1 \land v^3) = (u^2 \lor u^3, v^2 \land v^3) = (u^3 \lor u^4, v^3 \lor v^4) = (u^3, v^3), (u^1 \lor u^4, v^1 \land v^4) = (u^1, v^1), (u^2, v^2), (u^3 \lor u^4, v^3 \lor v^4) = (u^3, v^3); (u^1 \land u^2, v^1 \lor v^3) = (u^1 \land u^4, v^1 \lor v^4) = (u^2 \land u^4, v^2 \lor v^4) = (u^3 \land u^4, v^3 \lor v^4) = (u^4, v^4), (u^1 \land u^2, v^1 \lor v^3) = (u^1, v^1), (u^2 \land u^3, v^2 \lor v^3) = (u^2, v^2) \text{ and } (u^3 \land u^4, v^3 \lor v^4) = (u^4, v^4).\]

The $P$-optimal stable payoff is $(u^3, v^3)$ and the $Q$-optimal stable payoff is $(u^4, v^4)$. 

\[\square\]

Acknowledgements

This work was partially supported by the J.S. Guggenheim Memorial Foundation, FIPE and FAPESP-Brazil. I am grateful to Alvin Roth for many helpful suggestions.

References