Axioms for the outcomes of negotiation in matrix games

Stephen J. Willson*

Department of Mathematics, Iowa State University, Ames, IA 50011, USA

Received 1 May 1998; received in revised form 1 April 1999; accepted 1 May 1999

Abstract

Suppose that we are given the payoff matrix for a game $G$ of perfect information but with no side-payments. We seek to study the likely results if rational players negotiate a settlement involving only pure strategies. Axioms are presented for the set $X(G)$ of payoff vectors that might actually occur under these idealized circumstances. We also present a formal negotiation process analogous to the rules in the Theory of Moves that realizes such a set $X(G)$. The process shows that the axioms are consistent and therefore provide a possible indication of what should be simultaneously attainable by negotiation. Such a result could also provide a guide to arbitration. Among these axioms are Pareto-optimality of the payoff vectors in $X(G)$ and natural properties with respect to nontriviality, lower bounds, symmetry, strong dominance, floors, and default reduction. Examples show that $X(G)$ can be readily calculated, by hand in the case of 2 by 2 matrices, or on the computer in more complicated cases. The procedures work for any finite number of players. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Payoff matrix; Pure strategies; Axioms; Rational players

1. Introduction

This paper seeks to give an axiomatic analysis of the outcomes for $n$-person games in normal form. The games are not necessarily zero-sum. The context of our analysis is different from the context usually assumed for matrix games. We do not assume that all players simultaneously, independently, and irrevocably announce their choice of strategy. Instead, we allow them to change their strategies repeatedly until there is a final choice of strategy by each player. We intend thereby to model a process of negotiation by rational players, and we seek to determine the final outcomes that might arise in this
manner. We insist that side-payments are not allowed; thus the only possible outcomes are given by the game matrix. We also desire that the outcomes obtained reflect outcomes only from a single play of the game; ‘mixed’ outcomes in which outcomes from many plays of the game are averaged are excluded.

Assume that all the players are ‘rational’ with complete information about the payoff function. Under a process of negotiation, each player will ultimately select a pure strategy, yielding a certain final state $s$. Suppose that $p$ denotes the payoff function. The main issue in this paper is a study of which payoff vectors $p(s)$ are likely to arise as the final outcomes. To say that each player is ‘rational’ is to say that for each $i$, player $i$ is concerned only with his or her own profit; thus player $i$ is concerned only with maximizing $p_i(s)$. Player $i$ recognizes that all the other players have their own concerns. The economic difficulty, of course, is that usually there is not a single $s$ such that all $p_i(s)$ are simultaneously maximized at $s$. Strategic aspects of the game become crucial.

Since this analysis gives rise to a prediction about the outcomes if all players are rational, the analysis may be of theoretical use in arbitration or mediation. In appropriate circumstances it may suggest the outcome of negotiation in the absence of confusing passions. Of course, real parties to a conflict need not be rational and the underlying utility functions of the parties may not be known, even to the parties themselves. Candor by the parties may not necessarily be rewarded, and bluffing may be profitable. Issues such as these are the subjects of many studies, for example Schelling (1966), Myerson (1979), and Raiffa (1982). Such considerations could diminish the practical application of the results in this paper.

Since the outcomes model the results of idealized negotiation with enforcement of the rules for negotiation, these outcomes need not model the results of the same games under other rules. For example, in Prisoners’ Dilemma the solution obtained is that where each player cooperates rather than where each player defects.

We will use $X(G)$ to denote the set of all payoff vectors that we expect actually to arise in the game $G$ in certain circumstances. Various other theories propose some interesting sets $X(G)$ which might be given an analogous interpretation in some circumstances:

1. $X(G) =$ the set of all payoff vectors. This set is usually too numerous to be of much analytic use;
2. $X(G) =$ the set of all Pareto-optimal outcomes. In some games, there are a great many Pareto-optimal outcomes, and the principal issue is then to decide which of these outcomes are to be preferred for various reasons;
3. $X(G) =$ the set of all pure strategy payoffs that are Nash equilibria (Nash, 1950). Note that in this case $X(G)$ may be empty. Moreover, in some prominent problems such as Prisoners’ Dilemma, it is clear that Nash equilibria are often suboptimal;
4. $X(G) =$ the set of nonmyopic equilibria (NME) described by Brams (1994);
5. $X(G) =$ the set of ultimate outcomes described by Willson (1998).

In general, we expect that $X(G)$ will be nonempty, since some payoff vector will have to arise during the game. We do not expect that $X(G)$ is the set of all payoff vectors $p(s)$ since many choices of $s$ might be avoided by rational players. We hope that $X(G)$ will be
a ‘small’ set with nice properties. The ‘nicest’ situation would be where \(X(G)\) consists of a single element.

The intent of this paper is threefold: We wish to give

(a) axiomatic restrictions on a set \(X(G)\) with interesting and plausible properties; and
(b) a dynamic procedure (reminiscent of an iterative negotiating procedure) which results in the computation of \(X(G)\) satisfying the axioms; and
(c) a proof that the axioms uniquely determine \(X(G)\).

The axioms in part (a) will be presented in Section 2, with examples illustrating the meaning of these axioms. The details of the dynamic procedure promised in part (b) appear in Section 3, with a proof that the procedures realize the axioms given in Sections 5 and 6. A proof of uniqueness is given in Section 6. An example of the ‘play’ by two players following the rules is given in Section 7, while Section 8 gives an example of a game with three players.

Part (b) shows that the axioms are nontrivial; i.e. there exists a consistent solution concept satisfying the axioms. This fact will not be completely obvious from the axioms. For example, the axiom of strong dominance says that certain rows or columns can be ignored in computing \(X(G)\), yet it is not clear from the axiom that if we ignore rows or columns in a certain order and then alternatively ignore different rows or columns in some order that the answers obtained would be the same.

Part (b) is also of interest since it provides a formal model of negotiation which gives rise to \(X(G)\). In fact, the calculation procedure will resemble the rules for two players in the ‘Theory of Moves’ as described for example by Brams (1994) or Willson (1998). The procedure in part (b) is a method for picking the joint strategies \(s\). Our set \(X(G)\) will then consist of all \(p(s)\) where \(s\) arises in this manner. This procedure is in effect an artificially constrained negotiation. Some randomness exists in the procedure, and as a consequence there may be more than one element in \(X(G)\). We do not claim that the formal procedure mimics any current negotiation process in the real world. Rather, we show that the sets \(X(G)\) so obtained have many desirable properties. We find it interesting that a formal negotiating process gives rise to sets \(X(G)\) with such properties.

Part (c) indicates that the axioms as presented uniquely determine \(X(G)\). Not all readers will agree that all the axioms presented are obviously desirable. A reader might, for example, find the default reduction axiom questionable. Such a reader, however, may still note that our argument shows that those axioms which that reader accepts are still consistent with each other. Further research might then lead to a new set of axioms, more plausible to the reader, and a modified procedure to obtain these new axioms.

The axioms presented for \(X(G)\) yield different results from a number of other solution concepts for matrix games. There are 57 inequivalent 2 by 2 matrix games of conflict (see Brams (1994) for a list of the games with the numerical labels we utilize). For 17 of these games \(X(G)\) differs from the set of Nash equilibria; indeed, \(X(G)\) need not consist of Nash equilibria, even if such exist for \(G\). The solution concept of nonmyopic equilibria (NME) by Brams (1994) does not satisfy the Pareto-optimality axiom, as is seen in game 32; Brams’ NME fail to satisfy the strong dominance axiom, as seen in games 33, 34, 36–41, and they fail the floor axiom in games 33–41, 48, 56. Similarly
the set of ultimate outcomes in Willson (1998) fails Pareto-optimality, as is seen in game 50; it fails strong dominance in games 33, 34, 36, 37; and it fails the floor axiom in games 22, 33–37.

This paper greatly generalizes and extends Willson (1998). Whereas that paper dealt only with games with two players, the results in this paper apply to any finite number of players. The current rules of play yield proofs of more fundamental properties than in the previous paper. This is in part true because the current system allows a richer collection of possible moves, thereby permitting a richer choice of strategies by the players. Moreover, the new rules are arguably more realistic because they require that all players must agree before the game is over.

The rules of play given in part (b) fit within the tradition of the ‘Theory of Moves’ for which an extensive overview is presented in Brams (1994). Other prior analyses include related equilibria of Marschak and Selten (1978), the limited move equilibria of Zagare (1984), the equilibria for far-sighted players of Kilgour (1984), the far-sighted equilibria of Aaftink (1989), and the graph model for conflict resolution of Fang et al. (1993).

The ‘rules’ may also be interpreted as an arbitration procedure. The literature on arbitration is large. Some prominent studies include offer–counteroffer procedures (Stähl, 1972; Rubinstein, 1982); final-offer arbitration (Stevens, 1966; Wittman, 1986); combined arbitration (Brams and Merrill, 1986); sequential arbitration (Brams et al., 1991). Some overviews may be found in Young (1991). Most of these studies, however, deal with a different scenario; the system is modeled by a single numerical quantity (such as the wage scale in a labor-management conflict) rather than by a payoff matrix; and most deal only with the case of two players.

2. Axioms for \( X(G) \)

In this section we collect some of the desirable properties possessed by the sets \( X(G) \). These properties form axioms for the sets \( X(G) \). The intent is that \( X(G) \) represents the set of payoff vectors that might arise if rational players are playing the game exactly once. We thus seek axioms that eliminate payoff vectors which are implausible or for which players possess strategies rendering them impossible. At this stage it is not obvious that all the properties can be met.

Formally, a game \( G \) has \( n \) players, which will be denoted \( 1, 2, \ldots, n \). For each player \( i \), there is a set \( S_i \) of pure strategies. The set of all states is \( S = \prod S_i \); thus each state is an \( n \)-tuple \( s = (s_1, s_2, \ldots, s_n) \) in which for each \( i \) we have \( s_i \in S_i \) is the pure strategy selected by player \( i \). As a result of the selection of a strategy \( s_i \) by each player \( i \) the state \( s \) is determined, and then each player receives a profit or payoff. The payoff function \( p: S \to \mathbb{R}^n \) assigns to each state \( s \) the vector \( p(s) \) whose \( i \)th component \( p(s)_i = p_i(s) \) is the payoff received by player \( i \). Each player prefers a higher payoff over a lower payoff.

In some axioms, comparisons are needed between payoff vectors. We will write \( p(s) \leq_j p(v) \) to indicate that player \( j \) rates the payoff vector \( p(v) \) at least as well as the payoff vector \( p(s) \); when this happens it follows that player \( j \)'s profits satisfy \( p(s)_j \leq p(v)_j \) as well. We write \( p(s) \leq_{\text{all}} p(v) \) if \( p(s) \leq_j p(v) \) for all players \( j \).
1. **Form axiom.** Each element in \(X(G)\) has the form \(p(s)\) for some \(s\) in \(S\).

This axiom restates that only payoff vectors explicitly given by \(G\) can be obtained; no other payoffs or side-payments are allowed.

2. **Nontriviality axiom.** \(X(G)\) is nonempty.

This axiom reflects the intuition that for any given game, there has to be some outcome. Every game must be resolved in some way. Since \(X(G)\) is to represent all possible rational outcomes, \(X(G)\) must contain some vector.

3. **Pareto-optimality axiom.** If \(p(s)\) is in \(X(G)\), there does not exist any state \(s'\) in \(S\) such that for all \(i\) we have \(p(s)_i = p(s'_i)\), and in addition, for some \(i\), the inequality is strict.

The property reflects the hope that a good negotiation process will lead to final outcomes that are efficient. Note that it is possible that certain mixed states of outcomes (which do not satisfy the form axiom) would be preferred by all players over \(p(s)\) in \(X(G)\). Thus, perhaps, there might still exist \(s'\) and \(s''\) in \(S\) such that for all \(i\), \(p(s)_i < (1/2) p(s'_i) + (1/2) p(s''_i)\). Such a mixed strategy could not correspond to the outcome of a single play of the game.

4. **Agreement axiom.** If there exists a single state \(s\) such that, simultaneously for all players \(i\), \(p(s)_i\) is maximal, then \(X(G)\) contains only the vector \(p(s)\).

Thus, in the event of complete agreement as to which strategies lead to the best outcome, the axiom states that these strategies should be selected.

5. **Symmetry axiom.** The sets \(X(G)\) should not be biased in any way by the order in which the strategies are listed nor by the order in which the players are listed. This axiom has two aspects:

(a) \(X(G)\) should depend only on the game \(G\), hence on the sets \(S_i\) and the payoff function \(p: S \to R\). In particular, the order of description of the strategies in each \(S_i\) should be irrelevant.

(b) Suppose that \(\sigma\) is a permutation of the \(n\) players. We think of \(\sigma\) as merely giving a new name for each player in the game \(G\). We may obtain a new game \(G'\) in which the \(i\)th player is the same as the player \(\sigma(i)\) in the game \(G\), playing exactly as in the game \(G\). The sets \(T_i\) of strategy for \(G'\) satisfy \(T_i = S_{\sigma(i)}\). Let \(T = \prod T_i\). The payoff function \(p': T \to R\) satisfies \(p'(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)})_{\sigma(i)} = p (s_1, s_2, \ldots, s_n)\), for each \(i\) since strategy \(s_{\sigma(i)}\) played by player \(i\) in the game \(G'\) is the same as the strategy played by player \(\sigma(i)\) in the game \(G\). We require that the outcomes \(X(G')\) should correspond to the outcomes \(X(G)\) in exactly the same manner. Thus \((x_1, x_2, \ldots, x_n)\) lies in \(X(G)\) if and only if \((x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})\) lies in \(X(G')\).

6. **Lower bounds.** A **lower bound** for player \(i\) is a number \(L\) for which there exists a pure strategy \(a \in S_i\) such that for all \(s\) in \(S\) for which \(s_i = a\) it is true that \(p_i(s) \geq L\).

**Lower bound axiom.** If \(L\) is a lower bound for player \(i\), then for all \(p(s)\) in \(X(G)\) it is also true that \(p_i(s) \geq L\).

This axiom reflects the intuition that, if player \(i\) can guarantee himself or herself a profit
of at least $L$ by choosing some strategy $a$ (regardless of what strategies the other players choose), then all reasonable payoffs in $X(G)$ also give player $i$ at least $L$. A player $i$ can unilaterally prevent any outcome that does not provide a profit of at least $L$. (It does not follow that only such strategies $a$ will arise in the settlement. Maybe there will be agreement on another choice of strategies and player $i$ will get more than $L$ by playing a strategy other than $a$.)

7. Strong dominance. The pure strategy $s_1$ of player 1 is strongly dominated by the strategy $s_1'$ if for all strategies $s_2$ and $s_3$ in $S_2$, $s_3$ and $s_3'$ in $S_3$, ..., $s_n$ and $s_n'$ in $S_n$ we have $p(s_1', s_2, s_3, ..., s_n) \geq p(s_1, s_2, s_3, ..., s_n)$. This says that player 1 prefers the outcome from using strategy $s_1'$ over the outcome from using strategy $s_1$, no matter what the other players do, even if they change their strategies. Alternatively, no outcome when player 1 uses $s_1$ is preferred by player 1 over any outcome if player 1 uses strategy $s_1'$ instead.

It is natural to conjecture that in this circumstance player 1 will never need to play strategy $s_1$ because strategy $s_1'$ is preferable, even if the other players change their strategies. A ‘threat’ by player 1 to play strategy $s_1$ would not be credible. We therefore expect that the strategy $s_1$ may be completely ignored in the computation of the outcomes; we expect that there is no loss in replacing the game $G$ by a game $G'$ in which strategy $s_1$ does not exist. The notion generalizes in the obvious manner to each player $j$ in the game. More formally, we assume the following.

**Strong dominance axiom.** Suppose that a player (say player $j$) has a strategy $s_j$ that is strongly dominated. Let $G'$ denote the matrix game obtained by eliminating the strategy $s_j$ from $S_j$. Equivalently, let $S' = \prod S'_i$, where $S'_i = S_i$ for $i \neq j$, and $S'_j = S_j - \{s_j\}$; define $p': S' \to R$ by $p' = p|S'$, and let $G' = \{S', p'\}$. Then $X(G) = X(G')$.

Note that our notion of strong dominance is not the same as the property commonly called ‘dominance’ or ‘domination’ in which the inequality above is replaced by the inequality $p(s_1', s_2, s_3, ..., s_n) \geq p(s_1, s_2, s_3, ..., s_n)$ for all $s_2, s_3, ..., s_n$. The difference is that in ordinary dominance the other players are not allowed to change their strategies. We definitely do not assume that we may ignore strategies that are merely dominated, as opposed to strategies that are strongly dominated. For example, Prisoners’ Dilemma (analyzed in detail in Section 7) has as its solution the outcome $p(s)$ when both players cooperate; here $s$ contains a dominated (but not strongly dominated) strategy. Effectively, the strong dominance axiom lets us ignore strategies unless they can be seen as a plausible basis for some compromise settlement acceptable to all parties.

8. Floors. The floor axiom seeks to model the default outcomes in case negotiations break down. If negotiations break down, a reasonable model is to assume that in some order $\sigma$ the players will announce their irrevocable and final strategies. This order will be called the default permutation. If the players are in addition rational, then the resulting payoff vector will be determined merely by the order $\sigma$; we will call it the ‘floor’ and denote it $F(\sigma)$.

More formally, let $\sigma$ be a permutation of $\{1, 2, ..., n\}$. The floor $F(\sigma)$ is the payoff vector $p(s)$ that arises from the state $s = (s_1, s_2, ..., s_n)$ obtained as follows:
(1) Player $\sigma(1)$ announces a choice of pure strategy $s_{\sigma(1)} \in S_{\sigma(1)}$.

(2) Knowing that player $\sigma(1)$ has irrevocably selected $s_{\sigma(1)}$, player $\sigma(2)$ announces a choice of pure strategy $s_{\sigma(2)} \in S_{\sigma(2)}$.

(3) Knowing the choices of $s_{\sigma(1)}$ and $s_{\sigma(2)}$, player $\sigma(3)$ announces a choice of pure strategy $s_{\sigma(3)} \in S_{\sigma(3)}$.

... (k) Knowing the choices of $s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(k-1)}$, player $\sigma(k)$ announces a choice of pure strategy $s_{\sigma(k)} \in S_{\sigma(k)}$.

... (n) Knowing $s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n-1)}$, player $\sigma(n)$ announces a choice of pure strategy $s_{\sigma(n)} \in S_{\sigma(n)}$.

We assume that all players are ‘rational’. Thus $\sigma(1)$ makes a choice of $s_{\sigma(1)}$ that will maximize the resulting payoff to player $\sigma(1)$ under the assumption that each of the other players is rational. In general player $\sigma(i)$ makes a choice of the strategy $s_{\sigma(i)}$ that will maximize the resulting payoff to $\sigma(i)$, using knowledge of the previous choices of $s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(i-1)}$ and assuming that all the other players are rational. The floor $F(\sigma)$ is the outcome $p(x)$ that results from all these choices. Note that saying that each player $\sigma(i)$ makes the choice $s_{\sigma(i)}$ from $S_{\sigma(i)}$ means that each player $j$ makes the choice $s_j$ from $S_j$. The choices determine a state $s = (s_1, \ldots, s_n)$. Define $F(\sigma) = p((s_1, s_2, \ldots, s_n))$. A more explicit calculation of $F(\sigma)$ is given in Theorem 6.5.

We call $F(\sigma)$ the ‘floor’ since, given $\sigma$, $F(\sigma)$ tells the lowest result in the absence of any negotiation, assuming that all the players are rational. Any improvement in their payoffs over $F(\sigma)$ will be on the basis of negotiation (or by use of a different permutation). The vectors $F(\sigma)$ for different choices of $\sigma$ represent the fall-back or default payoffs in the event of breakdown of negotiations. The particular default permutation $\sigma$ which arises, however, is unclear. Maybe a player who feels threatened would initiate the breakdown, becoming $\sigma(1)$; or maybe some other player in anticipation of this would instead initiate the breakdown. A player $j$, however, has an incentive to break down the negotiations if he would benefit from $F(\sigma)$ for all possible $\sigma$. If a proposed outcome $p(v)$ does not represent an improvement over $F(\sigma)$ for some player $i$, then player $i$ can refuse further negotiation thereby forcing the outcome to be $F(\sigma)$ for some permutation $\sigma$ rather than $p(v)$. It is therefore natural to assume that each member $p(v)$ of $X(G)$ is at least as good as some $F(\sigma)$.

Conversely, any method of breaking down negotiations and therefore leading to an outcome $F(\sigma)$ for some permutation $\sigma$ provides a temptation to end negotiations leading to the outcome $F(\sigma)$. Since $X(G)$ must include all possible rational outcomes, it follows that $X(G)$ must contain a possible outcome $p(v)$ at least as good as $F(\sigma)$.

We therefore will assume that, if the default permutation is $\sigma$, then there will be a uniquely determined outcome $p(v)$ that arises at least as good for all players as $F(\sigma)$. Let $X(\sigma)$ denote this outcome $p(v)$.

Floor axiom. For each permutation $\sigma$ of the players there is a uniquely determined outcome $X(\sigma)$ in $X(G)$ such that $X(\sigma) \succeq_x F(\sigma)$. Moreover, every element of $X(G)$ arises in this manner for some permutation $\sigma$.

According to the floor axiom, $X(G)$ consists of all the payoff vectors $X(\sigma)$. It is quite possible that different default permutations $\sigma$ give rise to the same payoff vector $X(\sigma)$; hence it is still possible, for example, that $X(G)$ is a singleton set.
9. Reduction using the default outcomes.

Suppose that $\sigma$ is a permutation of the players $\{1,2,\ldots,n\}$. If the negotiations break down and players make final irrevocable choices of strategy in the order $\sigma$, then the outcome will be the floor $F(\sigma)$. For definiteness, let $X(\sigma; G)$ denote the $X(\sigma)$ that arises in game $G$ using the permutation $\sigma$, and similarly $F(\sigma; G)$ denotes the floor arising from game $G$ using $\sigma$. Thus $X(\sigma; G)$ is a payoff vector $p(v)$ for some state $v$, and $X(\sigma; G) \geq \forall F(\sigma; G)$.

Our intuition suggests that, given this $\sigma$ as the default order of play if negotiations fail, the only relevant payoff vectors $p(v)$ satisfy that $p(v) \geq \forall F(\sigma)$. No other $p(v)$ would be relevant since some player would instead refuse negotiation and instead the game would end up at $F(\sigma)$. It is therefore natural to define for each permutation $\sigma$ a modified game $G_{\sigma}$ as follows in which the irrelevant payoff vectors are replaced by $F(\sigma)$.

More explicitly, suppose $G$ is the game with strategies $S$ and payoff vectors $p: S \rightarrow R^n$. Let $G_{\sigma}$ be the game with strategies $S$ and payoff vectors $q: S \rightarrow R^n$ given by

\[
q(s) = p(s) \text{ if } p(s) \geq \forall F(\sigma; G);
\]
\[
q(s) = F(\sigma; G) \text{ otherwise}.
\]

We call $G_{\sigma}$ the game $G$ reduced by the permutation $\sigma$ because the possible payoffs for the game have been reduced using the default permutation $\sigma$.

Since only the payoffs in $G_{\sigma}$ are relevant to the game when the default permutation is $\sigma$, the players may as well utilize $G_{\sigma}$ rather than $G$ in determining $X(\sigma; G)$ if $\sigma$ is known. Hence $X(\sigma; G)$ may be determined from the game $G_{\sigma}$, and the outcome in $G$ given the default permutation $\sigma$ will be the same as that for $\sigma$ in the reduced game $G_{\sigma}$. This intuition is the basis for the next axiom.

Default reduction axiom. For each permutation

\[
\sigma, X(\sigma; G) = X(\sigma; G_{\sigma}).
\]

Remark. The axioms stated above are not independent. Some axioms imply others. For example, the Pareto-optimality axiom implies the agreement axiom, and the floor axiom implies the lower bound axiom. In this section such redundant properties are stated because the strongest axioms (such as the floor axiom) could be regarded as questionable; the reader might seek to reject some axioms and seek some weaker versions.

Following are several examples of matrix games with analyses using the axioms described above. The games have two players. Player 1 may be called Row or $R$ and selects the row of the matrix; player 2 is called Column or $C$ and selects the column of the matrix. The payoff vector from the $i$th row and the $j$th column has form $(a, b)$, where $a$ is the profit to player 1 and $b$ is the profit to player 2. Larger numbers are better.

Example 1. Use of the lower bound and Pareto-optimality axioms.

Consider the matrix

\[
\begin{pmatrix}
(3, 4) & (1, 1) \\
(2, 3) & (4, 2)
\end{pmatrix}
\]
Each $X$ in $X(G)$ must be Pareto-optimal, so the only possibilities are (3, 4) and (4, 2). The latter violates the lower bound axiom since Column can guarantee herself at least 3 by selecting the first column. This leaves (3, 4) as the only possibility. Since $X(G)$ is nonempty it follows that $X(G)=\{(3, 4)\}$.

**Example 2.** Use of the floor axiom.

\[
\begin{array}{cc}
(2, 4) & (3, 3) \\
(1, 2) & (4, 1)
\end{array}
\]

Row can guarantee herself at least 2 by selecting the first row, and Column can guarantee herself at least 2 by selecting the first column. Hence the lower bound axiom eliminates (1, 2) and (4, 1).

We next show that $F(CR) = (2, 4)$. If Column initiated a breakdown of negotiations and selected the first column irrevocably, then the result would be (2, 4) since Row would select the first row to get a better payoff. If, instead, Column selected the second column irrevocably, then Row would select the second row to obtain the better payoff (4, 1). Column prefers (2, 4) over (4, 1) and hence would select the first column when initiating a breakdown. Therefore $F(CR) = (2, 4)$. Similarly we can compute $F(RC) = (2, 4)$.

All breakdowns of the negotiations therefore lead to the payoff vector (2, 4), and the floor axiom eliminates (3, 3) as a possible outcome. Effectively, Column could benefit from a breakdown of negotiations unless (2, 4) were being considered. Hence $X(G)\neq\{(2, 4)\}$.

**Example 3.** Use of the strong dominance axiom.

\[
\begin{array}{cc}
(3, 4) & (4, 3) \\
(1, 2) & (2, 1)
\end{array}
\]

The strong dominance axiom says that the outcomes should be the same as for the matrix game $G'$ given by

\[
(3, 4) (4, 3)
\]

since the last row of $G$ is strongly dominated by the first. For $G'$, Column can guarantee herself at least 4 by choosing the first column, so the lower bound axiom then implies that the only possible outcome is (3, 4). Hence $X(G) = \{(3, 4)\}$.

**Example 4.** Use of the default reduction axiom.

\[
\begin{array}{cccc}
(1, 4) & (7, 1) & (4, 6) \\
(9, 2) & (2, 5) & (8, 3) \\
(3, 9) & (5, 8) & (6, 7)
\end{array}
\]

The lower bound axiom says that Row can guarantee herself at least 3 and Column can guarantee himself at least 3. The Pareto optimal outcomes that are consistent with this are precisely (8, 3), (3, 9), (5, 8), (6, 7). Since $F(RC) = (4, 6)$ while $F(CR) = (8, 3)$, the
floor axiom eliminates (3, 9) as a possibility. Since (8, 3) is Pareto-optimal and
\( X(CR) \succeq_{all} F(CR) \), it follows that \( X(CR) = (8, 3) \). The floor axiom, however, does not
determine \( X(RC) \). There are two choices of Pareto-optimal payoffs that each player
prefers over \( F(RC) = (4, 6) \), namely (5, 8) and (6, 7). The floor axiom is thus consistent
with either \( X(RC) = (5, 8) \) or \( X(RC) = (6, 7) \).

To decide among these, we utilize the default reduction axiom. Since \( F(RC) = (4, 6) \),
the reduced game \( G(RC) \) is obtained by replacing payoffs not \( \succeq_{all} (4, 6) \) by (4, 6). It
therefore has payoff matrix

\[
\begin{pmatrix}
(4, 6) & (4, 6) & (4, 6) \\
(4, 6) & (4, 6) & (4, 6) \\
(4, 6) & (5, 8) & (6, 7)
\end{pmatrix}
\]

In this matrix, the last row strongly dominates the first two rows, so the strong
dominance axiom reduces the analysis to the matrix

\[
(4, 6) (5, 8) (6, 7)
\]

and the lower bound axiom then leads to outcome (5, 8). Hence \( X(RC; G(RC)) = (5, 8) \),
whence, by the default reduction axiom, \( X(RC; G) = (5, 8) \).

It follows that \( X(G) = \{X(RC), X(CR)\} = \{(5, 8), (8, 3)\} \).

3. Rules of play

In this section we present the ‘rules of play’ for a dynamic determination of \( X(G) \). In
later sections we will show that the final outcomes obtained by these rules of play in fact
satisfy the axioms of Section 2.

1. Each player \( i \) selects an initial pure strategy \( s_i \in S_i \). Together, these strategies yield
an initial state \( s \), which becomes the first ‘current state’ of the game.

2. Randomly, a permutation \( \sigma \) of the players is selected. Assume that each of the \( n! \)
permutations has positive probability. This permutation \( \sigma \) gives the order in which the
players will make their moves for each round of the play.

3. A (sufficiently large) integer \( r \) is selected. This number \( r \) is the maximum number
of rounds that can take place, and it is announced to all players. (In fact, the arguments
in Theorem 5.4 give an easy way to estimate the minimum size \( t \) for \( r \). As long as \( r = t \),
the exact choice of \( r \) will have no effect on the outcome. The players need only to have
the possibility of playing at least \( t \) rounds.)

4. A ‘move’ by player \( i \) consists of a choice of a pure strategy from \( S_i \). Before the
move, there is a current state \( s \); after the move, there is a current state \( s' \) such that \( s'_j = s_j \)
for all \( j \neq i \), but \( s'_i \) is the state that player \( i \) selects during this move. It is possible that
\( s'_i = s_i \) and hence \( s' = s \) if player \( i \) reasserts the previous choice; it is also possible that
\( s'_i \) equals any other member of \( S_i \). Player \( i \) is completely free to choose any state from \( S_i \)
during the move. No other player \( j \) with \( j \neq i \) can change \( s_j \) during player \( i \)'s move.

5. A ‘round’ consists of \( n \) moves in the order given by the permutation \( \sigma \). Before the
round begins, there is a current state $s$. When the round begins, player $\sigma(1)$ makes a move, yielding a current state $s'$. When this move is completed, player $\sigma(2)$ makes a move, yielding a current state $s''$. When this move is completed, player $\sigma(3)$ makes a move. This process continues until player $\sigma(n)$ makes a move, completing the round at a current state $v$.

6. After each round, ending in a current state $v$, all players are asked whether they wish to stop the game. If all the players agree to stop the game, then the game ends with payoff vector $p(v)$. Thus player $i$ receives payoff $p_i(v)$, where $v$ is the current state. If the maximum number $r$ of rounds has been played, leading to the current state $v$, then the game likewise ends with payoff vector $p(v)$; no more rounds are allowed. If the maximum number $r$ of rounds has not yet been reached, and even just one player does not agree to stop the game after a certain round, then the game continues with another round.

It is clear that ultimately a final payoff vector $p(v)$ is obtained by this process since there are at most $r$ rounds and each consists of $n$ moves. We will let $X(G)$ denote the set of final payoff vectors $p(v)$ that will arise.

**Remark.** In fact, we will see in Section 6 that if rational players use these rules, then the outcome will not depend at all on the initial pure strategies selected in step 1 nor on the number $r$ selected in step 3, as long as $r$ satisfies a condition given in Section 6. For example, we will see that the possibly arbitrary choice of $r$ in step 3 could be replaced by the following deterministic step 3':

3'. Suppose that the number of pure strategies for player $i$ is $c_i = |S_i|$. Let $r = \prod c_i$. Announce to all players that there will be at most $r$ rounds.

Consequently the only randomness that may affect the final payoff vector is the choice of the permutation $\sigma$ in step 2.

**4. Rationality and strictness**

We make predictions about the outcomes only if each player is ‘rational’. (We cannot make predictions if the players play the game erratically or without thought.) This means that at each stage, each player will make a calculation of the outcome of any choice he or she makes and will choose the alternative that maximizes his or her payoff. When making this calculation, the player will assume that all the other players are also rational.

We shall call a game **strict** if any two payoff vectors are either identical or else differ in all components. For example, the following 2 by 2 game is strict:

$$(1, 3) \quad (3, 1)$$

$$(2, 2) \quad (1, 3)$$

even though the payoff $(1, 3)$ occurs twice. On the other hand, the following game is not strict:
(1, 3) (3, 3)
(2, 4) (4, 1)

since there are two different payoff vectors (1, 3) and (3, 3) that nevertheless agree in the second component.

If the game is strict, then whenever any player tries to compare two payoff vectors which are not identically the same, then that player will have an unambiguous preference because the vectors will differ in the payoff to him or her.

We assume throughout this paper that all games are strict. As a result, each player \( i \) has a complete ranking of all the payoff vectors. If \( p(v) \) and \( p(w) \) are distinct payoff vectors, then we write \( p(v) <_i p(w) \) if player \( i \) strictly prefers vector \( p(w) \) over vector \( p(v) \). To say that the ranking is complete is to assert that for all payoff vectors \( p(v) \) and \( p(w) \), either \( p(v) <_i p(w) \) or \( p(w) <_i p(v) \) or \( p(v) = p(w) \). We also write \( p(v) \preceq_i p(w) \) if either \( p(v) <_i p(w) \) or \( p(w) = p(v) \).

If the game is strict and \( p(v) \neq p(w) \) but \( p(v) <_i p(w) \) for all \( i \), then it follows that \( p(v) <_i p(w) \) for all \( i \).

Since each player has a complete ranking of the outcomes \( p(v) \), each player can tell the maximum of any finite nonempty set of these payoff vectors. We will write \( \max, U \) for the maximum (from the viewpoint of player \( j \)) of the payoffs in \( U \). For example, \( \max_2 \{(2, 3), (4, 1)\} = (2, 3) \).

5. Basic calculations and results

Consider the game \( G \). Suppose that the permutation is \( \sigma \), that the ceiling \( r \) gives the maximum number of rounds, and that the initial state is \( s \). Let \( O(\sigma; r, s) \) denote the payoff vector that results from play by rational players. In this section we will give a calculation of \( O(\sigma; r, s) \). We will assume that \( G \) is strict and that the players are all rational.

We will also require some other quantities: suppose that during the \( r \)th round (after which there will be at most \( r - 1 \) rounds to play), player \( \sigma(i) \) makes a move from the current state \( s \). Let \( M(\sigma; r, s, \sigma(i); G) \) denote the final payoff vector that will result when the game ends, assuming that all subsequent play is rational. For any player \( \sigma(i) \) except \( \sigma(n) \), it is clear that \( M(\sigma; r, s, \sigma(i)) = \max_{s' \in S} \{M(\sigma; r, t, \sigma(i + 1))\} \) where \( t \) ranges over all states to which a single move by \( \sigma(i) \) can lead starting from \( s \). This is because player \( \sigma(i) \) must make a move leading to some state \( t \), and by definition of \( M \), a move to \( t \) ultimately leads to the final outcome \( M(\sigma; r, t, \sigma(i + 1)) \). Since \( \sigma(i) \) is rational, \( \sigma(i) \) will select the move whose payoff will be the best as far as \( \sigma(i) \) is concerned.

We will abbreviate the fact that a single move from \( s \) by \( \sigma(i) \) can lead to \( t \) by the notation \( s [\sigma(i)] t \). Each possible \( t \) satisfies \( t_j = s_j \) for all \( j \neq \sigma(i) \), and there is one choice of \( t \) for each element of \( S_{\sigma(i)} \). Thus, more briefly,

\[
M(\sigma; r, s, \sigma(i)) = \max_{s' \in S} \{M(\sigma; r, t, \sigma(i + 1)) : s' [\sigma(i)] t \} \text{ for } i < n.
\]
Player $\sigma(n)$ uses slightly different logic because after his or her move the round ends. Suppose that the move leaves the current state at $t$. Then there are two possibilities: either (a) all players agree to stop the game at $t$, so that the payoff vector is $p(t)$; or else (b) some player wants to continue the game by playing another round. In case (b) the payoff is then $O(\sigma; r - 1, t)$. In case (a) the payoff is $p(t)$.

Define $U(\sigma; r - 1, t) = p(t)$ if $p(t) \succeq_{\text{all}} O(\sigma; r - 1, t)$; this case corresponds to (a). Otherwise, in case (b) there is some player $i$ such that $O(\sigma; r - 1, t) \succ_{i} p(t)$, and we define $U(\sigma; r - 1, t) = O(\sigma; r - 1, t)$. We thus obtain that if player $\sigma(n)$ moves to $t$, then the payoff vector will be $U(\sigma; r - 1, t)$. Hence $M(\sigma; r, s, \sigma(n)) = \max_{\sigma(i)} \{U(\sigma; r - 1, t): s [\sigma(n)] t\}$, because player $\sigma(n)$ will move to the possible state $t$ whose payoff is best from his or her point of view.

Finally, $O(\sigma; r, s) = M(\sigma; r, s, \sigma(1))$ since $\sigma(1)$ makes the first move of the $r$th round.

To summarize, we thus have the following relationships:

1. $O(\sigma; 0, s) = p(s)$.
2. $U(\sigma; r, s) = p(s)$ if $p(s) \succeq_{\text{all}} O(\sigma; r, s)$; otherwise $U(\sigma; r, s) = O(\sigma; r, s)$.
3. $M(\sigma; r, s, \sigma(n)) = \max_{\sigma(i)} \{U(\sigma; r - 1, t): s [\sigma(n)] t\}$.
4. $M(\sigma; r, s, \sigma(i)) = \max_{\sigma(j)} \{M(\sigma; r, t, \sigma(j)): s [\sigma(j)] t\}$ for $i = 1, 2, \ldots, n - 1$.
5. $O(\sigma; r, s) = M(\sigma; r, s, \sigma(1))$.

Together, these formulae yield a recursive calculation of all the quantities. When the context makes it desirable to extend the notation so as to include the game $G$ explicitly, we will write $O(\sigma; 0, s; G), U(\sigma; r, s; G), M(\sigma; r, s, \sigma(i); G)$ as needed.

The explicit recursive formulas given above are easily programmed. As a result, one can write a fast program to find these quantities by computer, at least when the number $n$ of players is small.

**Lemma 5.1.** For any $r \geq 1$, $O(\sigma; r, s)$ is independent of the choice of $s$.

**Proof.** For simplicity of notation, we give the proof only in the case where $n = 3$ and the order is $\sigma = (1, 2, 3)$. A state will be described by three strategies $abc$, in which $a$ is the pure strategy by player 1, $b$ is the pure strategy by player 2, and $c$ is the pure strategy by player 3. We will omit the $\sigma$ from the notation for $M$, $O$, and $U$. Assume $r = 1$.

Suppose that $abc$ and $abc'$ differ only in the strategy by player 3. Then $M(r, abc, 3) = \max_{\sigma} \{U(r - 1, t): abc [3] t\} = \max_{\sigma} \{U(r - 1, abc)\} + \max_{\sigma} \{U(r - 1, ab\bar{c})\} + \max_{\sigma} \{U(r - 1, \bar{a}bc)\}$. Since $M(r, abc, 3) = M(r, abc', 3)$, we have $M(r, abc, 3) = M(r, abc', 3)$ for all $a, b, c, c'$ since they are maxima over the same set of states.

Similarly $M(r, abc, 2) = \max_{\sigma} \{M(r, abc, 3): \bar{a} \in S_{1}\}$ while $M(r, abc', 2) = \max_{\sigma} \{M(r, abc', 3): \bar{a} \in S_{1}\}$. Hence $M(r, abc, 2) = M(r, abc', 2) = M(r, abc, 3) = M(r, abc', 3)$ for each $\bar{a}$ by the previous paragraph, it follows that $M(r, abc, 2) = M(r, abc', 2)$ for all $a, b, b', c, c'$.

Now $M(r, abc, 1) = \max_{\sigma} \{M(r, \bar{a}bc, 2): \bar{a} \in S_{1}\}$ while $M(r, a'b'c', 1) = \max_{\sigma} \{M(r, \bar{a}b'c', 2): \bar{a} \in S_{1}\}$. But $M(r, abc, 2) = M(r, abc', 2)$ for each choice of $\bar{a}$ by the previous paragraph. Hence $M(r, abc, 1) = M(r, a'b'c', 1)$ for all $a, a', b, b', c, c'$. Since $O(r, abc) = M(r, abc, 1)$ and $O(r, a'b'c') = M(r, a'b'c', 1)$, it then follows that $O(r, abc) = O(r, a'b'c')$ for all $a, a', b, b', c, c'$.

$\square$
As a consequence of the lemma, we may omit \( s \) from the notation for \( O \). Thus we may define \( O(\sigma; r) = O(\sigma; r, s) \) for any choice of initial state \( s \) since the choice of \( s \) is immaterial. We call \( O(\sigma; r) \) the \( r \)th outcome with order \( \sigma \).

**Lemma 5.2.** For all \( r \geq 1 \) and for all permutations \( \sigma \), \( O(\sigma; r + 1) \geq \text{all} \ O(\sigma; r) \).

**Proof.** We give the proof only in the case where the order is \( \sigma = (1, 2, 3, \ldots, n) \). We omit the \( \sigma \) from the notation for \( M, O, \) and \( U \).

Select an initial state \( s \). Then \( O(r + 1) = O(r + 1, s) = M(r + 1, s, 1) = M(r + 1, t(1), 2) \) [for some choice of state \( t(1) \), differing from \( s \) only in the pure strategy of player 1] = \( M(r + 1, t(2), 3) \) [for some choice of state \( t(2) \) differing from \( t(1) \) only in the pure strategy of player 2] = \( \cdots = M(r + 1, t(n - 1), n) \) [for some choice of state \( t(n - 1) \) differing from \( t(n - 2) \) only in the pure strategy of player \( n - 1 \)] = \( U(r, t(n)) \) [for some choice of state \( t(n) \) differing from \( t(n - 1) \) only in the pure strategy of player \( n \)]. But by definition of \( U, U(r, t(n)) \geq \text{all} \ O(r, t(n)) = O(r) \) [by Lemma 5.1]. The result follows. □

**Lemma 5.3.** If for some \( r \geq 1 \) we have \( O(\sigma; r + 1) = O(\sigma; r) \), then for all \( u \geq r \) we have also \( O(\sigma; u) = O(\sigma; r) \).

**Proof.** It is clear from the formulas that for \( u \geq 1 \), \( O(\sigma; u + 1, s) \) is a (complicated) function of only \( O(\sigma; u, s) \) and the various payoff vectors \( p(s) \). Hence the same data will always yield the same value of that function. If \( O(\sigma; r + 1, s) = O(\sigma; r, s) \) for all \( s \), then the same calculation will yield \( O(\sigma; r + 2, s) = O(\sigma; r + 1, s) \), and then \( O(\sigma; r + 3, s) = O(\sigma; r + 2, s) \). Repeating the argument as necessary, we see that \( O(\sigma, r) = O(\sigma; r + 1) = O(\sigma; r + 2) = \ldots = O(\sigma; u) \). □

**Theorem 5.4.** Suppose that the game is strict and the players are all rational. There exists a positive integer \( t \) such that for all \( r \geq t \) we have \( O(\sigma; r, s) = O(\sigma; t) \).

**Proof.** Let \( T \) denote the maximal possible integer \( t \) such that there are payoff vectors \( V_1, V_2, \ldots, V_t \) satisfying \( V_1 \leq \text{all} V_2 \leq \text{all} V_3 \leq \text{all} \ldots \leq \text{all} V_t \) and such that for each \( i \) we have that \( V_{i+1} \) is strictly preferred by some player over \( V_i \). Since the vectors \( V_i \) would need to be distinct, \( T \) is at most equal to the number of distinct payoff vectors, which is known to be finite since each player has only finitely many pure strategies.

Note that we know \( O(\sigma; 1) \leq \text{all} O(\sigma; 2) \leq \text{all} O(\sigma; 3) \leq \text{all} \ldots \leq \text{all} O(\sigma; r) \leq \text{all} O(\sigma; r + 1) \leq \text{all} \ldots \). If any two adjacent terms were identical, then by Lemma 5.3 all the terms to the right from that point on would also be identical. Hence there are at most \( T \) distinct terms \( O(\sigma; r) \). Thus there exists \( t = T \) satisfying the statement of the theorem. □

We call the smallest such integer \( t \) the transiency of the game. We have proved that \( t \leq T \), where \( T \) is as in the proof of the theorem. The payoff vector \( O(\sigma; r, s) \) for \( r \geq t \) then depends only on the permutation \( \sigma \).

Define \( O(\sigma) = O(\sigma; r, s) \) for any initial state \( s \) and any \( r \) at least as large as the
transiency $t$; we have seen that $O(\sigma)$ does not depend on the choice of $r$ (as long as $r \geq t$) or on the choice of $s$. We call $O(\sigma)$ the ultimate outcome for permutation $\sigma$.

We may note that $O(\sigma) = \lim_{r \to \infty} O(\sigma; r, s)$ for any choice of $s$. This statement is somewhat misleading, however, since in fact $O(\sigma) = \hat{O}(\sigma; t, s)$ for any choice of $s$; no true limiting process is really required.

**Definition.** The outcome set for the game $G$ is defined to be $X(G) = \{O(\sigma): \sigma$ is a permutation of $\{1, 2, \ldots, n\}\}$.

This outcome set $X(G)$ is the solution concept for the game $G$ discussed in this paper. It tells all possible ultimate outcomes $O(\sigma; r, s)$ for any $\sigma$ and any $s$, as long as $r$ is large enough. If the negotiation process takes place as described in Section 3, if the maximum number of rounds $r$ is large enough (i.e. $r$ is at least the transiency $t$ of the game), if the game starts at any state $s$, if the players make their choices successively in the order given by $\sigma$, and if the players are rational, then the outcome will be $O(\sigma)$. This outcome depends on the payoff function and $\sigma$, but it does not depend on $s$ or $r$.

**Remark.** In the proof of Theorem 5.4 we showed that the transiency $t$ satisfied that $t \leq T$. We may obtain an even simpler estimate for $t$ as follows: suppose that $|S_i| = c_i$ is the number of pure strategies that player $i$ has. Then clearly $T \leq \prod c_i$ since the product of the $c_i$ is the total number of states. Hence $t \leq \prod c_i$ as well.

For future reference, here is a more explicit computation of the floor $F(\sigma)$: given the permutation $\sigma$ suppose that player $\sigma(1)$ has selected strategy $s_{\sigma(1)}$ from $S_{\sigma(1)}$, $\sigma(2)$ has selected strategy $s_{\sigma(2)}$ from $S_{\sigma(2)}$, $\ldots$, player $\sigma(n-1)$ has selected $s_{\sigma(n-1)}$ from $S_{\sigma(n-1)}$.

Define $K_{n-1}(\sigma; s_{\sigma(1)}^n, \ldots, s_{\sigma(n-1)}) = \max_{\sigma(n)} \{p(t): t_{\sigma(1)} = s_{\sigma(1)}, \ldots, t_{\sigma(n-1)} = s_{\sigma(n-1)}\}$. Clearly this is the payoff vector that results if the assignments $s_{\sigma(1)}, \ldots, s_{\sigma(n-1)}$ have been made irrevocably and the rational player $\sigma(n)$ gets to select his or her irrevocable assignment so as to maximize his or her payoff.

Now assume that player $\sigma(1)$ has irrevocably selected $s_{\sigma(1)}$, $\ldots$, player $\sigma(n-2)$ has selected $s_{\sigma(n-2)}$. Define $K_{n-2}(\sigma; s_{\sigma(1)}^n, \ldots, s_{\sigma(n-2)}) = \max_{\sigma(n)} \{K_{n-1}(\sigma; s_{\sigma(1)}^n, \ldots, s_{\sigma(n-1)}): s_{\sigma(n-1)} \in S_{\sigma(n-1)}\}$. Clearly this is the payoff vector that results if the assignments $s_{\sigma(1)}, \ldots, s_{\sigma(n-2)}$ have been made irrevocably; player $\sigma(n-1)$ selects a strategy $s_{\sigma(n-1)}$ from $S_{\sigma(n-1)}$ so as to realize $K_{n-2}(\sigma; s_{\sigma(1)}^n, \ldots, s_{\sigma(n-2)})$.

This procedure leads to a recursive calculation of $F(\sigma)$ as follows:

**Lemma 5.5.** Suppose $s_{\sigma(1)} \in S_{\sigma(1)}$,..., $s_{\sigma(n-1)} \in S_{\sigma(n-1)}$. Define $K_{n-1}(\sigma; s_{\sigma(1)}^n, \ldots, s_{\sigma(n-1)}) = \max_{\sigma(n)} \{p(t): t_{\sigma(1)} = s_{\sigma(1)}, \ldots, t_{\sigma(n-1)} = s_{\sigma(n-1)}\}$. For each $i = n-2, \ldots, 1$, recursively define $K_i(\sigma; s_{\sigma(1)}^n, \ldots, s_{\sigma(i)}) = \max_{\sigma(i+1)} \{K_{i+1}(\sigma; s_{\sigma(1)}^n, \ldots, s_{\sigma(i+1)}): s_{\sigma(i+1)} \in S_{\sigma(i+1)}\}$. Then $F(\sigma) = \max_{\sigma(1)} \{K_1(\sigma; s_{\sigma(1)}): s_{\sigma(1)} \in S_{\sigma(1)}\}$.

**Proof.** The result is evident from the discussion. □
6. Proof that the outcome set satisfies the axioms

The purpose of this section is to prove the following result.

**Main Theorem 6.1.** There exists a unique function $X$ which assigns to each strict game $G$ a nonempty set $X(G)$ of payoff vectors from $G$ and which satisfies the axioms in Section 2. Moreover, if rational players negotiate under the rules of play of Section 3, then the set $X(G)$ consists precisely of all the possible final outcomes under these rules of play.

These rules of play thus give a formal procedure that realizes the set $X(G)$. Simulation of the play via these rules by rational players will give a computation of $X(G)$.

The proof of Theorem 6.1 is lengthy and will be broken into several pieces. Note that we assume throughout that the game of the play via these rules by rational players will give a computation of $X(G)$.

**Theorem 6.2.** There is at most one function $X$ satisfying the axioms.

**Proof.** Suppose that both $X$ and $Y$ satisfy the axioms. We know that $X(G)$ consists of all the vectors $X(\sigma; G)$ and $Y(G)$ consists of all the vectors $Y(\sigma; G)$ by the floor axiom. Hence it suffices to prove that for every permutation $\sigma$ and every game $G$ we have $X(\sigma; G) = Y(\sigma; G)$. The proof will be by induction on the number $k$ of distinct payoff vectors for the game $G$. If $k = 1$, then the result is trivial, since each $X(\sigma; G)$ and each $Y(\sigma; G)$ must be a payoff vector of $G$ by the form axiom; since $k = 1$ there is only one such vector, so they are equal. We now assume by induction the truth of the result for any game with fewer than $k$ distinct payoff vectors, and we prove the result for a game $G$ with exactly $k$ distinct payoff vectors, where $k > 1$.

By the default reduction axiom, $X(\sigma; G) = X(\sigma; G\sigma)$ and $Y(\sigma; G) = Y(\sigma; G\sigma)$. We show that $G\sigma$ has fewer than $k$ distinct payoff vectors; then $X(\sigma; G\sigma) = Y(\sigma; G\sigma)$ by induction, and the result follows. To see that $G\sigma$ has fewer than $k$ distinct payoff vectors, note that every payoff vector for $G\sigma$ is also a payoff vector from $G$, so $G\sigma$ has at most $k$ distinct payoff vectors. If any payoff vector $p(v)$ for $G$ fails to satisfy $p(v) \geq_{\text{all}} F(\sigma; G)$, then by the definition of $G\sigma$ it follows that $p(v)$ is replaced by $F(\sigma; G)$, already one of the payoff vectors, wherever it occurs; hence the payoff vector $p(v)$ does not occur in $G\sigma$, $G\sigma$ has strictly fewer than $k$ distinct payoff vectors, and by the inductive hypothesis, $X(\sigma; G\sigma) = Y(\sigma; G\sigma)$.

There remains only the case where every payoff vector $p(v)$ for $G$ satisfies $p(v) \geq_{\text{all}} F(\sigma; G)$. In this case, the definition of $G\sigma$ yields that the payoff matrix for $G\sigma$ is the same as the payoff matrix for $G$. Hence $F(\sigma; G\sigma) = F(\sigma; G)$. The minimal possible payoff is $F(\sigma; G)$ for all players simultaneously. All other payoffs are strictly better simultaneously for all the players, by strictness of $G$. Since $k = 2$, not all payoff vectors can equal $F(\sigma; G)$, and there exists a state $v$ for which $p(v) \neq F(\sigma; G)$.

We now utilize the calculation in Lemma 5.5 to obtain a contradiction. Since $F(\sigma; G)$ is the minimal payoff vector for $G\sigma$ and $F(\sigma) = \max_{\alpha(1)} \{K_i(\sigma; s_{\alpha(1)}); s_{\alpha(1)} \text{ lies in } S_{\alpha(1)}\}$,
it follows that for all \( s_{\alpha(1)} \) in \( S_{\alpha(1)} \), \( K_i(\sigma; s_{\alpha(1)}) = F(\sigma; G) \). Similarly, since \( K_i(\sigma; s_{\alpha(1)}) = \max_{s_{\alpha(2)}} \{ K_2(\sigma; s_{\alpha(1)}, s_{\alpha(2)}); s_{\alpha(2)} \text{ lies in } S_{\alpha(2)} \} \) and \( K_i(\sigma; s_{\alpha(1)}) \) is the minimal vector \( F(\sigma; G) \), it follows that \( K_i(\sigma; s_{\alpha(1)}, s_{\alpha(2)}) = F(\sigma; G) \) for all \( s_{\alpha(1)} \) in \( S_{\alpha(1)} \), and all \( s_{\alpha(2)} \) in \( S_{\alpha(2)} \). Repeating this argument inductively, we see that \( K_{n-1}(\sigma; s_{\alpha(1)}, \ldots, s_{\alpha(n-1)}) = F(\sigma; G) \) for all \( s_{\alpha(1)} \) in \( S_{\alpha(1)} \), all \( s_{\alpha(2)} \) in \( S_{\alpha(2)} \), \ldots, and all \( s_{\alpha(n-1)} \) in \( S_{\alpha(n-1)} \). In particular, if \( v \) is the state found above such that \( p(v) \neq F(\sigma; G) \), then \( K_{n-1}(\sigma; v_{\alpha(1)}, \ldots, v_{\alpha(n-1)}) = F(\sigma; G) \). But \( K_{n-1}(\sigma; v_{\alpha(1)}, \ldots, v_{\alpha(n-1)}) = \max_{\alpha(n)} \{ t_{\alpha(1)} = v_{\alpha(1)}, \ldots, t_{\alpha(n-1)} = v_{\alpha(n-1)} \} \) for all \( v \) if \( v \) is one of the candidates for \( t \). This proves \( F(\sigma; G) \geq \sum_{j} p(v) \) for all \( j \) since the game is strict.

In the remainder of this section we demonstrate that the sets \( X(G) \) defined in Section 5 satisfy the axioms given in Section 2. This will complete the proof of Theorem 6.1. The form and nontriviality axioms are immediately seen to be satisfied by the outcomes obtained by our rules of play. The rules of play clearly do not favor any particular strategies or any particular players, and it follows that the outcome sets \( X(G) \) obey the symmetry axiom.

We now prove the other properties.

6.1. Pareto-optimality

**Theorem 6.3.** (Pareto-optimality axiom). Each \( O(\sigma) \) is Pareto-optimal.

**Proof.** Suppose that the transiency is \( t \), so \( O(\sigma) = O(\sigma; t, s) = O(\sigma; t + 1, s) \) for any state \( s \). If \( O(\sigma) \) is not Pareto-optimal, then there exists a state \( w \) such that \( O(\sigma) \leq \sum_{j} p(w) \) and for some player \( i \), \( O(\sigma) < p(w) \). Hence \( U(\sigma; t, w) = p(w) \). Moreover, since \( p(w) \neq O(\sigma) \), it must follow that \( O(\sigma) < p(w) \) for all \( j \) since the game is strict.

For simplicity of notation, we give the proof only in the case where \( n = 3 \) and the order is \( \sigma = (1, 2, 3) \). A state will be described by three strategies \( abc \), in which \( a \) is the strategy by player 1, \( b \) is the strategy by player 2, and \( c \) is the pure strategy by player 3. We will sometimes omit the \( \sigma \) from the notation for \( M, O, \) and \( U \). Suppose that state \( w = ab \). Hence \( U(\sigma; t, ab) = p(w) > O(\sigma) \) for all \( i \).

Note \( M(t + 1, ab, 3) = \max_{\{ U(t, u) : abc \in \{ 3 \} u \}} \{ U(t, ab) : c \text{ in } S_3 \} \geq \sum_{j} U(t, abc) > O(\sigma) \). Hence \( M(t + 1, ab, 3) > O(\sigma) \). If \( M(t + 1, ab, 3) = U(t, u) \), then because \( U(t, u) \neq O(\sigma) = O(t,u) \), it follows that \( U(t, u) = p(u) \) whence \( U(t, u) > O(t, u) = O(\sigma) \) for all \( i \). We thus obtain that \( M(t + 1, ab, 3) > O(\sigma) \) and \( M(t + 1, abc, 3) > O(\sigma) \) for all \( i \).

Similarly \( M(t + 1, abc, 2) = \max_{\{ M(t + 1, abc, 2) \} : \tilde{b} \text{ in } S_2} \geq \sum_{j} M(t + 1, abc, 3) \). Since \( M(t + 1, abc, 2) > O(\sigma) \), by the previous paragraph, we see that \( M(t + 1, abc, 2) \neq O(\sigma) \). But \( M(t + 1, abc, 2) = M(t + 1, abc, 3) \) [for some \( \tilde{b} = U(t, u) \) [for some \( u \)] \( \geq \sum_{j} M(t + 1, abc, 2) \). Hence \( M(t + 1, abc, 2) > O(\sigma) \) for all \( i \).

Now \( M(t + 1, abc, 1) = \max_{\{ M(t + 1, abc, 1) \} : \tilde{a} \text{ in } S_1} \geq M(t + 1, abc, 2) > O(\sigma) \) by the preceding paragraph. Hence \( M(t + 1, abc, 1) \neq O(\sigma) \). But \( M(t + 1, abc, 1) =
$M(t + 1, \hat{abc}, 2) \{[\text{for some } \hat{a}] = M(t + 1, \hat{abc}, 3) \{[\text{for some } \hat{b}] = U(t, u) \{[\text{for some } u] \geq \forall i O(\sigma). \text{ Hence } M(t + 1, abc, 1) > O(\sigma) \text{ for all } i.$

Finally, $O(\sigma; t + 1) = M(t + 1, abc, 1)$. Hence we see that $O(\sigma; t + 1) > O(\sigma) = O(\sigma; t)$ for all $i$. This contradicts Theorem 5.4, which showed that $O(\sigma; t + 1) = O(\sigma; t)$. $\square$

The agreement axiom follows immediately from the Pareto-optimality axiom since in the perhaps rare situation where there is a single state $s$ that is simultaneously best for all players, it follows that $p(s)$ is the only Pareto-optimal payoff vector. Since $X(G)$ is non-empty by nontriviality, $X(G)$ must be exactly $\{p(s)\}$.

6.2. Lower bound axiom

Recall that a lower bound for player $i$ is a number $L$ for which there exists a pure strategy $a \in S_i$ such that for all $s$ in $S$ for which $s_i = a$ it is true that $p_i(s) \geq L$.

**Theorem 6.4. (Lower bound axiom)** Suppose that $L$ is a lower bound for player $i$ arising from the pure strategy $a \in S_i$. Then for each permutation $\sigma$, $O(\sigma) \geq L$.

**Proof.** Since $O(\sigma) \geq \forall i O(\sigma; 1)$ by Lemma 5.2, it suffices to prove $O(\sigma; 1) = L$ for all $i$. Suppose that $i = \sigma(j)$. Then $O(\sigma; 1) = M(\sigma; 1, s^{(1)}, \sigma(1)) \{[\text{for any state } s^{(1)}] = \max_{\sigma(1)} \{M(\sigma; 1, t, \sigma(2)); s^{(1)} \{[\text{for some state } s^{(1)}] = M(\sigma; 1, s^{(2)}, \sigma(2)) \{[\text{for some state } s^{(2)}] = M(\sigma; 1, s^{(3)}, \sigma(3)) \{[\text{for some state } s^{(3)}] = \ldots = M(\sigma; 1, s^{(j)}, \sigma(j)) \{[\text{for some state } s^{(j)}].$

There are two cases to consider. If $j < n$ then $M(\sigma; 1, s^{(j)}, \sigma(j)) = \max_{\sigma(j)} \{M(\sigma; 1, t, s^{(j)}); \sigma(j) = i\}$. Among the possible $t$ is some $t$ for which $t_i = a$. For this $t$ it follows that $M(\sigma; 1, t, \sigma(j + 1)) \geq L$ since $M(\sigma; 1, t, \sigma(j + 1))$ will be the payoff from a state with $i$th component $a$ [because player $i$ never thereafter changes the strategy again], and by the definition of $L$ all the resulting payoffs to player $i$ must be at least $L$. Hence the best $t$ from the viewpoint of $\sigma(j) = i$ must give payoff to $i$ at least $L$.

On the other hand, if $j = n$ then $M(\sigma; 1, s^{(j)}, \sigma(j)) = \max_{\sigma(j)} \{p(t); s^{(j)} \{[\text{since } \sigma(j) = i\} \text{ must also give payoff to } i \text{ at least } L.$

6.3. Floor axiom

**Theorem 6.5.** For any permutation $\sigma$, $F(\sigma) = O(\sigma; 1)$.

**Proof.** Let $s = (s_1, s_2, \ldots, s_n)$ be the state obtained in the computation of $F(\sigma)$, so that $F(\sigma) = p(s)$. When player $\sigma(n)$ makes the choice of strategy, $s_{\sigma(n)}$ is already determined for all $k \neq n$. Player $\sigma(n)$ thus chooses $s_{\sigma(n)}$ such that $p(s) = \max_{\sigma(n)} \{p(t); t \{\sigma(n)\} \}$ and for all $k \neq n$, $t_{\sigma(k)} = s_{\sigma(k)}$.

Hence $p(s) = M(\sigma; 1, u, \sigma(n))$ for any $u$ agreeing with $s$ in all positions except possibly in position $\sigma(n)$. In particular, $M(\sigma; 1, s, \sigma(n)) = p(s) = F(\sigma)$.
Thus, if player $\sigma(n - 1)$ chooses strategy $s_{\sigma(n - 1)}$, then the assumption that $\sigma(n)$ is rational means that the payoff will be $M(\sigma; 1, s, \sigma(n))$. Hence player $\sigma(n - 1)$ chooses $s_{\sigma(n - 1)}$ so as to maximize $M(\sigma; 1, s, \sigma(n))$, hence to obtain the payoff $\max_{s_{\sigma(n - 1)}} \{M(\sigma; 1, u, \sigma(n)) : s \in \sigma(n - 1)] u\}$. Each such $M(\sigma; 1, u, \sigma(n))$ equals $p(s)$ by the previous paragraph. This same quantity defines $M(\sigma; 1, s, \sigma(n - 1))$, whence $M(\sigma; 1, s, \sigma(n - 1)) = p(s) = F(\sigma)$.

Repeating this argument, we find that $M(\sigma; 1, s, \sigma(j)) = p(s)$ for each $j$. In particular, $O(\sigma; 1) = O(\sigma; 1, s) = M(\sigma; 1, s, \sigma(1)) = p(s) = F(\sigma)$. □

**Corollary 6.6.** For any permutation $\sigma$, $F(\sigma) \leq \forall \sigma O(\sigma)$.

**Proof.** This follows from Lemma 5.2 and the definition of $O(\sigma)$. □

**Corollary 6.7.** (Floor axiom). For each permutation $\sigma$ of the players there is a uniquely determined outcome $X(\sigma)$ in $X(G)$ such that $X(\sigma) \equiv \forall \sigma F(\sigma)$. Moreover, every element of $X(G)$ arises in this manner for some permutation $\sigma$.

**Proof.** Given any permutation $\sigma$, let $X(\sigma) = O(\sigma)$. Then $X(\sigma)$ satisfies $X(\sigma) \equiv \forall \sigma F(\sigma)$. Conversely, any $p(u)$ in $X(G)$ has the form $p(u) = O(\sigma)$ for some permutation $\sigma$. □

### 6.4. Strong dominance

**Theorem 6.8.** Strong dominance axiom: suppose that a player (say player $j$) has a strategy $s_j$ that is strongly dominated. Let $G'$ denote the matrix game obtained by eliminating the strategy $s_j$ from $S_j$. Thus, we let $S' = \prod S'_i$, where $S'_i = S_i$ for $i \neq j$, $S'_j = S_j \setminus \{s_j\}$. Define $p': S' \to R$ by $p' = p|S'$, and let $G' = \{S', p'\}$. Then $X(G) = X(G')$.

**Proof.** For simplicity of notation, we prove the result in the case where $j = 1$, $S_1 = \{1, 2, \ldots, m\}$, and strategy is strongly dominated by strategy 1 for player 1.

**Claim 1.** For each permutation $\sigma$, $F(\sigma, G)$ equals $F(\sigma, G')$ and also equals $p(u)$ for some state $v$ in $S'$.

**Proof of Claim 1.** Suppose that the order in which players announce their irrevocable strategies is $\sigma$. If player 1 is player $\sigma(k)$, we compute $F(\sigma, G) = O(\sigma; 1, s, G)$ [for any $s$, by Theorem 6.4, so we may select $s \in S'] = M(\sigma; 1, s, \sigma(1); G) = M(\sigma; 1, s^{(k)}; \sigma(2); G)$ [for some $s^{(2)} = \ldots = M(\sigma; 1, s^{(k)}; \sigma(k); G)$ [for some $s^{(k)}$].

In the case where $k = n$, then $F(\sigma; G) = \max_{s^{(n)}} \{U(\sigma; 0, t; G); s^{(n)} [\sigma(n)]t\} = F(\sigma, G')$. Suppose that the last quantity is maximized when $t = u$. If $u_1 \neq m$, then $u \not\in S'$ and clearly $F(\sigma, G')$ also equals $p(u)$ since the only change in the computation for $F(\sigma; G')$ would be that in finding $\max_{s^{(n)}} \{p(t); s^{(n)} [1] t\}$ we would need to restrict $t$ to being in $S'$, and $t = u$ would still be obtained. If instead $u_1 = m$, consider the state $v$ such that $v_1 = u_1$ for all $j \neq 1$, $v_1 = 1$. Then $s^{(n)} [1] v$. Moreover, by strong dominance $p(v) \geq _1 p(u)$ so $\max_{s^{(n)} [1] v}$ is also maximized when $t = v$. Thus we may
utilize v rather than u and \( F(\sigma; G) = p(v) \). Since \( v \in S' \) again \( F(\sigma; G') = p(v) = F(\sigma; G) \). This proves the result if \( k = n \).

If \( k < n \) then \( M(\sigma; 1, s^{(k)}, \sigma(k); G) = \max_{\sigma(k)} \{ M(\sigma; 1, u, \sigma(k + 1); G) \} \) for some \( u \) such that \( s^{(k)}[\sigma(k)]u = M(\sigma; 1, u, \sigma(k + 1); G) \) for some \( u \) such that \( s^{(k)}[\sigma(k)]u \). If \( u_1 \neq m \), then \( u \) lies in \( S' \). Moreover, \( M(\sigma; 1, u, \sigma(k + 1); G) = M(\sigma; 1, u^{(k + 2)}, \sigma(k + 2); G) \) for some \( u^{(k + 2)} \) such that \( u \) is chosen to maximize the expression, we also have \( s^{(k)}[\sigma(k)]u = M(\sigma; 1, u^{(k + 2)} \in S') = \ldots = M(\sigma; 1, u^{(n)}, \sigma(n); G) \) [for some \( u^{(n)} \) such that \( u \) is chosen to maximize the expression, we also have \( s^{(k)}[\sigma(n)]u = M(\sigma; 1, u^{(n)} \in S') \) [for some \( u \) such that \( u^{(n)} \) is chosen to maximize the expression, we also have \( s^{(k)}[\sigma(n)]u = M(\sigma; 1, u^{(n)} \in S') \)]. Thus if \( u_1 \neq m \), then \( F(\sigma; G) = p(v) \) for some \( v \in S' \), and again a consideration of each step in the parallel computation of \( F(\sigma; G') \) shows that \( F(\sigma; G') = p(v) \) for this same \( v \). This proves the result if \( u_1 \neq m \).

Otherwise, if \( u_1 = m \), define the state \( v \) such that \( v_1 = 1 \) and \( s^{(k)}[\sigma(k)]v \). We will show that \( M(\sigma; 1, v, \sigma(k + 1); G) \leq \max_{\sigma(k)} M(\sigma; 1, u, \sigma(k + 1); G) \). To see this, note that \( M(\sigma; 1, u, \sigma(k + 1); G) = M(\sigma; 1, u^{(k + 2)}, \sigma(k + 2); G) \) [for some \( u^{(k + 2)} \) such that \( u \) is chosen to maximize the expression, we also have \( s^{(k)}[\sigma(k)]u = M(\sigma; 1, u^{(k + 2)} \in S') = \ldots = M(\sigma; 1, u^{(n)}, \sigma(n); G) \) [for some \( u^{(n)} \) such that \( u \) is chosen to maximize the expression, we also have \( s^{(k)}[\sigma(n)]u = M(\sigma; 1, u^{(n)} \in S') \) [for some \( u \) such that \( u^{(n)} \) is chosen to maximize the expression, we also have \( s^{(k)}[\sigma(n)]u = M(\sigma; 1, u^{(n)} \in S') \)]. Similarly \( M(\sigma; 1, v, \sigma(k + 1); G) = p(v) \) for some state \( x \) satisfying \( x_1 = 1 \). But \( p(x) \leq \max_{\sigma} p(w) \) by strong dominance, showing that \( M(\sigma; 1, v, \sigma(k + 1); G) \leq \max_{\sigma} M(\sigma; 1, u, \sigma(k + 1); G) \). On the other hand, since \( u \) was chosen to maximize the expression, we also have \( M(\sigma; 1, v, \sigma(k + 1); G) \leq \max_{\sigma} M(\sigma; 1, u, \sigma(k + 1); G) \). Hence \( M(\sigma; 1, v, \sigma(k + 1); G) = M(\sigma; 1, u, \sigma(k + 1); G) = p(v) \). We see that if \( u_1 = m \), we may replace \( u \) by \( v \) in the computation because of a tie. Since \( u_1 \neq m \), the argument in the preceding paragraph then yields the result.

**Claim 2.** If \( u \) is a state for \( G \) such that \( u_1 = m \), then \( p(u) \leq p(\sigma, G) \).

**Proof.** We modify the argument of Theorem 6.4. Suppose \( 1 = \sigma(k) \). Then \( F(\sigma; G) = O(\sigma; 1; G) \) [by Theorem 6.3] = \( M(\sigma; 1, s^{(1)}, \sigma(1)) \) [for any state \( s^{(1)} \) = \( \max_{\sigma(1)} \{ M(\sigma; 1, t, \sigma(2)); s^{(1)}[\sigma(1)]t = M(\sigma; 1, s^{(2)}, \sigma(2)) \} \) [for some state \( s^{(2)} = M(\sigma; 1, s^{(3)}, \sigma(3)) \) [for some state \( s^{(3)} \) = \( \ldots = M(\sigma; 1, s^{(k)}, \sigma(k)) \) [for some state \( s^{(k)} \)]. We consider two cases. If \( k < n \) then \( M(\sigma; 1, s^{(k)}, \sigma(k)) \) = \( \max_{\sigma(k)} \{ M(\sigma; 1, t, \sigma(j + 1)); s^{(k)}[\sigma(k)]t \} \) [since \( \sigma(k) = 1 \). Among the possible \( t \) is some \( \tilde{t} \) for which \( \tilde{t}_1 = 1 \). For this \( \tilde{t} \) it follows that \( M(\sigma; 1, \tilde{t}, \sigma(k + 1)) \geq p(v) \) since \( M(\sigma; 1, \tilde{t}, \sigma(k + 1)) \) will be the payoff \( p(w) \) from a state \( w \) with first component 1 [because player 1 never thereafter changes the strategy again], and by strong dominance \( p(w) \geq p(u) \). Hence the best \( t \) from the viewpoint of player 1 must also yield that \( M(\sigma; 1, t, \sigma(k + 1)) \geq p(u) \), so that \( F(\sigma; G) \geq p(u) \) as well.

On the other hand, if \( k = n \) then \( M(\sigma; 1, s^{(k)}, \sigma(k)) \) = \( \max_{\sigma(k)} \{ p(t); s^{(k)}[\sigma(k)]t \} \) [since \( \sigma(k) = 1 \). Among the candidates \( t \) is some \( \tilde{t} \) for which \( \tilde{t}_1 = 1 \). Hence \( M(\sigma; 1, s^{(k)}, \sigma(k)) \) = \( p(\tilde{t}) \). By strong dominance we know \( p(\tilde{t}) \geq p(u) \), so that again \( F(\sigma; G) \geq p(u) \). This proves Claim 2.

**Claim 3.** \( O(\sigma; G) = O(\sigma; G') \) and is the payoff from a state in \( S' \).
Proof. Suppose that \( O(\sigma; G) = p(u) \) for some state \( u \) not in \( S' \); hence \( u_1 = m \). Suppose that the transiency is \( t \). Then \( p(u) = O(\sigma; G) = O(\sigma, t; G) \geq \) all \( O(\sigma, t - 1; G) \geq \) all \( \ldots \geq \) all \( O(\sigma, 1; G) = F(\sigma; G) \geq_1 p(u) \) by Lemma 5.2 and Claim 2. It follows that \( F(\sigma; G) = p(u) \). By Claim 1, however, \( F(\sigma; G) = p(v) \) for some \( v \) in \( S' \). Hence \( p(u) = p(v) \), so it is also true that \( O(\sigma; G) = O(\sigma; G') \).

Hence in the play of the game \( G \), no ultimate outcome needs to arise from a state \( s \) with \( s_1 = m \); at best the use of such a state merely ties the payoffs using a state from \( S' \). Clearly then player 1 can always make the same selection of strategies in the play of the game \( G' \), avoiding strategy \( m \), without compromising any payoff, so \( O(\sigma; G) = O(\sigma; G') \).

The theorem follows from Claim 3. □

Remark. The conclusion of the strong dominance axiom is false under the assumption of mere dominance, as may be seen as follows using Game 32. Suppose \( G \) is

\[
\begin{align*}
(2, 2) & \quad (4, 1) \\
(1, 4) & \quad (3, 3)
\end{align*}
\]

Here the first row dominates the second but does not strongly dominate it. If we could ignore any dominated row, then we would have \( X(G) = X(G') \) for \( G' \) with matrix

\[
\begin{align*}
(2, 2) & \quad (4, 1) \\
(2, 2) & \quad (4, 1)
\end{align*}
\]

Clearly \( X(G') = \{(2, 2)\} \), while one easily sees that \( X(G) = \{(3, 3)\} \).

Remark. Much of the length and complexity of the proof of Theorem 6.8 is to deal with the possibility that some state \( u \) with \( u_1 = m \) could have the same payoff as a state \( v \) with \( v_1 \neq m \). In the common circumstance when distinct states have distinct payoff vectors, such ties cannot occur and the argument can be simplified dramatically. For example, Claim 2 could be then strengthened to assert that \( p(u) <_1 F(\sigma; G) \) when \( u_1 = m \).

6.5. Default reduction

Theorem 6.9. Let \( G \) be the game with strategies \( S \) and payoff vectors \( p: S \to \mathbb{R}^n \). For each permutation \( \sigma \) let \( G_\sigma \) be the game with strategies \( S \) and payoff vectors \( q: S \to \mathbb{R}^n \) given by

\[
\begin{align*}
q(s) & = p(s) \text{ if } p(s) \geq_\text{all} F(\sigma; G); \\
q(s) & = F(\sigma; G) \text{ otherwise.}
\end{align*}
\]

Then \( O(\sigma; r + 1, s; G) = O(\sigma; r, s; G_\sigma) \) for all \( s \) and all \( r \geq 1 \); \( M(\sigma; r + 1, s, \sigma(j); G) = M(\sigma; r, s, \sigma(j); G_\sigma) \) for all \( s \), all \( r \geq 1 \), and all \( j \); and \( U(\sigma; r + 1, s, G) = U(\sigma; r, s; G_\sigma) \) for all \( r \geq 1 \) and all \( s \). In particular, \( O(\sigma; G) = O(\sigma; G_\sigma) \).

Proof. For all states \( s \), we know that \( O(\sigma; 1, s; G) = F(\sigma; G) \) by Theorem 6.5. Then \( U(\sigma; 1, s; G) = q(s) \) from the definition of \( U \). Hence \( M(\sigma; 2, s, \sigma(n); G) = \max_{\sigma(n)} \)

Proof. Proposition 6.11. by the rules of play, hence completes the proof of Theorem 6.1. 

Repeating the argument we find that for all \( j \) and for all \( s \) we have \( M(\sigma; 2, s, \sigma(n); G) = M(\sigma; 1, s, \sigma(n); G \sigma) \). Hence \( O(\sigma; 2, s; G) = O(\sigma; 1, s; G \sigma) \) for all \( s \).

Now \( U(\sigma; 2, s; G) = p(s) \) if \( p(s) \geq \max_{\sigma(n); G} O(\sigma; 2, s; G) \); or otherwise \( U(\sigma; 2, s; G) = O(\sigma; 2, s; G) \). Similarly, \( U(\sigma; 1, s; G \sigma) = q(s) \) if \( q(s) \geq \max_{\sigma(n); G} O(\sigma; 1, s; G \sigma) \); or otherwise \( U(\sigma; 1, s; G \sigma) = O(\sigma; 1, s; G \sigma) \).

Hence \( U(\sigma; 2, s; G) = U(\sigma; 1, s; G \sigma) \) unless either

(a) \( p(s) \geq \max_{\sigma(n); G} O(\sigma; 2, s; G) \), \( q(s) \geq \max_{\sigma(n); G} O(\sigma; 1, s; G \sigma) \), but \( p(s) \neq q(s) \); or
(b) \( p(s) \geq \max_{\sigma(n); G} O(\sigma; 2, s; G) \); for some \( i \), \( q(s) < \max_{\sigma(n); G} O(\sigma; 1, s; G \sigma) \), and \( p(s) \neq O(\sigma; 1, s; G \sigma) \); or
(c) for some \( i \), \( p(s) < O(\sigma; 2, s; G) \); for some \( j \), \( q(s) < \max_{\sigma(n); G} O(\sigma; 1, s; G \sigma) \), and \( O(\sigma; 2, s; G) \neq O(\sigma; 1, s; G \sigma) \); or
(d) for some \( i \), \( p(s) < O(\sigma; 2, s; G) \); \( q(s) \geq \max_{\sigma(n); G} O(\sigma; 1, s; G \sigma) \), and \( O(\sigma; 2, s; G) \neq q(s) \).

Case (c) cannot occur because of the calculation above. From case (a) it would follow that \( p(s) \geq \max_{\sigma(n); G} O(\sigma; 1, s; G) = F(\sigma; G) \), so \( q(s) = p(s) \) by the definition of \( q \), a contradiction showing that case (a) cannot occur. From case (b) it would follow that \( p(s) \geq \max_{\sigma(n); G} O(\sigma; 1, s; G) = F(\sigma; G) \), so \( q(s) = p(s) \); hence \( q(s) \geq \max_{\sigma(n); G} O(\sigma; 2, s; G) = O(\sigma; 1, s; G \sigma) \), contradicting that for some \( i \), \( q(s) < \max_{\sigma(n); G} O(\sigma; 1, s; G \sigma) \). From case (d) it would follow that \( q(s) \geq \max_{\sigma(n); G} O(\sigma; 1, s; G \sigma) = O(\sigma; 2, s; G) \), but \( O(\sigma; 2, s; G) \neq q(s) \); hence for some \( j \) we have \( q(s) > \max_{\sigma(n); G} O(\sigma; 1, s; G) = F(\sigma; G) \); hence \( q(s) \neq F(\sigma; G) \), whence \( q(s) = p(s) \) and \( p(s) \geq \max_{\sigma(n); G} F(\sigma; G) \) by the definition of \( q \); now \( p(s) \geq \max_{\sigma(n); G} O(\sigma; 1, s; G \sigma) \), hence none of cases (a), (b), (c), or (d) can occur, whence \( U(\sigma; 2, s; G) = U(\sigma; 1, s; G \sigma) \) for all \( s \).

We now repeat the paragraphs above with obvious modifications in order to yield an inductive proof of the result. □

**Corollary 6.10.** (Default reduction axiom). For each permutation \( \sigma \), \( X(\sigma; G) = X(\sigma; G \sigma) \).

**Proof.** \( X(\sigma; G) = O(\sigma; G) \) while \( X(\sigma; G \sigma) = O(\sigma; G \sigma) \). □

Corollary 6.10 completes the argument that all the axioms are satisfied by \( X(G) \) given by the rules of play, hence completes the proof of Theorem 6.1. □

**Proposition 6.11.** The axioms other than default reduction completely determine the outcomes on all strict 2 by 2 games with no repeated payoff vectors.

**Proof.** If the game \( G \) has a single strategy that is best for both players, then this outcome
is the only member of $X(G)$ since it is the only Pareto-optimal outcome. Otherwise, the game is equivalent to one of the 57 games of strategy given in Brams (1994). For each one, we can readily verify that these axioms determine the outcome, as computed in the manner at the end of Section 2. At no stage is the default reduction axiom required.

7. Example of the ‘play’ of a game

In this section we illustrate by a simple example how the rules of the game might actually be utilized in the play of a game. Consider the following payoff matrix for Prisoners’ Dilemma (Game 32 in Brams, 1994) with players $R$ (Row) and $C$ (Column):

\[
\begin{array}{cc}
(2, 2) & (4, 1) \\
(1, 4) & (3, 3)
\end{array}
\]

Each $X$ in $X(G)$ must be Pareto-optimal, so the only possibilities are $(1, 4)$, $(4, 1)$, and $(3, 3)$. The first two violate the lower bound axiom. This leaves only $(3, 3)$, so $X(G) = \{(3, 3)\}$ by the nontriviality axiom. The axioms thus show that $X(G) = \{(3, 3)\}$, but intuition suggests that the Nash equilibrium $(2, 2)$ should commonly arise. Why should $(2, 2)$ be eliminated by our rules of play?

A possible play of the game is as follows: initially, each player picks the first option, so the initial current state has payoff $(2, 2)$. By the toss of a coin, the order of players is chosen to be $RC$. The transiency of the game is easily seen to be 2. Thus the number $r$ of rounds must be at least 2. Suppose that $r = 6$.

In the first round, $R$ picks the second row, and then $C$ picks the second column. Now the current state has payoff $(3, 3)$. Each player is asked whether they are willing to end the game at this point; and they agree to end the game. The final payoff is then $(3, 3)$.

The complete argument why the game would end with payoff $(3, 3)$ follows from Section 6. Here we present an informal justification. First, note that if the game continued into the sixth round, then it would end at $(2, 2)$. This is because at the beginning of the sixth and last round (last because $r = 6$) Row would choose the first row; otherwise if Row had chosen the second row, Column would clearly choose the first column and the game would end with payoff $(1, 4)$, much to the disadvantage of Row. Since Row chooses the first row, Column will choose the first column. Thus all players know that the game will end with $(2, 2)$ unless they can both agree to end the game before the sixth round.

In any round except the sixth, it will not be the case that both players would agree to end the game at $(4, 1)$ or $(1, 4)$; one or the other player would know that they could hold out for the sixth round and improve their payoff. On the other hand, both players can agree to end the game with payoff $(3, 3)$, since both are better off than at the default payoff $(2, 2)$.

We may contrast this analysis with that of Brams’ Theory of Moves (1994) for the same game, starting at $(2, 2)$. Brams predicts that the final outcome in this case will be $(2, 2)$ because neither player can afford to make the offer of changing to the dominated strategy.

Note that even though there were six rounds possible, in this scenario only one round
was actually played. The number 6 played no real role; we would obtain the same result using any $r = t$, such as $r = 15$. What was essential was that there was the possibility of at least $t = 2$ rounds. If there were only 1 round, then clearly the outcome would instead have been $(2, 2)$. Similarly, the initial choice of strategies before any rounds were played had no role in the outcome.

8. An example with three players

All our preceding examples had only two players. Here we give an example with three players. Player 1 selects a row, player 2 selects a column, and player 3 selects a box. Payoffs are given in the form $(a, b, c)$ where $a$ is the profit for Row, $b$ the profit for Column, and $c$ the profit for Box.

If Box option = 1 the payoffs are

- $(5, 1, 3)$
- $(3, 3, 4)$

If Box option = 2 the payoffs are

- $(2, 5, 1)$
- $(6, 8, 5)$

Row can guarantee herself at least 2 by selecting the first row; Column can guarantee herself at least 2 by selecting the second column. Box can guarantee himself at least 2 by selecting the first box. The lower bound axiom thus gives the possibilities $(4, 4, 2), (3, 3, 4), (8, 2, 7), (6, 8, 5), (7, 6, 6)$. Pareto-optimality reduces it further to $(8, 2, 7), (6, 8, 5), (7, 6, 6)$. There is no strong dominance.

The floors are somewhat lengthy to compute by hand. We compute $F(123)$ as an example: if Row plays the first row and Column the first column, then Box having the choice between $(5, 1, 3)$ and $(2, 5, 1)$ will select the first box for $(5, 1, 3)$. If Row plays the first row and Column the second column, then Box having the choice between $(4, 4, 2)$ and $(8, 2, 7)$ will select $(8, 2, 7)$. Hence in the order $\sigma = (1 2 3)$, if Row plays the first row, Column will have the choice between $(5, 1, 3)$ and $(8, 2, 7)$ and will select the second column to ensure $(8, 2, 7)$. Similarly if Row plays the second row and Column the first column, Box will ensure that the outcome is $(6, 8, 5)$; if Row plays the second row and Column the second column, then Box will ensure that the outcome is $(1, 7, 8)$. Hence in the order $(1 2 3)$, if Row plays the second row, then Column will select the first column to ensure $(6, 8, 5)$. Hence Row will prefer the first row in order to ensure $(8, 2, 7)$, and $F(123) = (8, 2, 7)$. Since $(8, 2, 7)$ is Pareto-optimal and $X(123) \supseteq \forall i F(123)$, it follows that $X(123) = (8, 2, 7)$ and $(8, 2, 7)$ lies in $X(G)$.

Similar arguments show $F(132) = (4, 4, 2); F(231) = (6, 8, 5); F(213) = (6, 8, 5); F(312) = (6, 8, 5); F(321) = (6, 8, 5)$. Since $(6, 8, 5)$ is Pareto-optimal, $X(G)$ must contain $(6, 8, 5)$. There remains to compute $X(132)$; for this we utilize the default reduction axiom.

The payoffs for $G(132)$ are given below.
If Box option $= 1$ the payoffs are

$$(4, 4, 2) \quad (4, 4, 2)$$

$$(4, 4, 2) \quad (4, 4, 2)$$

If Box option $= 2$ the payoffs are

$$(4, 4, 2) \quad (4, 4, 2)$$

$$(6, 8, 5) \quad (7, 6, 6)$$

Note that Box option 2 strongly dominates Box option 1, eliminating Box option 1; thereafter the second row strongly dominates the first, eliminating the first row, and then the first column strongly dominates the second. Thus $X((132); G(132)) = (6, 8, 5)$, not $(7, 6, 6)$, and it follows that $X((132); G) = (6, 8, 5)$ as well. Hence $X(G) = \{(6, 8, 5), (8, 2, 7)\}$.

9. Conclusions

Suppose that we are given the payoff matrix for a game $G$ of perfect information. We seek to find the set $X(G)$ of payoff vectors that might actually occur under idealized circumstances in which rational players negotiate repeatedly. We have selected a set of axioms for $X(G)$ that seem plausible, and we have also presented a formal process (the ‘rules of play’) that realizes such a set $X(G)$. It is of interest that all the axiomatized properties, including Pareto-optimality and strong dominance, can be simultaneously satisfied. The rules of play can easily be programmed, and as a result $X(G)$ can be found by a fast computer program. Alternatively, many examples are easy to work by hand. The procedures work for any finite number of players.

The analysis of the games presented here is different from the type of analysis common for zero-sum games as in, for example, von Neumann and Morgenstern (1947). The sequential selection of strategies by the players in the rules of play fits instead within the tradition of the Theory of Moves, as in Brams (1994) or Willson (1998). For many games $G$ the solution sets $X(G)$ are identical with the solutions found in those previous papers. Nevertheless, these previous solutions did not satisfy the axioms on all games; one of the principal advantages of the current system is that the axiomatized properties are all satisfied.

Moreover, the current rules are arguably more realistic since typically all the players must simultaneously be in agreement if the game is to end before the final deadline. The rules exploit commonality of interests in the players. Consequently they could form a guide to reasonable compromises in some disputes subject to arbitration or mediation.

Acknowledgements

I wish to thank Steven J. Brams for introducing me to the Theory of Moves and providing helpful criticisms and suggestions.
References


