The representative Nash solution for two-sided bargaining problems

Haruo Imai, Hannu Salonen

*Kyoto Institute of Economic Research, Sakyo, Kyoto 606, Japan
bDepartment of Economics, University of Turku, 20014 Turku, Finland

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Abstract

An n-person bargaining situation is two-sided when participants of bargaining are divided into two groups and their preferences over bargained outcomes are exactly opposite to each other. This is so when the issue on the bargaining table is represented by a one dimensional set and people’s preferences are monotonically increasing in one group and monotonically decreasing in the other group. In this paper a solution for two-sided problems called the Representative Nash solution is introduced and axiomatized. A strategic bargaining model is constructed such that the unique stationary subgame perfect equilibrium outcome corresponds to the Representative Nash solution.

Keywords: Two-sided bargaining; Representative Nash solution

1. Introduction

Suppose the owners of a firm and its workers bargain over the wage level. Each worker prefers a high wage level, while each owner prefers a lower level. Thus the interests of the participants of bargaining are diametrically opposite. The class of bargaining games we consider consists of this sort of situations. An n-person bargaining situation is two-sided when participants of bargaining are divided into two groups and their preferences over bargained outcomes are exactly opposite of each other. This is so when the issue on the bargaining table is represented by a one dimensional set and people’s preferences are monotonic.

Many simple bargaining situations involve just one issue like negotiation over by
which year a certain environmental standard should be achieved or by which level the national budget should be cut, although not every single-issue bargaining situation is two-sided. Also, one should add the case where bargaining over more complicated issues may be broken into subproblems, and two-sided situation takes place in these subproblems. Similarly, an $n$-person problem may be reduced to bargaining between two already formed coalitions, inside which an allocation procedure (probably some bargaining solution) has been established already. For example, if inside a university department, it has been an established custom that any increase in the departmental budget is shared according to a fixed proportion by each faculty member, then the allocation problem of budgetary surplus involving two departments may appear two-sided. This is the case where a reduced-form problem is two-sided.

The reason why this class is of interest lies in the fact that it is a subclass of the standard $n$-person bargaining problems but its special structure of preferences suggests a reasonable deviation from the standard model. In fact, several difficulties raised against axioms adopted in the literature concern pervasive use of powerful requirements like independence of irrelevant alternatives. Restricting domain of the solution is one direction to alleviate such difficulties. Further, we define the class of problems below through an outcome space (although unlike Rubinstein et al. (1992), we retain the classical definition as well), which also help to localize the interpretation of axioms employed.

The notable feature of the two-sided problems is the dichotomy of interests which makes the problem close to the two-person problem. Moreover, since a unanimous agreement is required in a pure bargaining game, what matters in any conceivable bargaining procedure is the player who agrees under the most stringent request (the point made by Ponsati and Sakovicz (1996) in the context of bargaining over two alternatives). Thus we are led to a solution which is determined solely by the toughest player on each side, and differ from standard $n$-person solutions, especially the ones motivated by normative principles.

Another problem arising in this context is the effect of population size on the solution. Since usual symmetry requirement does not apply in general for this class of problems, there seems to be no particular principle which would single out a solution in which population affects the outcome. A noncooperative approach which we use as a guiding device for our solution does not help much in this matter. Different bargaining procedures yield different solutions reflecting differing power of causing friction. Below, we characterize an extreme solution in which this effect is minimal.

Related issue is the role of coalitions in bargaining, and the issue raised above can be seen as reflecting the joint bargaining paradox (Harsanyi, 1977). However, as was mentioned above as an example, more relevant questions may be that starting out from more general bargaining situation, would there be a force toward or away from utilizing the two-sided problem as a sort of implementation device (for sub-problems or for reduced-form problems). We do not give an answer to this question here, but we believe that the result here can be viewed as a step towards such a research programme.

In this paper we introduce and study a solution called the Representative Nash Solution. The solution is given an axiomatic characterization (Section 2), and we also construct a noncooperative bargaining model (Section 3) to provide some complementary motivation and justification to the model.
As for the axiomatic characterization of the solution, we take the approach extending
the two-person solution through axioms relevant to the class of two-sided problems. The
crucial point is that while the Representative Nash solution satisfies independence of
irrelevant alternatives in two-sided problems, it doesn’t satisfy this axiom in the whole
class of \( n \)-person bargaining problems. The axiom incorporating the idea that only the
toughest players’ preferences matter (the Tough Player Principle) is fairly straight-
forward, and it can be applied to any other two-person solution. To obtain the
Representative Nash solution, we also need an axiom (Population Invariance) requiring
the independence of the solution from a change in the population.

In Section 3 we show that utilizing the standard sequential bargaining game, the
stationary perfect equilibrium outcome does not yield the \( n \)-person Nash solution. As has
been noted by Shaked (reported in Sutton, 1986), \( n \)-person extensions of the Rubinstein-
stein’s sequential bargaining game suffer from a plethora of subgame perfect equilibrium
outcomes (Shaked and Sutton, 1984; Jun, 1987; Chae and Yang, 1988; Osborne and
Rubinstein, 1990; Krishna and Serrano, 1991). While there were several criticism against
the use of stationary equilibrium as a solution concept, the stationary subgame perfect
equilibrium outcome still yields the \( n \)-person Nash solution in this game. (This fact
motivated several attempts to find another refinement and to search for a rule which
implements the same outcome as the unique subgame perfect equilibrium outcome. Also
one should point out that there are attempts to motivate stationary equilibrium as a
solution concept, and also in coalitional bargaining literature, the use of stationary
solution is quite common.) Here, we note that even this solution deviates from the Nash
solution in the class under consideration.

The stationary perfect equilibrium outcome obtained exhibits the above-mentioned
property that only the preference of the toughest player matters in identifying the
outcome. The toughness in this context is naturally obtained from the comparison of the
measure of boldness (which appeared in literature like Aumann and Kurz, 1977; Svenjar,
1986; Roth, 1989; Chae, 1993; see also Harsanyi, 1977), a measure corresponding to the
derivative of the logarithm of the utility function. For the particular form of the solution,
weights given to the population of each side matter, i.e. depending on who has the right
to make a counter offer, population may or may not matter. The two rules yielding the
same stationary equilibrium outcome for the general splitting-a-dollar game produce
different results. Below, we mainly focus upon the rule which gives no account for the
population difference and we call the obtained solution as the Representative Nash
solution because it gives the two-person Nash solution when each side is represented by
the “toughest” player.

2. The axiomatic model

The problems we consider consist of the set of outcomes and two classes of players,
\( N^+ \) and \( N^- \), whose preferences are diametrically opposite over agreeable outcomes.
Thus agreeable outcomes can be represented by a linearly ordered set like \( \{ (w, \ 1 - w) \in \mathbb{R}, \ 0 \leq w \leq 1 \} \) and one class of players (\( N^+ \)) care for the first component while
the others (\( N^- \)) only care about the second component. So, preferences are expressed by
utility functions \( u^+(w) \) for \( N^+ \) players and \( u^-(1-w) \) for \( N^- \) players, which are increasing in their arguments.

Consequently a problem could be given by specifying \( N^+, \{u_i^+\}_{i \in N^+}, N^-, \) and \( \{u_i^-\}_{i \in N^-} \) (together with a disagreement outcome). However, we find it more convenient to represent agreement outcome by \([0, 1]\) which is identified with the first component of the above set, and to treat players’ preferences on this set is either strictly increasing (for \( N^+ \)) or strictly decreasing (for \( N^- \)). As a byproduct, we could distinguish two types of players implicitly through their preferences. This, in return, would occasionally cause some loss of straightforwardness in expression, and to complement, we reemploy notation like \( u^- \) for such cases.

Denote by \( U^+ \) the set of all von Neumann-Morgenstern utility functions \( u: \{0, 1\} \times \{d\} \rightarrow \mathbb{R} \), such that (i) \( u \) is non-negative, continuous, concave, strictly increasing on \([0, 1]\), and continuously differentiable on \((0, 1)\), and (ii) \( u(d) = 0 \). To save notation, let \( X = \{0, 1\} \times \{d\} \). Define \( U^- \) in the same way, except that functions in \( U^- \) are strictly decreasing on \([0, 1]\). A pair \((N, \{u_i\}_{i \in N}) = B \) is a bargaining problem, if the following conditions are satisfied:

\[
N \text{ is finite, and } N = N^+ \cup N^- \text{ such that } N^+, N^- \neq \emptyset \tag{1}
\]

\[
i \in N^+ \text{ iff } u_i \in U^+, i \in N^- \text{ iff } u_i \in U^- \tag{2}
\]

Let \( B \) denote the set of all bargaining problems, i.e., all pairs \( B = (N, \{u_i\}_{i \in N}) \) satisfying Eqs. (1) and (2). Note that the specification of a bargaining problem includes the information that the set of “real” outcomes is \( X \). The number and identity of players is allowed to vary across problems. A solution is a function \( f: B \rightarrow \Delta X \), where \( \Delta X \) denotes the set of all probability measures over \( X \).

The interpretation is the following. In a problem \( B \), there are \( n = |N| \) agents bargaining about which outcome \( w \in [0, 1] \) should be selected. If no unanimous agreement is reached, then the bargaining results in conflict, \( d \), which gives zero utility to each player. In principle, they could use a random device to select \( w \), and this mechanism (or contract) could also choose \( d \) with a positive probability. All kinds of contracts can be made binding at no cost. The solution \( f \) tells what kind of contract is written by the players in any particular bargaining situation.

There exists no \( w \in [0, 1] \) such that \( u_i(w) = 0 \ (= u_i(d)) \) for every \( i \in N \), since for \( i \in N^- \), \( u_i(w) = u_i(1-w) = 0 \) implies \( w = 1 \). We may then view the set \( X \) as a compact metric space, and so the expected utility functions \( u_i(p) = \int u_i(x) \, dp \), \( p \in \Delta X \) (with some abuse of notation) are continuous on \( \Delta X \) with respect to the topology of weak convergence (for details, see e.g. Dudley (1989)).

Given a problem \( B = (N, \{u_i\}_{i \in N}) \), there is a unique classical bargaining model \((S, c)\) corresponding to it, where \( S \subset \mathbb{R}^n \) is the set of feasible utility allocations among players in \( N \), and \( c = (0, \ldots, 0) = 0^* \), the origin of \( \mathbb{R}^n \). To see this, fix some ordering of the players in \( N \), and denote by \( u \) the (ordered) list \( \{u_i\} \). Then let \( S = \{ u(p) | p \in \Delta X \} \), and \( 0^* = c = u(d) \). Now \( S \) is compact and convex subset of the non-negative orthant of \( \mathbb{R}^n \) and \( S \) contains the origin, since \( X \) is compact and utilities are continuous von M-side utility functions.
2.1. The axioms

A contract \( p \in \Delta X \) is Pareto optimal in a problem \( B = (N, \{u_i\}_{i \in N}) \), if there is no alternative contract \( q \in \Delta X \) such that \( u_i(q) \geq u_i(p) \) for all \( i \in N \), and \( u_i(q) > u_i(p) \) for some \( i \in N \). If \( p \in \Delta X \) is Pareto optimal in a problem \( B = (N, \{u_i\}_{i \in N}) \), then there is a unique riskless alternative \( w^p \in [0, 1] \) such that \( u_i(p) = u_i(w^p) \) for all \( i \in N \). Namely, \( p([d]) = 0 \) if \( p \) is Pareto optimal, since \( u_i \) is strictly monotone on \([0, 1]\), and \( u_i(w) \neq 0 \) if \( u_i \) is concave and \( p \) is Pareto optimal. Now \( w^p \) is unique, since the players in \( N^+ \) and \( N^- \) have exactly opposite preferences over \( w \in [0, 1] \).

The set of efficient utility allocations in \( S \) is then given by \( P(S) = \{(u_i(w))_{i \in N} | 0 \leq w \leq 1 \} \), and we can write \( S = co(P(S), \emptyset) \), where \( co \) denotes the convex hull.

Let us now write down the first axiom any solution should satisfy.

**PARETO OPTIMALITY (PO)** \( f(B) \) is Pareto optimal, for all \( B \in \mathcal{B} \).

Next, the solution should be scale invariant: since vNM-utility functions are unique (representations of preferences over \( \Delta X \)) up to affine transformations only, \( u_i \) and \( a \cdot u_i + b_i \) represent the same preferences, when \( a, b \in \mathbb{R}, a > 0 \). Since the choice between equivalent utility representations is completely arbitrary, we don’t want the solution to depend on this choice. We have already fixed the zeros of utility functions, \( u_i(d) = 0 \), so we need to consider only scale transformations \( a \cdot u_i \). Given a problem \( B = (N, \{u_i\}_{i \in N}) \), and \( a_i > 0, i \in N \), denote by \( B^a = (N, \{a_i u_i\}_{i \in N}) \) the problem that is obtained from \( B \) just by applying scale transformations to utility functions.

**SCALE INvariance (SI)** \( f(B) = f(B^a) \), for all \( B \in \mathcal{B} \).

Two problems \( B \) and \( B' \) may be considered as equivalent also in some other situations. For example, let \( B = (N, \{u_i\}_{i \in N}) \) and \( B' = (M, \{v_i\}_{i \in M}) \) be such that \( |N^+| = |M^+| \) and \( |N^-| = |M^-| \). In particular, the number of players is equal to \( n = |N| \) in both problems. Suppose there is a bijection \( \pi : N \rightarrow M \) such that \( \pi(N^+) = M^+ \), \( \pi(N^-) = M^- \), \( u_i = v_{\pi(i)} \) for all \( i \in N \). So to any \( i \in N \), there is a player \( j \in M \) \( (j = \pi(i)) \) who has exactly the same preferences as \( i \). In this example, it is reasonable to assume that \( f(B) = f(B') \), if players’ names (the indices \( i \)) have no particular informational content. If names should play no role, there are many other bijections \( \pi \) we could consider (including a switch of + side and − side), and demand that the solution satisfies some kind of “anonymity” axiom. Here we will be content in formulating the following simple symmetry axiom.

**SYMMETRY (SY).** Suppose \( B = (N, \{u_i\}_{i \in N}) \) is such that \( |N^+| = |N^-| \), \( u_i(w) = w \) for all \( i \in N^+ \), and \( u_i(w) = 1 - w \) for all \( i \in N^- \). Then \( u_i(f(B)) = 0.5 \) for all \( i \in N \).

This form of symmetry is actually quite weak. It says only that utility levels at the solution should be equal, when utilities are linear on \([0, 1]\), and there are equally many
members in the groups $N^+$ and $N^-$. The contract $f(B)$ satisfying $SY$ could in principle be any probability measure $p$ on $[0, 1]$ such that $\int dx \, dp = 0.5$.

The following axiom is well known, but more controversial than the other axioms mentioned so far.

**INDEPENDENCE OF IRRELEVANT ALTERNATIVES (IIA).** Suppose $(S, 0^*)$ and $(T, 0^*)$ are the classical models representing $B = (N, \{u_i\}_{i \in N})$ and $B' = (M, \{v_i\}_{i \in M})$, respectively, with $n = |N| = |M|$. If $S \subset T$ and $v(f(B')) \in S$, then $u(f(B)) = u(f(B'))$.

It will be shown that in two-person problems, $PO$, $SI$, $SY$, and $IIA$ characterize a class of solutions $F$ in the following sense. Given any two-person problem $B = (N, \{u_i\}_{i \in N})$ and $f \in F$, $u(f(B))$ is the Nash solution of the classical problem $(S, 0^*)$ corresponding to $B$. Recall that the Nash solution of $(S, 0^*)$ is the utility vector $z \in \mathbb{R}^n$ that solves the problem $\max_{\Pi_i, x_i}$ s.t. $x_i \in S$. There is no difficulty in defining the Nash solution on $B$. Let us call a solution $F: \mathcal{B} \to \Delta x$, $f(B) \in [0, 1]$ the Nash solution on $B$, if for all $B = (N, \{u_i\}_{i \in N}), f(B)$ solves the problem $\max_{\Pi_i, u_i(y)}$ s.t. $y \in [0, 1]$. Note that in this definition, the Nash solution always selects a pure, non-random element from the interval $[0, 1]$.

It is well known that axioms $PO$, $SI$, $SY$, and $IIA$ uniquely characterize the Nash solution in the domain of $n$-person classical problems $(S, 0^*)$, when these axioms are properly adapted to this domain. So the result that we can discover the Nash solution in two-person cases may not come as a big surprise. It may be a bit more surprising that these axioms are not strong enough to characterize a solution uniquely on $B$ (uniquely in the sense that all solutions satisfying these axioms result in the same utility allocation). In particular, the $n$-person Nash solution is not uniquely characterized by these axioms.

We will here define a solution on $B$, satisfying the axioms $PO$, $SI$, $SY$, and $IIA$, whose utility allocation coincides with the Nash solution in two-person cases, but whose utility allocation in $n$-person games may be different. To do this, we need a bit more notation.

Given $u \in U^+$, let $Du(w)$ be the gradient of $u$ at $w \in (0, 1)$: $Du$ is decreasing and continuous. At $w = 1$ and $w = 0$, define $Du$ by taking appropriate limits. For functions $u$ in $U^-$, we let $Du(w)$, $w \in (0, 1)$, denote the absolute value of the gradient of $u$ at $w$, i.e. $-Du(1 - w)$. $Du$ is increasing and continuous. Again extend $Du$ to $w = 0$ and 1 by taking appropriate limits. (In this paper, we limit utility functions to be continuously differentiable functions mainly for the sake of an ease of exposition. Extension to the non-differentiable functions can be done easily by evaluating a set of super-gradients.)

Given a bargaining problem $B = (N, \{u_i\}_{i \in N})$ and any player $i \in N$, let $b_i(x) = (\max Du_i(x))/u_i(x)$, whenever the right hand side is a real number, and otherwise set $b_i(x) = \infty$. Then $b_i(x)$ measures the boldness of player $i$ at $x$ (see Roth, 1989). If $i \in N^+$, then $b_i$ is a strictly decreasing continuous function (except that $b_i(0)$ could be infinite). If $i \in N^-$, then $b_i$ is a strictly increasing continuous function (except that $b_i(1)$ could be infinite).

For all $x \in [0, 1]$, denote $b^+(x) = \sup\{b_i(x) | i \in N^+\}$ and $b^-(x) = \sup\{b_i(x) | i \in N^-\}$. Then $b^+$ is strictly decreasing and continuous, and $b^-$ is strictly increasing and continuous (with the provisions at 0 and 1 as mentioned above).

We say that player $i$ is bolder than player $j$ at $x$, if $b_i(x) > b_j(x)$. Player $i$ is bolder than $j$, if $b_i(x) > b_j(x)$ for all $x \in [0, 1]$ (note that $b_i = b_j$ is allowed). Sometimes we may also
say that utility function \( u \) is bolder than \( v \), if we don’t want to talk about specific players.

We are now ready to define a new solution, the “representative Nash solution” \( G \).

**Definition 1.** \( G \) is a solution such that: (i) \( G(B) \in [0, 1] \) for all problems \( B \); (ii) \( G(B) = 1 \), if \( b^+(x) > b^-(x) \) for all \( x < 1 \); (iii) \( G(B) = 0 \), if \( b^+(x) < b^-(x) \) for all \( x > 0 \); (iv) \( 0 < G(B) < 1 \), if \( b^+(G(B)) = b^-(G(B)) \).

It is easy to check that \( G \) is a well-defined solution. In two-person problems, the solution \( G \) and the Nash solution \( F \) coincide. This is shown in the lemma below, where we also give an alternative definition of \( G \) in terms of Nash solutions of two-person games. Given a problem \( B = (N, \{u_i\}_{i \in N}) \), and \( i \in N^+ \), \( j \in N^- \), let \( B(i, j) = (\{i, j\}, \{u_i, u_j\}) \). So \( B(i, j) \) is a two-person problem played by \( i \) and \( j \), who also were players in \( B \).

**Lemma 0.** Given a problem \( B = (N, \{u_i\}_{i \in N}) \), \( G(B) = \max_{i \in N^+} \min_{j \in N^-} F(B(i, j)) = \min_{i \in N^-} \max_{j \in N^+} F(B(i, j)) \), and \( F(B(i, j)) = G(B(i, j)) \).

**Proof.** Let us first show that \( G(B(i, j)) = F(B(i, j)) \). Roth (1979, p. 50) shows that this is true in the case (iv). It is straightforward to verify the claim also in cases (ii) and (iii).

Now let \( z = G(B) \). Choose \( i \in N^+ \), \( j \in N^- \) such that \( b^+(z) = b_i(z) \) and \( b^-(z) = b_j(z) \). Then by definition, \( z = G(B(i, j)) \). Further, \( z = z(i, j) = F(B(i, j)) \) by the previous paragraph. Recall that \( b_h \) is strictly decreasing for \( h \in N^+ \) and strictly increasing for \( h \in N^- \). Then by definition of \( G \), \( z(i, j) \leq z(i, k) \) for all \( k \in N^- \), and \( z(h, j) \leq z(i, j) \) for all \( h \in N^+ \). This completes the proof. \( \Box \)

The \textit{minmax}-characterization of \( G \) in terms of two-person Nash solutions is one reason we call \( G \) the “representative Nash solution”. Suppose in a problem \( B \) it is not practical for all players in \( N^+ \) (in \( N^- \)) to participate in the bargaining process. Instead, each party selects a single representative, and these representatives then bargain. Suppose further that the result of these two-person bargaining situations will always be the two-person Nash solution (this is approximately the case, if the representatives are engaged in the Rubinstein alternating offers game and the discount factors are very close to one). Then prior to the bargaining, the parties \( N^+ \) and \( N^- \) anticipate the bargaining outcome for each pair of representatives \( (i, j) \), namely \( F(B(i, j)) = z(i, j) \). The whole situation then looks like a two-person normal form game, in which strategy sets are \( N^+ \) for player 1 and \( N^- \) for player 2. The utility of player 1 could be any function \( u \) that is the restriction of \( u^* \) to \( \{z(i,j)\}_{i,j} \), where \( u^* \) in turn is some concave strictly increasing function on \([0, 1] \). Similarly for player 2, except that her function must be strictly decreasing. Otherwise the specification of utility functions is not crucial, since the \textit{minmax}-characterization of \( G \) implies that the equilibrium outcome of every Nash equilibrium must be \( G(B) \). In particular, if there are players \( i \) and \( j \) in each side who have all the power to nominate their representatives (and players’ don’t get any (dis)utility from being elected as representatives), then their choices will lead to \( G(B) \). Of course, things get more complicated with more possible outcomes, if the representa-
tives are elected by applying some kind of voting method. Then whether or not $G(B)$ occurs, depends on the ability of each side to solve coordination problems.

Given a problem $B = (N, \{u_i\}_{i \in N})$, we say that $u \in U^+ \cup U^-$ solidly represents players in $A \subseteq N$, if $u_i = u$ for all $i \in A$. Consider problems $B = (N, \{u_i\}_{i \in N})$ and $B' = (M, \{v_i\}_{i \in M})$. Then we say that players in $A \subseteq N$ and $A' \subseteq M$ are the same, if there is a bijection $\pi$: $A \rightarrow A'$ such that $u_i = v_{\pi(i)}$ for all $i \in A$. Obviously $A \subseteq N^+$ iff $A' \subseteq M^+$.

**Definition 2.** Given a solution $f$, and utility functions $u$, $u' \in U^+ \cup U^-$, $u$ is tougher than $u'$ against $\{v_i\}_{i \in A} \subseteq U^+ \cup U^-$, if $u(f(B)) \geq f(B')$ and $u'(f(B')) \geq u'(f(B'))$, where bargaining problems $B = (N, \{u_i\}_{i \in N})$ and $B' = (M, \{u'_i\}_{i \in M})$ are such that

1. $u$ and $u'$ solidly represent players in $N^+$ and $M^+$ (or $N^-$ and $M^-$), respectively, and \(|N'| = |M'| = |N^+| = |M^+|\);  
2. players in $N^-$ and $M^-$ (or $N^+$ and $M^+$) are the same, and $\{v_i\}_{i \in A} = \{u_i\}_{i \in N}$ (or $\{v_i\}_{i \in A} = \{u_i\}_{i \in M^+}$).

The contract selected by $f$ is $f(B')$ in the problem $B'$, although all players $i \in M^+$, say, at least weakly prefer $f(B)$ over $f(B')$. In the problem $B$, all players $i \in N^+$ also weakly prefer $f(B)$ over $f(B')$, but, unlike in the problem $B'$, they also get the preferred contract $f(B)$ instead of $f(B')$. Since players in $N^-$ and $M^-$ are the same, it makes sense to say that “$u$ is tougher than $v$” (the definition allows $u = v$). On the other hand, players in $N^+$ and $M^+$ are required to be solidly represented by $u$ and $v$, since otherwise there could exist some very “tough” player in $N^+$, not having the utility function $u$, but whose presence could explain why $f(B)$ and not $f(B')$ is chosen as the contract in the game $B$. Further, there are equal number of players in $N^-$ and $M^-$. This allows us to separate the effects that individual preferences on one hand and population size on the other hand have on the bargaining power of a party.

Consider now the following axiom.

**TOUGH PLAYER PRINCIPLE (TPP).** Suppose problems $B = (N, \{u_i\}_{i \in N})$ and $B' = (M, \{v_i\}_{i \in M})$ are such that

1. for some $i, j \in N^+$ (or $i, j \in N^-$), $u_i$ is tougher than $u_j$ against $\{u_k\}_{k \in N}$ (or against $\{u_k\}_{k \in N^+}$), given the solution $f$;
2. $\{u_i\}_{i \in M^+} = \{u_i\}_{i \in N^+} \cup \{u_j\}$ (or $\{u_i\}_{i \in M^-} = \{u_i\}_{i \in N^-} \cup \{u_j\}$);
3. $|N^+| = |M^+| = |N^-| = |M^-|$, and players in $N^-$ and $M^-$ (or in $N^+$ and $M^+$) are the same. Then $f(B) = f(B')$.

All preferences represented in the problem $B$ are also represented in the problem $B'$, except $u_j$. However, since $u_i$ is tougher than $u_j$, and $u_i$-preferences are represented in the problem $B'$, the solution doesn’t change. So the opponents of $i$ in the problem $B$ are not able to exploit to their advantage the weakness of player $j$, who in $B$ is one of player $i$’s partners.
There are many solutions that satisfy all the axioms mentioned so far. Some of them are discussed in Section 4. The reason for this multiplicity is that the number of players on each side could have a tremendous impact on the solution outcome. The axioms don’t say how the solution should change, if one player is added in the game. The following axiom says something about the circumstances under which such a population change should have no impact on the solution.

**POPULATION INVARIANCE (PI).** Suppose problems $B = (N, \{u_i\}_{i \in N})$ and $B' = (M, \{u_i\}_{i \in M})$ are such that

1. $u$ solidly represent players in $N^+$ and $M^+$ (or $N^-$ and $M^-$), respectively;
2. $|M^+| = |N^+| + 1$ (or $|M^-| = |N^-| + 1$);
3. players in $N^-$ and $M^-$ (or in $N^+$ and $M^+$) are the same.

Then $f(B) = f(B')$.

The only difference between games $B$ and $B'$ is that there is one player more in $M^+$ than in $N^+$. This “new” player has exactly the same preferences than the “old” players in $N^+$. PI then dictates that adding such a player should have no effect on the solution. It is clearly conceivable that in some problems this kind of increase in the population $N^+$ would also increase their share of the cake, so we are not claiming that PI is the only reasonable way to incorporate variations in the population sizes in the model.

To motivate the situation in question under this axiom, think of the case where a two-sided bargaining takes place within a committee which has a right to determine issues valid to an entire organization including non-members of the committee. Then the axiom asks whether an addition of a new member to the committee having the same preference as the one of the existing members affects the bargained outcome.

The PI axiom could be compared with the Population Monotonicity axiom (see Thomson (1997) for a good discussion). The latter axiom is designed for classical multi-person bargaining problems with variable population. This axiom says roughly the following: if a new player is added into the problem but total resources are otherwise kept constant, then (at the solution) all the old players should be made (weakly) worse off. In our framework the intuition behind this axiom is difficult to formalize: it is not possible that (assuming Pareto optimality) adding one player would make none of the old players better off, and at least one of these players would become strictly worse off. So the only possibilities are that the well-being of the old players doesn’t change at all (the PI axiom), or that $N^+$ players loose and $N^-$ gain, or vice versa.

We shall give a characterization of the solution $G$ by means of the axioms listed so far in the next section. To see the structure of the proof, it may be convenient to give another way of interpreting the solution $G$ in terms of a surrogate utility function. A surrogate utility function $V^+$ for $N^+$ is a function so that $DV^+(w)/V^+(w) = b^+(w)$ for each $w$. One can construct such upon integrating $b^+$ and then taking its anti-logarithm. Similarly, $V^-$ for $N^-$ can be picked. Then the solution $G$ is the Nash bargaining solution for a two person problem between $V^+$ and $V^-$. Based on these definitions, we can
roughly state our result as follows. With a standard set of axioms, one obtains the Nash bargaining solution for two-person problems. By PI axiom, every solidly represented problems shares the solution with a two-person problem. Therefore, for any problem, TPP implies each side is represented by the surrogate utility functions defined above.

More generally, one could view axioms PI and TPP as a device to reduce multi-person problems into two-person problems, and so if different set of axioms yielding different solutions for two-person problems, then different representative solutions based on appropriate surrogate utility functions would be obtained.

2.2. The result

We are now ready to prove the main result of this section. We start by the following.

Lemma 1. Suppose f is any solution satisfying PO, SI, SY, and IIA. If \( B = (N, \{u_i\}_{i \in N}) \in B \) is a two-person problem, then \( u_i(f(B)) = u_i(G(B)) \), for all \( i \in N \). If \( (S, 0^+) \) is the classical bargaining problem corresponding to \( B \), then the utility of player \( i \) at the Nash solution of \( (S, 0^+) \) is \( u_i(G(B)) \), for all \( i \in N \).

Proof. Even though we adopt a slightly different formulation of the problems, with a restriction on the class (i.e. smoothness of utility functions, 0-normalization of disagreement payoffs, and the set of contracts being limited to those obtained from convex combinations of disagreement payoffs and Pareto optimal payoffs), essentially the classical proof remains valid, and so the proof is omitted.

In our proof of the theorem below, we show how this two person result is extended, especially with two axioms TPP and PI. Next lemma establishes one key relationship for that purpose.

Lemma 2. Suppose \( B = (N, \{u_i\}_{i \in N}) \) and \( B' = (M, \{v_i\}_{i \in M}) \) are such that \( u \) and \( v \) solidly represent the players in \( N^+ \) and \( M^+ \), respectively, \( |N^+| = |M^+| \), and players in \( N^- \) and \( M^- \) are the same. Given the solution \( G \), \( u \) is tougher than \( v \) against \( \{u_i\}_{i \in N^-} \), iff \( u \) is bolder than \( v \) at \( G(B) \).

Proof. If \( u \) is tougher than \( v \) against \( \{u_i\}_{i \in N^-} \), then \( G(B) \succeq G(B') \). By the definition of \( G \), \( u \) must then be bolder than \( v \) at \( G(B) \). Conversely, suppose \( u \) is bolder than \( v \) at \( G(B) \). By the definition of \( G(B) \), it must be true that \( G(B) \succeq G(B') \). Since \( u, v \in U^+ \), we have \( u(G(B)) \succeq u(G(B')) \) and \( v(G(B)) \succeq v(G(B')) \), which implies that \( u \) is tougher than \( v \) against \( \{u_i\}_{i \in N^-} \), given \( G \).

Lemma 2 has an obvious dual for functions \( u, v \in U^- \), which is not stated explicitly here. Next lemma establishes another half of the theorem.

Lemma 3. \( G \) has the properties PO, SI, SY, IIA, PI, and TPP.
Proof. It is easy to verify that $G$ has the properties $PO$, $SI$, $SY$, and $PI$. Let us show that $G$ satisfies $IIA$. Suppose $(S, 0^s)$ and $(T, 0^t)$ are the classical models representing $B = (N, \{u_i\}_{i \in N})$ and $B' = (M, \{v_j\}_{j \in M})$, respectively, with $n = |N| = |M|$, and $|N^+| = |M^+|$. Suppose $T \subseteq S$ and $u(G(B)) \in T$. We have to show that $u(G(B')) = v(G(B))$. We may set w.l.o.g. $N^+ = \{1, \ldots, k\} = M^+$, $N^- = \{k + 1, \ldots, n\} = M^-$. By the definition of the Nash solution we must then have that in the problem $(\text{the classical bargaining games corresponding to problems } v_i, v_j)$, where $B(i, j)$ is the two-person game between $i, j \in N$, and $F(B(i, j)) \subseteq [0, 1]$ denotes the contract selected by the Nash solution. If $i^* \in N^+$ is the boldest in $N^+$ at $G(B)$, and $j^* \in N^-$ is the boldest in $N^-$ at $G(B)$, then $G(B) = F(B(i^*, j^*)) = G(B(i^*, j^*))$ by Lemma 0. Similarly $G(B') = \max_{i \in N^+} \min_{j \in N^-} F(B'(i, j))$, where $B'(i, j)$ is the two-person game between $i$ and $j$ having utility functions $v_i$ and $v_j$. The solution $G$ picks always a deterministic Pareto optimal contract, and the Pareto set of $B$ and $B'$ are $\{u(z)|0 \leq z \leq 1\}$ and $\{v(z)|0 \leq z \leq 1\}$.

Suppose $u_i(G(B')) > u_i(G(B))$ for some $i \in N^+$. Then since $u_i(G(B))$ is Pareto optimal in $B'$, $v_i(G(B')) > v_i(G(B))$ for all $i \in N^+$ and $v_i(G(B')) = v_i(1 - G(B')) < u_i(G(B)) = u_i'(1 - G(B'))$ for all $i \in N^+$. The maxmin-property of $G$ implies that in game $B$ there exists $j \in N^-$ such that $F(B(i, j)) \geq G(B)$ for all $i \in N^+$, and the equality holds for some $i$. Hence $u_i(F(B(i, j)) = u_i(G(B))$ for all $i \in N^+$. We may assume that $j = n$. Similarly, the maxmin-property of $G$ implies that there exists $i \in N^+$ such that $G(B') = F(B'(i, j))$ for all $j \in N^-$, and the equality holds for some $j$. Hence $v_i(G(B')) = v_j'(1 - G(B')) \geq v_j(F(B'(i, j))) = u_j'(1 - F(B'(i, j)))$ for all $j \in N^-$. We may assume $i = 1$.

Now consider the games $B(1, n)$ and $B'(1, n)$ between players $1$ and $n$. We have $u_1(F(B(1, n))) \leq u_i(G(B)) < v_i(G(B')) \leq v_i(F(B'(1, n)))$ and $u_i(F(B(1, n))) \geq u_i(G(B)) > v_i(G(B')) \geq v_i(F(B'(1, n)))$. Let $A$ and $A'$ denote the sets of feasible utility allocations in the classical bargaining games corresponding to problems $B(1, n)$ and $B'(1, n)$, respectively. It is easy to verify that $T \subseteq S$ implies that $A = \text{proj}_{x_n} S$ and $A' = \text{proj}_{x_n} T$, and therefore $A' \subseteq A$ (\text{proj}_{x_i} maps each vector $(x_1, \ldots, x_n)$ to the vector $(x_i, x_n)$). Now $(u_1(G(B)), u_n(G(B)))$ is Pareto optimal in $A$ and $A'$. Since $u_1(G(B)) < v_1(F(B'(1, n)))$, and the right-hand side of this equality is the utility level of player 1 at the Nash solution of the problem $B'(1, n)$, we must have that $u_1(G(B))u_n(G(B)) < v_1(F(B'(1, n)))v_n(F(B'(1, n)))$. By the definition of the Nash solution we must then have that $u_i(F(B(1, n)))u_j(F(B(1, n))) < v_i(F(B'(1, n)))v_j(F(B'(1, n)))$. But then the definition of the Nash solution gives a contradiction since $A' \subseteq A$ and $(v_i(F(B'(1, n))), v_j(F(B'(1, n)))) \in A'$ and $(u_1(F(B(1, n))), u_n(F(B(1, n)))) \in A$. This means that the solution $G$ has the property $IIA$.

Let then $B = (N, \{u_i\}_{i \in N})$ and $B' = (M, \{v_j\}_{j \in M})$ be any problems satisfying the requirements of $TPP$, and assume w.l.o.g. $u_i$ is tougher than $u_j$ for $i, j \in N^+$, given $G$. Let $k$ be the boldest player in $N^+$ at $G(B)$. Let $B(i)$, $B(j)$, and $B(k)$ be problems otherwise identical to $B$, except that $u_i(u_j, u_k)$ solidly represents players in $N^+$. Then by Lemma 1, $u_k$ is tougher than $u_i$ and $u_j$. This implies that some player in $M^+$ has the utility function $u_k$, and therefore $k$ is the boldest player in $M^+$ at $G(B)$. Since players in $N^+$ and $M^-$ are the same, player $h$ is the boldest in $N^-$ at $G(B)$, if and only if $h$ is the boldest in $M^-$ at $G(B)$. By the definition of $G$, we must have $G(B) = G(B')$, and therefore $G$ has the property $TPP$. □

The main result of this section is the following.
Theorem 1. $G$ has the properties PO, SI, SY, IIA, PI, and TPP. If $f$ is a solution on $B$ having these properties, then $u_i(f(B)) = u_i(G(B))$, for all $B = (N, \{u_i\}_{i \in N})$, $i \in N$.

Proof. By Lemma 3, $G$ has all the desired properties, and by Lemma 1, the Theorem holds for two-person games. Let’s divide the proof in two steps: in Step 1 we show that the theorem is valid if either $|N^+| = 1$, or $|N^-| = 1$. Step 2 contains the proof of the general case.

**Step 1.** $u_i(f(B)) = u_i(G(B))$, for all $B = (N, \{u_i\}_{i \in N})$, if $|N^+| = 1$ or $|N^-| = 1$.

Proof. Suppose $B$ is such that $|N^-| = 1$, and let $f$ be any solution having the desired properties. We may w.l.o.g. set $N^+ = \{1, \ldots, n\}$, and $N^- = \{n + 1\}$. Given $i \in N^+$, denote by $B(i)$ the problem which is otherwise the same as $B$, except that the players in $N^+$ are solidly represented by $u_i$, and denote by $w(i)$ the certain outcome in $[0,1]$ such that $u_i(w(i)) = u_i(f(B(i)))$. We may w.l.o.g. assume that $w(1) \geq w(2) \geq \cdots \geq w(n)$. This means that $u_i$ is tougher than $u_j$, against $\{u_{n+1}\}_{i \neq n+1}$, given $f$, when $1 \leq i \leq j \leq n$.

Applying TPP on players in $N^+$ implies that $f(B(1)) = f(B)$. Let $A(i) = (\{1, n + 1\}, \{u_i, u_{n+1}\})$ for $i = 1, \ldots, n$. Applying PI inductively gives us that $f(A(i)) = f(B(i))$, for $i = 1, \ldots, n$, and in particular $f(A(1)) = f(B) = f(B(1))$. By Lemma 1, $u_i(f(A(i))) = u_i(G(A(i)))$, $i = 1, \ldots, n$. Since $G$ has all the desired properties, we have $G(B) = G(B(1)) = G(A(1)) = w_i$, and therefore $u_i(f(B)) = u_i(G(B))$, $i = 1, \ldots, n$. The case $|N^-| = 1$ is analogous.

**Step 2.** $u_i(f(B)) = u_i(G(B))$, for all $B = (N, \{u_i\}_{i \in N})$.

Proof. Pick a problem $B = (N, \{u_i\}_{i \in N})$ arbitrarily, and let $f$ be any solution having the desired properties. We may w.l.o.g. set $N^+ = \{1, \ldots, n\}$, and $N^- = \{n + 1, \ldots, n + m\}$. Given $i \in N^+$, denote by $B(i)$ the problem which is otherwise the same as $B$, except that the players in $N^+$ are solidly represented by $u_i$, and denote by $w(i)$ the certain outcome in $[0,1]$ such that $u_i(w(i)) = u_i(f(B(i)))$. We may w.l.o.g. assume that $w(1) \geq w(2) \geq \cdots \geq w(n)$. This means that $u_i$ is tougher than $u_j$, against $\{u_{n+1}\}_{i \neq n+1}$, given $f$, when $1 \leq i \leq j \leq n$. In the same fashion we may define $B(j)$ and $w(j)$ for $j \in N^-$, and we may say that $u_i$ is tougher than $u_j$, against $\{u_{n+1}\}_{i \neq n+1}$, given $f$, when $n + 1 \leq i \leq j \leq n + m$.

Applying TPP on players in $N^+$ implies that $f(B(1)) = f(B)$, and applying TPP on players in $N^-$ implies that $f(B) = f(B(n + 1))$. Let $B' = (M, \{u_i\}_{i \in M})$ be a problem such that $M^+ = \{1\}$, and players in $M^-$ are the same as in $N^+$. Let $B' = (P, \{u_i\}_{i \in P})$ be a problem such that $P^- = \{n + 1\}$, and players in $P^+$ are the same as in $N^-$. Applying PI inductively implies that $f(B') = f(B(1))$, and $f(B') = f(B(n + 1))$. Therefore $f(B') = f(B) = f(B')$.

Consider the two-person problem $A = (\{1, n + 1\}, \{u_i, u_{n+1}\})$ played by $1$ and $n + 1$. By Step 1, $f(A) = f(B') = f(B')$, and $u_i(f(B')) = u_i(f(B')) = u_i(f(A)) = u_i(G(A)) = u_i(G(B')) = u_i(G(B'))$. Since $G$ satisfies all the assumptions, $G(B') = G(B') = G(B)$, and therefore $u_i(f(B)) = u_i(G(B))$. □
3. The noncooperative model

Let $B = (N, \{u_i\}_{i \in N})$ be a bargaining problem as defined in the previous section, and assume w.l.o.g. $N = \{1, \ldots, n\}$. Consider the following noncooperative game based on $B$. There are infinitely many periods $t \in \mathbb{N}$, and then players 2, 3, \ldots, $n$ (in that order) reply by accepting or rejecting the offer. If all players accept, then the agreement is $w$ and the game is over. If player $i \geq 2$ rejects the offer $w$, the process moves to period 1, player $i$ makes an offer $w'$, and players $i + 1 \mod n$, $n, 1, \ldots, i - 1$ (in that order) reply by accepting or rejecting the offer. If all players accept, then $w'$ becomes the agreement and the game ends. Otherwise, if player $j$ rejects $w'$, then the process moves to period 2, in which player $j$ makes an offer, and players $j + 1, \ldots, j - 1$ respond, and so on. When player $i$ has to make a move, he knows all the previous choices, so information is perfect. We suppose that the time interval between any periods $t$ and $t + 1$ is $T > 0$, and that between each period $t$ and $t + 1$, the opportunity to bargain disappears with probability $1 - (1 - p)^T$, $0 < p < 1$. If the game ends without agreement, each player receives 0 utility. This is also the utility level if the game lasts infinitely many periods but the players cannot reach an agreement (of course, this event has zero probability). An agreement $w$ in any period gives $u_i(w)$ for player $i$. Let us denote this game by $G^T$.

We analyze the stationary subgame perfect equilibria (s.s.p.e.) of this game, and in particular the s.s.p.e. outcomes, as the length of time interval $T$ goes to zero. Stationarity means that player $i$ always makes the same offer, and always accepts (rejects) the same offers, no matter what has happened in the past. This game is an extension of the two-person bargaining games considered by Binmore (1987) and Rubinstein (1982), and in particular the one analyzed by Binmore et al. (1986). The $n$-person games examined by Shaked (in Sutton (1986)) and Herrero (1984) are also of this type (but for the order by which a counter proposal is made). The main result of this section is the following.

**Theorem 2.** For any $T > 0$, the game $G^T$ has a unique s.s.p.e. outcome, in which the agreement is reached immediately. As $T$ tends to 0, the equilibrium offer converges to the representative Nash solution $G(B)$.

**Remark 1.** While the s.s.p.e. outcome is unique, the equilibrium strategies are not. The reason is that there is no time cost between moves within a given period. Uniqueness could be restored by adding such a cost.

The basic logic behind the result is very simple. Suppose for simplicity that in each set $N^+$ and $N^-$ there is a unique player who is the toughest in the sense that whenever another player on the same side prefers to reject an offer, this player also wants to reject but not vice versa. Then these “toughest” players essentially have the veto power and the preferences of other players become irrelevant. Each player makes his offer considering the toughest player’s preference. Further, since the right to make the next offer is given to the player who first rejects, the game is more or less transformed into a two-player game where the players are the toughest players on each side. However, the identity of the toughest player may depend on both $T$ and $w$, and therefore we have to
construct a proof. In the proof, we utilize the existing results on two person games, and so the proof is divided into four steps. In Step 1, it is shown that in s.s.p.e. the agreement is reached immediately. Based on the Step 1, we note that the equivalence to the two-person game among “representative players”, in Step 2, the uniqueness, and the existence of the s.s.p.e. outcome is obtained from the known results on two-person games. Existence of an s.s.p.e. is shown in Step 3, while for the convergence result, we need a separate proof which is carried out in Step 4.

Step 1 (Immediate agreement).

Proof. Suppose there exists an s.s.p.e., and choose arbitrarily such an equilibrium. Let \( v_i \) be the equilibrium payoff of player \( i \) in the subgame starting from \( i \)’s offer. If there is \( w \in [0, 1] \) such that \( u_i(w) = (1 - p)^i v_j \), then denote it by \( w_i^j \). Otherwise, set \( w_i^j = 0 \) for \( i \in N^+ \) and \( w_i^j = 1 \) for \( i \in N^- \). Then let \( w_i^+ = \max\{w_i^j | i \in N^+ \} \) and \( w_i^- = \min\{w_i^j | i \in N^- \} \).

If all the subgames starting from any player \( i \)’s offer result in a delay, then perpetual disagreement results, which cannot be an equilibrium. Thus there must be player \( i \) whose offer is accepted in the corresponding subgame. Suppose w.l.o.g. that this player \( i \) belongs to \( N^+ \). For player \( i \)’s offer to be accepted, it must be at most \( w_i^- \), whereas it cannot be lower than \( w_i^+ \) since it is an equilibrium offer. Thus player \( i \) offers \( w_i^- \) which is accepted by the other players. In particular, this implies that \( u_i(w_i^-) \geq (1 - p)\sum_{j} v_j \) for all \( j \in N^+ \), and therefore \( w_i^+ = w_i^- \). Note that the offer \( w_i^- \) must be the equilibrium offer for any player \( j \in N^+ \) whose offer is accepted in this s.s.p.e. In the same fashion it can be concluded that if \( i \in N^- \) makes an equilibrium offer that is accepted, then this offer must be \( w_i^- \).

Let us show next that no delay will take place in any subgame in which some player \( i \in N^+ \) makes an offer. Suppose to the contrary that there is \( j \in N^+ \) such that the subgame starting with \( j \)’s offer ends up with a delay. Then \( v_j \) is at most \( (1 - p)\sum_{j} v_j \) for any \( i \in N^+ \), and \( w_i^+ < w_i^- \). Thus there is an offer slightly below \( w_i^- \) (since \( w_i^- > 0 \) in this case) which is accepted by all other players and which makes player \( j \) better off, a contradiction.

To complete the argument, it suffices to show that there is player \( j \in N^+ \) whose offer is accepted in equilibrium (then, as in the previous paragraph, it follows that equilibrium play involves no delay in any subgame). If no such player \( j \in N \) exists, then an agreement is reached only with an offer \( w_i^- \). Therefore \( v_j \leq (1 - p)^i u_j(w_i^-) = (1 - p)^i (1 - w_i^-) \), and so \( u_j(w_i^-) = u_j(1 - w_i^-) > v_j \) for each \( j \in N^- \). Thus by making an offer slightly below \( w_i^- \) (which is accepted by all players), player \( j \)’s payoff can be increased. Hence a delay is not possible.

Step 2 (Uniqueness of s.s.p.e offers).

Proof. For each \( i \in N^+ \), define a function \( W_i(w) = \arg\min\{u_i(w') | u_i(w') \geq (1 - p)^i u_j(w)\} \), and let \( W_i^+(w) = \max\{W_i(w) | i \in N^+ \} \), \( 0 \leq w \leq 1 \). For each player \( i \in N^- \), let \( W_i(w) = \arg\max\{u_i(w') | u_i(w') = u_i(1 - w') \geq (1 - p)^i u_j(w) = (1 - p)^i u_i(1 - w)\} \), and
let \( W^{-T}(w) = \min\{W^T_i(w) | i \in N^-\} \), \( 0 \leq w \leq 1 \). Note that \( W^{+T} \) and \( W^{-T} \) are nondecreasing functions, and \( w - W^{+T} \) and \( W^{-T} - w \) are strictly increasing. The argument in Step 1 indicates that in any s.s.s.e. the offers \( w^{+T} \) and \( w^{-T} \) must satisfy \( W^{+T}(w^{-T}) = w^{+T} \) and \( W^{-T}(w^{+T}) = w^{-T} \). This means that the s.s.s.e. coincides with that of the two-person bargaining game between “representative players” with \( W^{+T} \) and \( W^{-T} \). Although, \( W^{+T} \) and \( W^{-T} \) may not be derived from the class of preferences in this paper, the theorem in Rubinstein (1982) applies to such players (with an obvious modification for the restriction on the set of agreeable outcomes and for the game with risk preferences and a chance of breakdown). In particular, we conclude that equilibrium offers \( w^{+T} \) and \( w^{-T} \) are unique.

**Step 3 (Existence).**

**Proof.** Consider the following strategies. All players \( i \in N^+ (i \in N^-) \) offer \( w^{-T}(w^{+T}) \), whenever \( i \) has to make an offer. Player \( i \in N^+ (i \in N^-) \) accepts an offer \( w \), if and only if \( w \geq W^T_i(w^{-T}) \) (\( w \leq W^T_i(w^{+T}) \)). Clearly, these strategies form an s.s.s.e.

**Step 4 (Convergence).**

**Proof.** We need to show that \( w^{-T} \) and \( w^{+T} \) converge to \( G(B) \), as \( T \) goes to zero. Assume \( T > 0 \) is so close to zero that \( 0 < W^{+T}(1) \), and \( W^{-T}(0) < 1 \). Let \( M^T = \max\{1 - W^{+T}(1), W^{-T}(0)\} \). Then \( |w^{+T} - w^{-T}| < 2M^T \), because \( w - W^{+T}(w) \leq 1 - W^{+T}(1) \) and \( W^{-T}(w) - w \leq W^{-T}(0) \). Now \( M^T \) goes to zero as \( T \) goes to zero, and therefore \( |w^{+T} - w^{-T}| \) goes to zero.

Consider the function \( X^T_i(w) = (1 - (1 - p)^T)/(|w - W^T_i(w)|) \), \( i \in N \), \( 0 < w < 1 \). Recall that for \( i \in N^+ \), \( W^T_i(w) \) is the least number \( w^* \) satisfying \( u_i(w^*) \geq (1 - p)^T u_i(w) \). Given \( w > 0 \), \( u_i(W^T_i(w)) = (1 - p)^T u_i(w) \) for small values of \( T \). So given \( w > 0 \), and \( T \) sufficiently small, we have \( X^T_i(w) = (u_i(w) - u_i(W^T_i(w))) / u_i(w)(w - W^T_i(w)) \) for \( i \in N^+ \). Obviously \( w - W^T_i(w) \) converges to zero, as \( T \) tends to zero. Therefore, given \( w > 0 \), \( i \in N^+ \), \( X^T_i(w) \) converges to \( b^*_i(w) = Du_i(w) / u_i(w) \). In the same fashion it can be concluded that given \( w < 1 \), and \( i \in N^- \), \( X^{T}_i(w) \) converges to \( b^*_i(w) \) as \( T \) goes to zero.

Given \( 0 < w < 1 \), let \( X^{+T}(w) = (1 - (1 - p)^T)/(|w - W^{+T}(w)|) \) and \( X^{-T}(w) = (1 - (1 - p)^T)/(|w - W^{-T}(w)|) \). As \( T \) tends to zero \( X^{+T}(w) \) converges to \( b^*_i(w) = \max\{b_i(w) | i \in N^+ \} \), and \( X^{-T}(w) \) converges to \( b^*_i(w) = \max\{b_i(w) | i \in N^- \} \). The s.s.s.e. offers satisfy \( 0 < w^{+T} < w^{-T} < 1 \), and \( W^{+T}(w^{-T}) = w^{+T} \). \( W^{+T}(w^{+T}) = w^{-T} \). Note also that for fixed \( T \), \( X^{+T}(w) \) is strictly decreasing in \( w \), while \( X^{-T}(w) \) is strictly increasing in \( w \).

Denote \( G(B) = w^* \), and suppose first that \( 0 < w^* < 1 \). By definition, \( w^* \) satisfies \( b^*_i(w) > b^*_i(w^*) > b^*_i(w) \) for \( w < w^*\), and \( b^*_i(w) < b^*_i(w) < b^*_i(w^*) \) for \( w > w^* \). Fix \( x, y \) such that \( 0 < x < w^* < y < 1 \). By the pointwise convergence result above, there exists \( T' > 0 \) such that \( T < T' \) implies \( X^{-T}(x) > X^{+T}(w) > X^{-T}(x) \), and \( X^{+T}(y) < X^{-T}(w^*) < X^{+T}(y) \). On the other hand, \( X^{+T}(w^*) > X^{+T}(w^{-T}) = X^{+T}(w^{+T}) < X^{-T}(w^{-T}) \). This implies that \( w^{+T} < w^* < w^{-T} \) for \( T < T' \). Since \( |w^{+T} - w^{-T}| \) converges to zero as \( T \) goes to zero, it is true that both \( w^{+T} \) and \( w^{-T} \) converge to the solution \( G(B) = w^* \).
Suppose then that \( w^* = 0 \). Then by definition of \( G \), \( b^+(w) < b^+(0) \leq b^-(0) < b^-(w) \) for all \( w > 0 \). Given that \( 0 < y < 0.5 \), there exists \( T' > 0 \) such that \( T < T' \) implies \( X^+(y) < b^+(0) \leq b^-(0) < X^-(y) \), and \( |w^{+T} - w^{-T}| < y \). If \( w^{-T} \geq 2y \), then \( w^{+T} > y \). Since \( X^+(w) \) is strictly decreasing and \( X^-(w) \) is strictly increasing in \( w \), we have \( X^+(w^{-T}) < X^+(y) < X^-(y) < X^-(w^{+T}) \). But this is a contradiction, since \( X^+(w^{-T}) = X^-(w^{+T}) \). Therefore \( w^{-T} < 2y \). Since \( y \) can be chosen to be arbitrarily close to zero, this means that \( w^{+T} \) converges to 0. In the same way it can be shown that \( w^{-T} \) converges to 1, if \( 1 = w^* = G(B) \). 

4. Discussion

By Theorem 1, \( PO, SI, SY, IIA, PI, \) and \( TPP \) characterize the Representative Nash solution. In this characterization, \( PI \) plays an important role. If this axiom is dropped, there are several solutions compatible with properties \( PO, SI, SY, IIA, \) and \( TPP \). For example, consider the following solution. Given a problem \( B = (N, \{u_i\}_{i \in N}) \), let \( a = |N^+|/|N| \). Given players \( i \in N^+ \) and \( j \in N^- \), denote by \( y(i, j) \in [0, 1] \) the unique maximizer of the asymmetric Nash product \( u_i(w)y_j(u_j(w)^{1-a}) \), \( 0 \leq w \leq 1 \). Let \( f \) be a solution such that \( f(B) = \max \min y(i, j) \). It can be shown that \( f \) satisfies the properties \( PO, SI, SY, IIA, \) and \( TPP \), but obviously not \( PI \). This solution favours the players in \( N^- \), if the number \( |N^-| \) grows, ceteris paribus. The representative Nash solution \( G \) is neutral against such changes. One could argue that \( f \) is a more reasonable solution than \( G \) in situations, where decision making processes within \( N^+ \) get slower the larger is the membership in this group. Then if \( N^+ \) is very large, it takes a long time to handle the proposal made by the party \( N^- \), or prepare a counter offer. In a Rubinstein-style bargaining game this would result in a larger pie for the players in \( N^+ \).

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References