Maximax, leximax, and the demanding criterion

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Abstract

Following the characterizations provided by Barberá and Jackson [Barberà, S., Jackson, M.O., 1988. Journal of Economic Theory 46, 34–44] for the maximin, the leximin, and the protective criterion, we examine the consequences of replacing convexity in their list of axioms by concavity, which means considering risk-loving instead of risk-averse agents. This yields characterizations of the maximax, the leximax and a new criterion which we term demanding criterion. We concentrate on the latter and demonstrate the independence of the axioms used for its characterization.

Keywords: Complete uncertainty; Risk-loving agents; Protective criterion

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1. Introduction

The protective criterion first appears in Barberà and Dutta (1982) where it is used in implementation theory and compared to Moulin’s concept of prudent behavior (cf. Moulin, 1981). The same authors employ the protective criterion in two sequel papers, clarifying the relation of their approach to the general practice in the implementation literature (cf. Barberà and Dutta, 1986) and applying it to matching models (cf. Barberà and Dutta, 1995). Recently Fiestras-Janeiro et al. (1998) examine it in the context of finite games in strategic form and their mixed extensions. Barberà and Jackson (1988) give an axiomatic characterization of the protective criterion along with a comparison with minimax and leximax behavior which they also characterize.

Our interest in a characterization of the polar criterion which we term demanding...
behavior stems from its application to implementation in Merlin and Naeve (1998) who are particularly concerned with the use of their results in the context of voting rules.

The organization of the paper is quite similar to Barberà and Jackson (1988). We first introduce some notation and define the three criteria. In listing the axioms, we take care that the definition of the axioms make sense without having to invoke other axioms, which becomes important when we consider their independence. Then we give the characterization results. Though the propositions and their proofs are basically adaptations of the theorems of Barberà and Jackson (1988), we chose to present them in full, both to make the exposition easier to follow and to be able to clarify some minor points that are not made explicit in the original proofs. Given the parallelism of our paper to Barberà and Jackson’s, our demonstration of the independence of the axioms immediately yields the same for theirs. In the concluding remarks, we discuss how our criteria compare to the ones presented by Barberà and Jackson (1988).

As to the interpretation, we prefer to think of the vectors in $\mathbb{R}$ describing utility consequences of an action in different states of the world. Indeed, we feel, that certain axioms are less suited for the alternative interpretation as consequences (measured in real numbers) a social arrangement has for different members of a society. We defer a more detailed discussion of this point to the concluding remarks.

2. Notation and definitions

Let $\mathbb{N}$ denote the set of natural numbers, not including 0, let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^I$ denote the set of finite dimensional vectors of real numbers. It is useful to be more explicit at this point. Barberà and Jackson (1988) implicitly assume that for a vector $x$ with dimension $d(x)$, the set of indices is $\{1, \ldots, d(x)\}$. We will take a slightly different approach, which avoids ambiguities later on (cf. Remark 5) and consists in explicitly defining the index sets. Let:

$$\mathcal{I} = \{I \subset \mathbb{N}; \ 1 \leq |I| < \infty\}$$

where $|I|$ denotes the cardinality of the set $I$. Then we define:

$$\mathbb{R} = \bigcup_{I \in \mathcal{I}} \mathbb{R}^I$$

where $\mathbb{R}^I = \{x : I \rightarrow \mathbb{R}\}$. Following the usual notation for vectors, for $x \in \mathbb{R}^I$, and $i \in I$, we will denote the image of $i$ under $x$ as $x_i = x(i)$. We can then define $x_{-i} = x_{\setminus \{i\}}$, i.e. as the restriction of $x$ to $\setminus \{i\}$ and write $x = (x_i, x_{-i})$. Given $x \in \mathbb{R}$, we will denote the index set of $x$, by $I_x$, i.e. $x \in \mathbb{R}^{I_x}$. The dimension of $x \in \mathbb{R}$ is $d(x) = |I_x|$. We use the following shorthand notations. Let $x \in \mathbb{R}$, then:

$$\min(x) = \min\{x_i; i \in I_x\}$$

and

$$\max(x) = \max\{x_i; i \in I_x\}$$

For $a \in \mathbb{R}$ and $x \in \mathbb{R}$ let $J(a,x) = \{i \in I_x; x_i \geq a\}$
What we are looking for is a binary relation $>$ on $R$ which ranks the vectors and has certain desirable properties we will list as axioms.

**Definition 1.** Given such a binary relation $>$ on $R$, we will define the following two associated binary relations, namely the indifference relation $\sim$ by:

$$x \sim y :\Leftrightarrow [I_x = I_y \text{ and neither } x > y \text{ nor } y > x]$$

and the relation $\succeq$ by:

$$x \succeq y :\Leftrightarrow [x > y \text{ or } x \sim y]$$

Of course, this definition is intended to be applied to relations which also satisfy Axiom 0 below, thus the restriction to vectors with the same index set. We give it at this stage in order to make sure, that Axiom 5 below is well defined also in absence of Axiom 1.

**Remark 1.** It is important to notice, that without further assumptions on $>$, the relations $\sim$ and $\succeq$ may fail to have many nice properties (cf. also Remark 3 after Axiom 1). However, $\sim$ will be symmetric, and $\succeq$ restricted to each $R$ complete by construction.

We now proceed to define the three criteria we are going to consider in this paper. They correspond to the minimax, the leximax, and the protective criterion of Barbera and Jackson (1988). In addition we formally define the relation $s$ on $R$ which is weak Pareto dominance if we were thinking in terms of social consequences. We prefer the notation $s$ over $\bowtie$ as used by Barbera and Jackson (1988) for two reasons: It stresses the fact that we deal with a relation on $5$ and it allows us to refer to the associated relations $\sim$ and $\succeq$.

**Definition 2.** The relation $>_{p}$ is defined by:

$$x >_{p} y \Leftrightarrow [I_x = I_y, x \neq y, \text{ and } x_i > y_i, \forall i \in I_x]$$

**Definition 3.** The maximax criterion $>_{m}$ is defined by:

$$x >_{m} y \Leftrightarrow [I_x = I_y, \text{ and } \exists a \in R: (J(a,x) \neq \emptyset \text{ and } J(a,y) = \emptyset)]$$

**Definition 4.** The leximax criterion $>_{l}$ is defined by:

$$x >_{l} y \Leftrightarrow I_x = I_y \text{ and } \exists a \in R: [|J(a,x)| > |J(a,y)| \text{ and } \forall b \succ a, |J(b,x)| = |J(b,y)|]$$

**Definition 5.** The demanding criterion $>_{d}$ is defined by:

$$x >_{d} y \Leftrightarrow I_x = I_y \text{ and } \exists a \in R: [J(a,x) \sup J(a,y) \text{ and } \forall b \succ a, J(b,x) = J(b,y)]$$

where $\sup$ denotes proper containment.
We illustrate the definitions by an example. Let \( I = \{1, 2, 3\} \), \( x = (5, 4, 3) \) and \( y = (5, 2, 4) \). Then we have for \( a \in \mathbb{R} \):

\[
J(a, x) = \begin{cases} 
0 & \text{if } a > 5 \\
\{1\} & \text{if } a \in [5, 4) \\
\{1, 2\} & \text{if } a \in [4, 3) \\
I & \text{if } a \leq 3
\end{cases}
\quad \text{and} \quad
J(a, y) = \begin{cases} 
0 & \text{if } a > 5 \\
\{1\} & \text{if } a \in [5, 4) \\
\{1, 3\} & \text{if } a \in [4, 2)
\end{cases}
\]

Then obviously, \( x \sim_y y \). Furthermore, \( x \sqsupset y \). To see that \( x \succeq y \), consider \( a = 3 \) with \( |J(3, x)| = 3 > 2 = |J(3, y)| \) and:

\[
|J(b, y)| = \begin{cases} 
0 & \text{if } b > 5 \\
1 & \text{if } b \in [5, 4) \\
2 & \text{if } a \in [4, 2)
\end{cases}
\]

For the demanding criterion, however, we have \( J(3, x) = I \cup \{1, 3\} = J(3, y) \) but \( J(4, x) = \{1, 2\} \neq \{1, 3\} = J(4, y) \), so \( x \not\sim_y y \).

It is easy to see that for all \( x, y \in \mathbb{R} \), \( x \succ y \Rightarrow [x \succ y \text{ and } x \not\sim y] \). Also, \( x \succ y \Rightarrow [x \succ y \text{ and } x \not\sim y] \), and \( x \not\sim y \Rightarrow x \succ y \).

2.1. Axioms

**Axiom 0.** \( x \succ y \Rightarrow I_x = I_y \).

This axiom has a clear interpretation in the context of choice under uncertainty: It only makes sense to compare two vectors of consequences if they are consequences in different states in the same underlying set of possible states.

**Remark 2.** Strictly speaking, our Axiom 0 is stronger than that of Barbera and Jackson (1988). This is an artifact of our more explicit modelling of the index sets, however.

**Axiom 1.** \( \succ \) is irreflexive, transitive, and asymmetric.

This axiom states that \( \succ \) should have the properties of a strict preference ordering.

**Remark 3.** Note that both the relations \( \sim \) as \( \succeq \) defined in Definition 2 may fail to be transitive even if \( \succ \) satisfies Axiom 1 (cf. the remark after Theorem 3 in Barbera and Jackson, 1988). Consider, for example, a relation \( \succ \) on \( \mathbb{R} \) which is defined by \( 1 \succ 2 \) and \( 1 \succ 3 \). This relation is irreflexive, transitive and asymmetric. Then we have \( 3 \succ 4 \) and \( 4 \sim 1 \), but \( 3 \not\sim 1 \), and in the same way \( 3 \succeq 4 \) and \( 4 \succeq 1 \), but \( 3 \not\succeq 1 \). Both \( \sim \) and \( \succeq \) are reflexive, however.

**Axiom 2 (Symmetry).** For \( x, y \in \mathbb{R} \) with \( I_x = I_y \), let \( \sigma \) be a permutation on \( I_x \). Define \( \sigma(x) \) by \( \sigma(x)_i = x_{\sigma^{-1}(i)} \). Then we have \( x \succ y \Leftrightarrow \sigma(x) \succ \sigma(y) \).

Symmetry states that the labeling of states within one index set should not matter in the
ranking of actions. It is justified if we assume that the agents are ignorant as to the states. Symmetry would have no appeal, however, if we would assume that agents had a priori non-uniform assessments of the probabilities of different states.

Remark 4. In our context, one could also formulate a stronger version of Axiom 2. Again this has to do with the explicit modelling of index sets. The stronger axiom considers changing the index set, rather than just relabeling within one set.

Axiom 2’ (Neutrality). For \( x, y \in \mathcal{R} \) with \( I_x = I_y \), let \( \sigma : I_x \to I' \) be a bijection to some other index set \( I' \) with the same cardinality. Define \( \sigma(x) \) by \( \sigma(x)_j = x_{\sigma^{-1}(j)} \) for all \( j \in I' = I_{\sigma(j)} \). Then we have \( x > y \iff \sigma(x) > \sigma(y) \).

The characterization results will show, however, that Axiom 2’ follows from Axiom 2 in conjunction with the other axioms used, which means, that always using initial segments of the natural numbers actually does not constitute any loss of generality.

Axiom 3 (Domination). \( x \succ y \implies x > y \)

The ranking should be monotonic with respect to the vector order. If in all states the consequences of action \( x \) are at least as good as those of action \( y \), and in some states strictly better, then \( x \) should be strictly preferred to \( y \).

Axiom 4 (Independence of Duplicated States). Let \( x, y \in \mathcal{R} \) be such that for some \( i, j \in \mathbb{N} \), \( x_i = x_j \) and \( y_i = y_j \). Then:

\[
x > y \iff x_{-i} > y_{-i} \iff x_{-j} > y_{-j}
\]

Again, for this axiom to make sense, it is important to assume that agents do not have any type of a priori assessment of probabilities of the states.

Remark 5. Barberà and Jackson (1988) remark in a footnote that without symmetry, there is the need to make explicit how the operation of removing one component of a vector is to be interpreted exactly. We have to address this point explicitly, since we have to consider Axioms 4 and 6 without Axiom 2 in Proposition 4, where we demonstrate the independence of the axioms. It turns out, however, that working with explicit index sets instead of only considering the dimension of a vector, there is no ambiguity. Let \( x \in \mathcal{R} \) with index set \( I_x \). Consider \( x_{-i} \). Then \( I_{x_{-i}} = I_x \setminus \{i\} \), and \( (x_{-i})_j = x_j \) for all \( j \in I_{x_{-i}} \).

Axiom 5 (Concavity). Let \( x, y, z \in \mathcal{R} \) be such that \( I_x = I_y = I_z \), then: (i) \( x \succ y \) and \( x \succ z \) \( \implies x \succ \frac{1}{2}(y + z) \); (ii) \( x \succeq y \) and \( x \succeq z \) \( \implies x \succeq \frac{1}{2}(y + z) \).

This is the only axiom that differs from those given by Barberà and Jackson (1988), other than by the formal presentation. Instead of risk-aversion captured by convexity, we assume that agents are risk-loving. If an action is preferred to two others, it will also be preferred to their average.
Remark 6. Axiom 5 actually consists of two parts which in the terminology of Debreu (1959, pp. 59f.) are concavity (i) and weak concavity (ii). Weak concavity follows from concavity only under additional assumptions (cf. Debreu, 1959, (1), p. 60). By Lemma 2 below, restricting Axiom 5 to concavity in Debreu’s sense alone is not enough to obtain the characterization results.

Axiom 6 (Independence of Identical Consequences). For \( x, y \in \mathbb{R} \), if \( x_i = y_i \) we have \( x > y \Rightarrow x_{-i} > y_{-i} \).

Comparison of actions which have identical consequences in some states are based on the other states in which they differ.

Axiom 7 (Continuity). Let \( x^k \) and \( y^k \) be two sequences of vectors in \( \mathbb{R} \).

\[
\left[ x^k > y^k, \forall k \right] \Rightarrow \lim_{k \to \infty} x^k \geq \lim_{k \to \infty} y^k
\]

provided the limits exist.

This axiom requires that small changes in the consequences should not reverse the ranking of actions.

Axiom 8 (Shuffling). Let \( I = \mathbb{I} \). Let \( \sigma \) and \( \varphi \) be two permutations of \( I \). Then:

\[
x > y \Rightarrow \sigma(x) > \varphi(y)
\]

This is a much more demanding form of Axiom 2, which allows for different ways of relabeling of states for two actions. Again one could formulate a stronger version in the spirit of Axiom 2'.

Before considering general pairs of elements in \( \mathbb{R} \), we restrict attention to a particular subset of \( \mathbb{R} \times \mathbb{R} \). Let \( \mathcal{I} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: \max(x) \neq \max(y)\} \). \( \mathcal{I} \) is the set of pairs of vectors the maximal coordinates of which are different. On that set, all three criteria defined above coincide. For any two vectors with the same dimension, the one with the largest maximum is preferred.

Lemma 1. The maximax, the leximax, and the demanding criterion agree on \( \mathcal{I} \). They are characterized (on \( \mathcal{I} \)) by Axioms 0–5 (on \( \mathcal{I} \)).

Proof. The agreement of the three criteria on \( \mathcal{I} \) is obvious.

Let \( > \) be any binary relation on \( \mathcal{I} \) satisfying Axioms 0–5. We proceed in two steps. To avoid unnecessarily complicated notation, we take an index set \( I = \{i,j\} \) and write \((a,b)\) to denote the vector \( x \) with \( x(i) = a \) and \( x(j) = b \). We shall follow this approach of simply listing the components of a vector without explicit reference to the indices for vectors of arbitrary dimension, as long as no confusion is to be expected.

Step 1. For \( a,b,c,d \in \mathbb{R} \) we show that:
This is proven by contradiction. Suppose, it was otherwise. Then we would have \( a > b, a > c, \) and \( a > d \) but \( (c,d) \geq (a,b) \). Choose \( \epsilon \in \mathbb{R}_+ \) such that \( a - \epsilon > b, a - \epsilon > c, \) and \( a - \epsilon > d \). Then by Axiom 3, \( (a - \epsilon, a - \epsilon) > (c,d) \) and by Axiom 1 (transitivity), \( (a - \epsilon, a - \epsilon) \geq (a,b) \). By Axiom 4, \( (a - \epsilon, a - \epsilon, a - \epsilon) \geq (a,a,b) \), and by Axiom 2, \( (a - \epsilon, a - \epsilon, a - \epsilon) \geq (a,b,a) \). Then concavity (Axiom 5) yields \( (a - \epsilon, a - \epsilon, a - \epsilon) \geq (a, (a + b/2), (a + b/2)) \). Using Axiom 4 again, we have \( (a - \epsilon, a - \epsilon) \geq (a, (a + b/2)) \).

Repeating the same argument \( n \) times, we find for any \( n \in \mathbb{N} \):

\[
(a - \epsilon, a - \epsilon) \geq \left( a, \frac{(2^n - 1)a + b}{2^n} \right)
\]

Note that:

\[
\lim_{n \to \infty} \frac{(2^n - 1)a + b}{2^n} = a
\]

Therefore, as \( a - \epsilon < a \), we can find \( N \) large enough, such that:

\[
a - \epsilon < \frac{(2^N - 1)a + b}{2^N}
\]

But for this \( N \) we have:

\[
(a - \epsilon, a - \epsilon) \geq \left( a, \frac{(2^N - 1)a + b}{2^N} \right)
\]

which contradicts Axiom 3. Therefore, the implication (1) must hold.

**Step 2.** Now consider a pair \( (x,y) \in \mathcal{S} \), with \( I_x = I_y \). Let \( a = \max(x) \) and \( b = \max(y) \). W.l.o.g., assume \( a > b \). Let \( d = (a + 2b)/3 \) and \( e = (2a + b)/3 \). Then, by Axiom 3, \( (d,d, \ldots d) > y \). Let \( c = \min(x) \). Then by Axiom 3, \( x > (e, e, \ldots , e) \). Now if \( e = c \), we have \( (e,c) > (d,d) \) by Axiom 3, otherwise (1) of Step 1 yields \( (e,c) > (d,d) \). Then by Axiom 4, \( (e,c \ldots , c) > (d,d, \ldots , d) \), and thus by Axiom 1 (transitivity), \( x > y \).

Hence we have shown, that if \( I_x = I_y \) and \( \max(x) > \max(y) \), we have \( x > y \). By Axiom 1 (Asymmetry), we know that it cannot be the case that \( y > x \) at the same time.¹ Thus, we have an ordering ordering on \( \mathcal{S} \) that coincides with maximax, leximax and the demanding criterion.

While all three criteria coincide on \( \mathcal{S} \), this is not so on \( \mathcal{R} \times \mathcal{R} \). We present the characterizations of the three criteria in Propositions 1–3 which are analoga of Theorems 1–3 in Barberà and Jackson (1988), and present the proofs to enhance readability. We concentrate on the characterization of the demanding criterion in Proposition 4, however, where we demonstrate the independence of the axioms used in the characterization.

¹This argument is not made explicit by Barberà and Jackson (1988).
Proposition 1. The maximax criterion on $\mathcal{R} \times \mathcal{R}$ is characterized by Axioms 0–5 on $\mathcal{I}$ and Axioms 0 and 7 on $\mathcal{R} \times \mathcal{R}$.

Proof. It is easy to check that $>_M$ satisfies the axioms. Indeed, all axioms but Axiom 3 are satisfied on the whole domain $\mathcal{R} \times \mathcal{R}$. An example for the violation of Axiom 3 on $\mathcal{R} \times \mathcal{R}$ is $(1.1) \sim_M (1.0)$.

By Lemma 1, it suffices to show that for $x, y \in \mathcal{R}$, with $I_x = I_y$ and $\max(x) = \max(y)$, the axioms imply $x \sim y$, to establish the proposition. So take two such vectors. Let $(\epsilon^i)_{i \in \mathbb{N}}$ be a sequence of real numbers with $\lim_{i \to \infty} \epsilon^i = 0$. Let $x^i$ be defined by $(x^i)_i = x_i + \epsilon^i$ for all $i \in I_x$, and $x^k$ by $(x^k)_i = x_i - \epsilon^k$. For all $k \in \mathbb{N}$, the pairs $(x^k, y)$ and $(x^k, y)$ belong to $\mathcal{I}$, and by Lemma 1 we have $x^k > y$ and $y > x^k$. Thus, Axiom 7 implies $x \sim y$ and $y \sim x$. Then by Definition 1, $x \sim y$, i.e. neither $x > y$ nor $y > x$.

Proposition 2. The lexicmax criterion on $\mathcal{R} \times \mathcal{R}$ is characterized by Axioms 0 and 3–5 on $\mathcal{I}$, and Axioms 0, 6 and 8 on $\mathcal{R} \times \mathcal{R}$.

Remark 7. Since Axiom 8 obviously implies Axiom 2, the conditions of Lemma 1 are satisfied.

Proof. It is easy to check that $>_L$ satisfies the axioms. Indeed, all axioms but Axiom 4 are satisfied on the whole domain $\mathcal{R} \times \mathcal{R}$. An example for the violation of Axiom 4 on $\mathcal{R} \times \mathcal{R}$ is the following. Let $x, y \in \mathcal{R} \times \mathcal{R}$, with $I_x = I_y = \{1, 2, 3\}$. Let $x = (2.2, 4)$ and $y = (3, 3, 2)$. Then $x >_L y$, but $y_1 >_L x_1$ and $y_2 >_L x_2$.

Now let $x, y \in \mathcal{R}$, with $I_x = I_y$, since otherwise Axiom 0 excludes $x > y$ and $y > x$. Let $|I_x| = n$. By Axiom 8, we can assume $I_x = I_y = \{1, \ldots, n\}$, and moreover, $x_i = y_{i+1}$, and $y_i \geq y_{i+1}$, for all $i \in \{1, \ldots, n-1\}$. Remove all indices $i$ for which $x_i = y_i$ and denote the resulting vectors by $x'$ and $y'$. By construction $(x', y') \in \mathcal{I}$. Therefore, by Lemma 1 $x' > y' \Leftrightarrow \max(x') > \max(y')$. But $\max(x') > \max(y')$ is equivalent to:

$$\exists a \in \mathbb{R}: |J(a,x)| > |J(a,y)| \text{ and } \forall b > a, |J(b,x)| = |J(b,y)|$$

Thus, $>_L$ is the lexicmax criterion.

Proposition 3. The demanding criterion is characterized on $\mathcal{R} \times \mathcal{R}$ by Axioms 0–6.

Proof. To see that $>_D$ satisfies Axioms 0–4, and 6 is straightforward. It remains to demonstrate that it also satisfies concavity, i.e. Axiom 5.

Suppose $x >_D y$ and $x >_D z$. Let $x'$, $y'$ and $z'$ denote the vectors we get by elimination of all states for which $x$, $y$ and $z$ agree. By Axiom 6, $x' >_D y'$ and $x' >_D z'$. Let $a = \max(x)$. By the definition of $>_D$, we know that for any $l$ with $x'_l \neq a$, we have $a > y'_l$ and $a > z'_l$. For $k$ such that $x_k = a$, we know $a \geq y'_k$ and $a \geq z'_k$. By construction of $x'$, $y'$ and $z'$, not all three can have the same component, and hence $a > y'_k$ or $a > z'_k$. 


Therefore, for all $i \in I_x$, we have $a > 1/2(y'_i + z'_i)$. Hence $x' > _D 1/2(y' + z')$. Since $>_D$ satisfies Axiom 6, it follows that $x > _D 1/2(y + z)$. The argument for $\geq_D$ is similar.

Now we must show, that any relation $>$ satisfying the axioms must be $>_y$. By Axiom 0, only vectors with the same dimension are in the relation. We start by proving some facts for two-dimensional vectors.

**Fact 1.** Let $a,b \in \mathbb{R}$, $a > b$. Then neither $(a,b) > (b,a)$, nor $(b,a) > (a,b)$.

**Proof of Fact 1.** Assume $(a,b) > (b,a)$. Then by Symmetry (Axiom 2), $(b,a) > (a,b)$. But then Axiom 1 cannot hold, since transitivity would imply $(a,b) > (a,b)$ contradicting irreflexivity. The same argument can be made if $(b,a) > (a,b)$.

**Fact 2.** Let $a,b,c \in \mathbb{R}$, $a > b$, $a > c$. Then neither $(b,a) > (a,c)$, nor $(a,c) > (b,a)$.

**Proof of Fact 2.** If $b = c$ the result follows from Fact 1. So we assume $b \neq c$. W.l.o.g., let $b > c$.

Assume $(a,c) > (b,a)$. By Axiom 3, $(b,a) > (c,a)$. Then by transitivity (Axiom 1), $(a,c) > (c,a)$, which contradicts Fact 1.

So assume $(b,a) > (a,c)$. By Axiom 6, $(b,a,a) > (a,c,a)$, and by Axiom 2, $(b,a,a) > (a,a,c)$. Then by concavity $(b,a,a) > (a,(a + c/2),(a + c/2))$. By Axiom 4 it follows that $(b,c) > (a,(a + c/2))$. This argument can be repeated $n$ times to arrive at $(b,a) > (a,(2^n - 1)a + c/2^n))$. For $n$ large enough, we have $((2^n - 1)a + c/2^n) > b$, and hence by Axiom 3 $(a,(2^n - 1)a + c/2^n)) > (a,b)$. But then Axiom 1 (transitivity) yields $(b,a) > (a,b)$ which again contradicts Fact 1.

We can extend Fact 2 to comparisons of vectors with arbitrary dimension.

**Fact 3.** Let $x$ and $y$ be such that $I_x = I_y$, $\max(x) = \max(y) = M$, and such that there exists a pair $(i,j) \in I_x \times I_x$ with $x_j = M > y_j$ and $y_i = M > x_i$. Then, neither $x > y$, nor $y > x$.

**Proof of Fact 3.** Suppose $y > x$. Let $\min(x) = m$, and let $k \in I_x$ be such that $x_k = m$. Define the vectors $\tilde{x}$ and $\tilde{y}$ with the same index set by:

\[
\tilde{x}_i = \begin{cases} M & \text{if } i = j \\ m & \text{if } i \in I_x \setminus \{j\} \end{cases} \quad \text{and} \quad \tilde{y}_i = \begin{cases} y_i & \text{if } i = j \\ M & \text{if } i \in I_x \setminus \{j\} \end{cases}
\]

By Axiom 3, we have $x > \tilde{x}$ or $x = \tilde{x}$. Also $\tilde{y} > y$ or $\tilde{y} = y$. Axiom 1 (transitivity) yields $\tilde{y} > \tilde{x}$, which by Axiom 4 implies $(y_j,M) > (M,m)$, contradicting Fact 2. Thus, $y > x$ cannot hold. A similar argument shows that $x > y$ also is not possible.

Now consider any two vectors $x$ and $y$ with $I_x = I_y$. Denote the vectors derived by elimination of all components which coincide for $x$ and $y$ by $x'$ and $y'$. By Axiom 6, $x > y \iff x' > y'$. Let $\max(x') = c$, and $\max(y') = d$.

If $c = d$, Fact 3 applies, and we have neither $x' > y'$, nor $y' > x'$, and hence also...
neither \(x > y\) nor \(y > x\). So let \(c \neq d\). Then \((x', y') \in \mathcal{J}\), and we know that \(x' > y'\) if \(c > d\), and \(y' > x'\) if \(d > c\). Since \(c > d\) iff:

\[
\exists a \in \mathbb{R}: [J(a,x) \supset J(a,y) \text{ and } \forall b > a, J(b,x) = J(b,y)]
\]

(take \(a = c\)), we have for all \(x, y \in \mathbb{R}\), \(x > y \Rightarrow [x > d, y]\). Axiom 1 (Asymmetry) then yields that \(>\) coincides with \(\succsim\).

**Proposition 4.** The Axioms 0–6 are independent on \(\mathcal{R} \times \mathcal{R}\).

**Proof.** We present a list of six relations, each satisfying all but one of the seven axioms. We will always just state the axiom that is not satisfied.

**Axiom 0:** Consider the relation \(>_0\) defined as follows. For \(x, y \in \mathbb{R}\):

\[
x >_0 y :\Leftrightarrow [x > d, y \text{ or } d(x) > d(y)]
\]

**Axiom 1:** Here we can actually go one step further and present three examples of relations each satisfying Axioms 0, and 2–6, one violating irreflexivity while satisfying transitivity and asymmetry, a second violating transitivity while satisfying irreflexivity and asymmetry, and a third violating asymmetry while satisfying irreflexivity and transitivity. Therefore, each of the three parts of Axiom 1 are independent of each other in conjunction with the rest of the axioms.

**Irreflexivity:** Consider the relation \(>_{\text{i}}\) defined as follows. For \(x, y \in \mathbb{R}\):

\[
x >_{\text{i}} y :\Leftrightarrow [x > d, y \text{ or } x = y]
\]

**Transitivity:** Consider the relation \(>_{\text{b}}\) which is defined as follows. For \(x, y \in \mathbb{R}\), first define:

\[
x >_{\text{b}} y :\Leftrightarrow [x > d, y \text{ or } (x, y) = ((3,4),(2,6))]\]

This relation clearly is not transitive, since for example \((2,6) >_{\text{b}} (1,5)\) but not \((3,4) >_{\text{b}} (1,5)\). However, it also fails to satisfy Axioms 4 and 6. Therefore we need to consider an extension which satisfies these two axioms by making sure that all pairs \((x, y) \in \mathcal{R} \times \mathcal{R}\) which can be transformed to \(((3,4),(2,6))\) by the operations considered in Axiom 4 or Axiom 6, respectively, are in the relation as well. We will formalize this by the relation \(>_{\text{b}*}\). So this relation is defined for \(x, y \in \mathcal{R}\) by:

\[
x >_{\text{b}*} y :\Leftrightarrow I_x = I_y \text{ and } \exists (i, j) \in I_x \times I_y:
\]

\[
\begin{align*}
x_i &= 3, x_j = 4, y_i = 2, y_j = 6 \text{ and } \\
\forall k \in I \setminus \{i, j\}, \\
x_k &= y_k \text{ or } \\
&[ (x_k = 3 \text{ or } x_k = 4) \text{ and } (y_k = 2 \text{ or } y_k = 6) ]
\end{align*}
\]

For \(x, y \in \mathcal{R}\) we finally define \(>_{\text{b}}\) by:
Asymmetry: Consider the relation \( >_c \) defined as follows. For \( x, y \in R \):

\[ x >_c y \iff [x >_D y \lor y >_D x] \]

Axiom 2: For \( x \in R \), define \( f(x) \) by \( f(x)_i = ix_i \). Then \( >_2 \) is defined by:

\[ x >_2 y \iff f(x) >_D f(y), \text{ for } x, y \in R \]

Axiom 3: Consider the relation \( >_3 \) defined as follows. For \( x, y \in R \):

\[ x >_3 y \iff -x >_D -y \]

Axiom 4: Consider the relation \( >_4 \) defined as follows. For \( x, y \in R \):

\[ x >_4 y \iff \left[ I_x = I_y \text{ and } \sum_{i=1}^{d(x)} x_i > \sum_{i=1}^{d(y)} y_i \right] \]

Axiom 5: Consider the protective criterion of Barberà and Jackson (1988).

Axiom 6: Consider the relation \( >_6 \) defined as follows. For \( x, y \in R \):

\[ x >_6 y \iff [x >_M y \text{ or } (x >_M y \text{ and } x >_p y)] \]

**Corollary 1.** Axioms 0–5 are independent on \( \mathcal{F} \).

**Proof.** In view of Lemma 1, obviously Axiom 6 is implied by Axioms 0–5. The examples in the proof of Proposition 4 show, that Axioms 0–5 are independent on \( \mathcal{F} \), as well. As we noted in the introduction, Proposition 4 and Corollary 1 immediately yield the independence of the axioms used by Barberà and Jackson (1988) in their characterizations of the protective criterion on \( R \times R \) and on \( \mathcal{F} \), respectively.

We close this section by demonstrating that part 2, the weak concavity, of Axiom 5 is essential in the characterization results of Lemma 1 and Proposition 3.

**Lemma 2.** The relation \( >_p \) satisfies Axioms 0–4, Axiom 6, and part 1 of Axiom 5. It does not satisfy part 2 of Axiom 5.

**Proof.** That \( >_p \) satisfies Axioms 0–4 and Axiom 6 is easy to see. We show that \( >_p \) is concave, but not weakly concave. Let \( x >_p y \) and \( x >_p z \). Consider \( t = 1/2(y + z) \). Clearly \( I_t = I_y \). Also \( x_i \geq t_i \) for all \( i \in I_y \) follows immediately, since \( x_i \geq y_i \) and \( x_i \geq z_i \) for all \( i \in I_y \). Since \( x >_p y \), there exists \( j \in I_y \) such that \( x_j > y_j \). But then, \( x_j > 1/2y_j + 1/2z_j = t_j \). Hence \( x \neq t \) and thus \( x >_p t \).

To see that \( >_p \) fails to satisfy weak convexity, it suffices to present an example. We have \((2,2) \nRightarrow (1,5)\) and \((2,2) \nRightarrow (5,1)\). Hence \((2,2) \nRightarrow (1,5)\) and \((2,2) \nRightarrow (5,1)\). However, \(1/2((1,5) + (5,1)) = (3,3) >_p (2,2)\).
3. Concluding remarks

As we have seen, the difference between the demanding criterion and the protective criterion of Barbera and Jackson (1988) can be attributed solely to the assumptions of concavity and convexity, respectively. Thus, the demanding criterion seems appropriate if agents are assumed to be risk-loving, while the protective criterion should be employed with risk-averse agents. Clearly, the assumption of risk-aversion is much more common in economics. There may well be situations for which assuming agents to be risk-loving could be appropriate, though. In the context of decisions under uncertainty, both possibilities are usually allowed for (cf., for example, Arrow and Hurwicz, 1972; Kelsey, 1993; Barrett and Pattanaik, 1994).

Comparing the results for implementation under protective behavior of Barbera and Dutta (1982) with those on implementation under demanding behavior of Merlin and Naeve (1998), especially their application to voting rules, also suggests that there may be some room for using demanding behavior. It turns out that protective behavior is linked to the anti-plurality rule, while demanding behavior is linked to the plurality rule. Therefore, the pervasive use of plurality voting could be taken as an argument in favor of the demanding criterion.

To conclude, we discuss why we feel that the second interpretation Barbera and Jackson (1988) suggest for their model may not be as convincing in our case. While in the decision under uncertainty interpretation, convexity, i.e. risk-aversion, and concavity, i.e. risk-loving, are clearly two extreme cases of possible assumptions on agents behavior, it seems much harder to defend concavity as a normative criterion for the comparison on the consequences of different social regimes. Convexity here means that averaging is seen as something positive, that is it captures the idea that some sort of equality in the society is deemed desirable. Concavity would say that instead averaging can only worsen the situation, in other words that extreme distributions in the society are seen as positive. This might seem a good description of social values held by capitalist societies to some critics of capitalism, but surely would not seem a normative criterion one would seriously put forward.

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