Optimal control of a simple manufacturing system with restarting costs

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Abstract

We consider the optimal control of an unreliable manufacturing system with restarting costs. In 1986 and 1988, Akella and Kumar (for the infinite horizon discounted cost) and Bielecki and Kumar (for the infinite horizon average expected cost) show that the optimal policy is given by an optimal inventory level ("hedging point policy"). Inspired by these simple systems, we explore a new class of models in which the restarting costs are explicitly taken into account. The class of models discussed often allow complete analytical discussions. In particular, the optimal policy exhibits an \((s; S)\) type form. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we consider the problem of controlling the production rate of a simple manufacturing system. Akella and Kumar [1] obtained the optimal policy for a single failure prone machine for an infinite horizon discounted cost criterion. They show that the optimal policy depends on an optimal inventory level \(z^*\) ("hedging point"). Under this policy the production rate is chosen to be maximum when the stock level \(X(t)\) is below \(z^*\), zero when \(X(t) > z^*\) and equal to the demand rate when \(X(t) = z^*\). Bielecki and Kumar [2] obtained a similar optimal policy which minimizes the long term average expected cost. In both of these models the cost function \(g(x)\) depends only on the stock level, and is defined by

\[
g(x) = C^+ x^+ + C^- x^- ,
\]

where \(x^+ = \max(x, 0)\) and \(x^- = \max(-x, 0)\).

In this paper we will restrict the production rate of the machine, which is the controllable variable, to be dichotomous \(\{0, M\}\), i.e. the machine can only produce at a maximum speed or can be put in a stand-by state where it is not producing. We then introduce a restarting cost \(\delta\) to be paid each time we decide to switch from the stand-by state to the producing one. We will show that adding this restarting cost leads to a “bang–bang” \((s; S)\) type optimal policy. The optimal policy \(u(x, \xi)\) is quite simple and the control depends only on the stock position \(x\) and on the state of the machine \(\xi(t)\). In the producing state, it consists in continuing producing if \(x < b\) and switching to the

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stand-by state if not. The stand-by state remains selected as long as \( x > a \); as soon as \( x \leq a \) a cost \( \delta \) is incurred and the production is switched on at the rate \( M \) (we clearly have \( b > a \)).

This paper is structured as follows: in the following section, we will describe the dynamic of the model and in Sections 3 and 4 we give the optimal policies which respectively minimizes the average and the discounted cost in infinite horizon for a perfect machine (without failures). In Sections 5 and 6 we take into account random failures and a random repair time and we give the optimal policy for the same costs as in Sections 3 and 4. Finally in Section 7, we show that the \((s; S)\) policy is optimal.

The results obtain in Sections 3–5 are limiting case of the general situation presented in Section 6. More precisely the control which minimizes the average cost is the limit of the control which minimizes the discounted cost for a discount \( \alpha \to 0 \) [4]. Similarly, the deterministic models are obtained for small values of the indisponibility factor \( I = \lambda/\mu \), where \( \lambda^{-1} \) is the mean time to failure and \( \mu^{-1} \) the mean time to repair. We nevertheless give the results of the simpler models for two main reasons: explicit expressions are only tractable for the deterministic models and in Sections 5 and 6 the methods used for the discounted and the average cases are different. For the sake of clarity, in Sections 5 and 6, we will expose the method but we give only explicit results for particular values of the parameters. General analytic formulas can easily be computed with the help of a program like Maple but the expressions obtained are too large to be presented. Moreover, the values of the parameters \( a \) and \( b \), which defines the optimal control, are at the end, given in terms of solutions of transcendental equations.

2. Description of the model

We consider a failure prone machine producing a single product. There is a constant demand \( d \) for this product and we can choose, whenever the machine is operating, to produce at rate \( M \) or not to produce, here the production rate cannot take intermediate values. The machine is subject to failures and can be repaired. We will suppose that the time to failure and the time needed to repair a failure are independent, exponentially distributed random variables with parameters \( \lambda \) and \( \mu \). Let \( X(t) \) be the stock level of the product at time \( t \), given by the difference between the cumulative production and the demand up to time \( t \). Remark that \( X(t) \) can be negative, i.e. we admit backlog. \( X(t) \) evolves as

\[
\frac{dX(t)}{dt} = u_i M - d,
\]

where \( u_i \in \{0, I(\xi(t) \neq 0)\} \) and where \( \xi(t) \) is defined by the continuous-time Markov chain sketched in Fig. 1, and where

\[
I(\xi(t) \neq 0) = \begin{cases} 0 & \text{if } \xi(t) \text{ is in off state at time } t, \\ 1 & \text{if not}. \end{cases}
\]

There are now three possible states, the on state: \( dX(t)/dt = M - d \), the stand-by state: \( dX(t)/dt = -d \) and the off state: \( dX(t)/dt = -d \). The control variable \( u_i \) triggers the transition from the on to the stand-by state and conversely. We assume that the machine cannot fail while it is in the stand-by state. From now on we will, respectively, note “on” (or 1), “stand by” (1*) and “off” (0) for these states. The transitions on \( \to \) off and off \( \to \) stand by are governed by exponential distributions of parameters \( \lambda \) and \( \mu \). The process of interest is thus \( \{X(t), \xi(t)\} \), where \( X(t) \in \mathbb{R} \) and \( \xi(t) \in \{0, 1, 1^*\} \). The “deterministic cases” in Sections 3 and 4 are given by the limit \( \lambda \to 0 \) and \( \mu \to \infty \).

The cost due to inventory is given by \( g(x) = C^+ x^+ + C^- x^- \), where \( x^+ = \max(x, 0) \) and \( x^- = \max(-x, 0) \). A restarting cost \( \delta \) is incurred each time we decide to switch from the stand-by to the on state. For the discounted case, the aim is to find the control which
minimizes
\[ J(x_0, \xi_0) = E \left( \int_0^\infty e^{-zt} g(X(t)) \, dt \right) \]
\[ + \sum_{\tau_i} e^{-zt_0} \delta |X(t = 0) = x_0; \xi(t = 0) = \xi_0) , \]
where \( \tau_i \) are the times where we choose to go from the stand-by state \( u_t = 0 \) to the on state \( u_t = 1 \) and \( E \) denotes the expectation over the realizations of the stochastic process \( \xi(t) \).

And for the average case:
\[ J = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left( \int_0^T g(X(t)) \, dt + \sum_{\tau \cap \{t \leq T\} \in T} \delta \right) . \]

Remark that for the average case, \( J \) does not depend on the initial conditions any more.

3. Deterministic system, average cost

As the machine cannot fail, we only have two states (on and stand-by) and the problem is deterministic. The dynamic is periodic and the control is given by two parameters \( a \) and \( b \):
\[ u_t = \begin{cases} 1 & \text{if } x(t) \leq b \text{ and } x(t) = 1, \\ 1 & \text{if } x(t) \leq a \text{ and } x(t) = 1^+, \\ 0 & \text{otherwise.} \end{cases} \]

We sketch in Fig. 2 a typical trajectory of \( x(t) \) for this control. The cost for one period \( J_1 \) is given by
\[ J_1(a, b) = -C^- \int_0^{-(a-b)(M-d)} (a + (M-d)t) \, dt \]
\[ + C^+ \int_{-(a-b)(M-d)}^{-(a-b)d+(b/d)} (a + (M-d)t) \, dt \]
\[ + C^+ \int_{-(a-b)d+(b/d)}^{-(a+b)(M-d)} (a + (M-d)t) \, dt \]
\[ \times \left( - \left( t - \frac{-a + b}{M-d} \right) d \right) \, dt \]

The long-term average expected cost \( J \) is then given by the cost over one period \( J_1 \) divided by the cycle time:
\[ J(a, b) = \frac{J_1}{(a+b)(d^{-1} + (M-d)^{-1})}. \]

The minimum of \( J(a, b) \) (\( \partial J(a, b)/\partial a = 0 \) and \( \partial J(a, b)/\partial b = 0 \)) can be expressed in a simple form:
\[ a = - \sqrt{2 \frac{(M-d) \, dC^+ \delta}{MC^-(C^+ + C^-)}} \]
\[ b = \sqrt{2 \frac{(M-d) \, dC^- \delta}{MC^+(C^+ + C^-)}} . \]

Remark that
\[ \frac{-a}{b} = \frac{C^+}{C^-} \]
and that \( ab \) is proportional to \( \delta \) and \( (C^+ + C^-)^{-1} \)
\[ -ab = 2 \frac{(M-d) \, d\delta}{M(C^+ + C^-)} . \]

We show in Fig. 3 the value of \( a \) and \( b \) as a function of the restarting cost \( \delta \).
4. Deterministic system, discounted cost

For a deterministic discounted system, the cost depends on the initial conditions, but the control does not. Indeed we can find the two parameters $a$ and $b$ by minimizing the expected discounted cost for the initial condition $X(t_0) = 0$ and the machine in the producing state. We start by calculating the cost $J_1(0, \text{on})$ incurred during one cycle:

\[
J_1(0, \text{on}) = C^+ \int_0^{b/(M-d)} e^{-zt} (M-d) t \, dt + C^+ \int_{b/(M-d)}^{b/(M-d)+(b/d)} e^{-zt} \times \left( - \left( t - \frac{b}{M-d} \right) d + b \right) \, dt + C^- \int_{b/(M-d)+(b/d)}^{(b/(M-d))+(b-a)/d} e^{-zt} \times \left( - \left( t - \frac{b}{M-d} \right) d + b \right) \, dt + e^{-\pi(b/(M-d)+(b-a)/d)\delta} - C^- \int_{(b/(M-d)+(b-a)/d)}^{(b/(M-d))+(b-a)/d} e^{-zt} \times \left( (M-d) \left( t - \frac{b}{M-d} - \frac{b-a}{d} \right) + a \right) \, dt
\]

and we sum over all cycles taking into account the elapsed time:

\[
J(0, \text{on}) = J_1(0, \text{on}) \sum_{i=0}^{\infty} e^{-i\pi(b-a)\delta^{-1} + (M-d)^{-1}} J_1(0, \text{on}) = \frac{J_1(0, \text{on})}{1 - e^{-\pi(b-a)\delta^{-1} + (M-d)^{-1}}}.
\]

Again we find $a$ and $b$ by solving $(\partial/\partial a)J(0, \text{on}) = 0$ and $(\partial/\partial b)J(0, \text{on}) = 0$. This time the expressions for $a$ and $b$ are slightly more complicated:

\[
a = \frac{d}{\pi z_1},
\]

where $z_1$ is the solution of

\[
(M-d) \ln \left( \frac{Ke^{\omega_1/(M-d)} + Le^{\omega_1}}{MC^+} \right) + d \ln \left( \frac{K + Le^{\omega_1}}{MC^+} \right) - \omega_1 M = 0
\]

and

\[
b = \frac{M-d}{\pi z_2},
\]

where $z_2$ is the solution of

\[
(M-d) \ln \left( \frac{Ke^{\omega_2/(M-d)} - MC^+e^{\omega_2}}{-L} \right) + d \ln \left( \frac{Ke^{\omega_2/(M-d)} - MC^+e^{\omega_2}}{-L} \right) - \omega_2 M = 0
\]

where

\[
K = M(C^+ + C^-),
\]

and

\[
L = \delta z^2 - MC^-.
\]

Note that for large values of $\delta$, the control is not of the same type, $a$ goes to minus infinity. This happens for example when the restarting cost is bigger than the one which occurs by not producing at all. Eqs. (16) and (18) can be simplified when $M = 2d$ to second order polynomials. Defining $z_1 = e^{\omega_1}$ and $z_2 = e^{\omega_2}$, we
have \( a = (d / z) \ln(z_1) \) and \( b = [(M - d) / z] \ln(z_2) \) where \( z_1 \) and \( z_2 \) are, respectively, the positive solutions of

\[
z_1^2 + \frac{L^2 + K^2 - (MC)^2}{KL} z_1 + 1 = 0 \quad (21)
\]

and

\[
z_2^2 + \frac{L^2 - K^2 - (MC)^2}{KMC} z_2 + 1 = 0. \quad (22)
\]

As expected, in the limit \( z \to 0 \) we find again the values of \( a \) and \( b \) (Eqs. (9) and (10)) for the average case. We show in Fig. 4 the values of \( a \) and \( b \) as a function of the discount rate \( \delta \).

5. Stochastic system, average cost

For this case, we will proceed in two steps. First, we will find the stationary distribution and then we will calculate its associate cost. The minimization over the parameters \( a \) and \( b \) will give us the optimal control law.

5.1. Stationary distribution

Let us denote by \( P_{0a}(x) \, dx, P_{1a}(x) \, dx \) and \( P_{1^-*a}(x) \, dx \) the probabilities of finding the stock level in \( [x, x + dx] \), respectively, in the state off, on and stand-by for \( x < a \). We denote the same probabilities for the region \( a < x < b \) by \( P_{0b}(x), P_{1b}(x) \) and \( P_{1^-*b}(x) \). The stationary measure of the stochastic process \( \{X(t), \zeta(t)\} \) subject to the previous control obeys, for the zone \( x \leq a \), a balance equation of the form [3]:

\[
d \frac{\partial}{\partial x} P_{0a}(x) = - \lambda P_{1a}(x) + \mu P_{0a}(x), \quad (23)
\]

\[
(M - d) \frac{\partial}{\partial x} P_{1a}(x) = - \lambda P_{1a}(x) + \mu P_{0a}(x), \quad (24)
\]

\[
P_{1^-*a}(x) = 0. \quad (25)
\]

Similarly for the zone \( a < x \leq b \), we have

\[
d \frac{\partial}{\partial x} P_{0b}(x) = - \lambda P_{1b}(x) + \mu P_{0b}(x), \quad (26)
\]

\[
(M - d) \frac{\partial}{\partial x} P_{1b}(x) = - \lambda P_{1b}(x), \quad (27)
\]

\[
d \frac{\partial}{\partial x} P_{1^-*b}(x) = - \mu P_{0b}(x). \quad (28)
\]

These equations are linear partial differential equation of the first order, it is easy to calculate their solutions. In order to fix the integration constants, we then need to use the boundary conditions Eqs. (29)–(33) and the normalization Eq. (34). If, in the mean, we can satisfy the demand, we have

\[
P_{1a}(-\infty) = 0. \quad (29)
\]

When \( x = b \), we transit from the on state to the stand-by state:

\[
P_{1^-*b}(b) = P_{1b}(b)(M - d). \quad (30)
\]

It is not possible to be at position \( x = b \) with the machine down:

\[
P_{0b}(b) = 0. \quad (31)
\]

The probability of being at \( x = a \) with the machine in the on state is the sum of the probability of coming from \( x < a \) and the probability of coming from the right in the stand-by state:

\[
P_{1b}(a)(M - d) = P_{1^-*b}(a) + P_{1a}(a)(M - d). \quad (32)
\]

The probability of being in the off state is continuous at \( a \):

\[
P_{0a}(a)d = P_{0b}(a)d. \quad (33)
\]
Fig. 5. Stationary distribution for the values $d=1$, $M=1.9$, $\mu=\frac{1}{2}$, $\lambda=\frac{1}{2}$, $\delta=20$, $C^+=20$, $C^-=60$.

\[
\int_{-\infty}^{a} (P_{1a}(x) + P_{0a}(x)) \, dx
+ \int_{a}^{b} (P_{1b}(x) + P_{1\cdot b}(x) + P_{0b}(x)) \, dx = 1. \quad (34)
\]

Using Maple or a similar program, we easily find analytically all the integration constants.

5.2. Cost associated to the stationary distribution

Here we have to distinguish the case $a > 0$ from the case $a \leq 0$, we will only show the equations for $a > 0$. The cost associated with the limit distribution is the sum of the cost due to the stock population plus the restarting cost at $a$ and the restarting cost for $x < a$:

\[
J = \int_{-\infty}^{0} -C^- x(P_{1a}(x) + P_{0a}(x)) \, dx
+ \int_{0}^{a} C^+ x(P_{1a}(x) + P_{0a}(x)) \, dx
+ \int_{a}^{b} C^+ x(P_{1b}(x) + P_{0b}(x) + P_{1\cdot b}(x)) \, dx
+ \delta P_{1\cdot b}(a) + \int_{-\infty}^{a} \delta P_{0a}(x) \mu \, dx. \quad (35)
\]

The optimal control is again given by minimizing the cost $J$ over $a$ and $b$. This minimization leads to cumbersome expressions. We only give here the result for the special values $d=1$, $M=2$, $\mu=\frac{1}{2}$, $\lambda=\frac{1}{4}$, $\delta=20$, $C^+=20$, $C^-=60$. We find $a=1.71$, and $b=4.06$ for a “hedging point”, which would have been (if $\delta=0$) $z^*=3.9$. We can verify that for the value $\mu=2$ and $\lambda=\frac{1}{10}$, the result of the deterministic problem is approached. In this case, we find $a=-0.263$ and $b=0.911$, while in the deterministic case we have $a=-0.288$ and $b=0.866$. We show in Fig. 5 the stationary distribution of the process (with $M=1.9 \rightarrow a=2.55, b=4.85$ to avoid superposition of the curves) and in Fig. 6 the cost as a function of $a$ and $b$.

6. Stochastic system, discounted cost

We modify the problem assuming that the transition (control) from state 1 to 1$^*$ (respectively, from 1$^*$ to
1) is no more instantaneous, but that when we decide to transit we have a probability $\eta \, dt$ (respectively, $\gamma \, dt$) to transit in a time $dt \ll 1$ (exponential distribution). We also modify the restarting cost $\delta$ and we now assume that we have a cost $\gamma \delta$ per unit time when we try to transit from the stand-by (1$^*$) state to the on (1) state.

The original problem will result from taking the limit $\gamma \to \infty$ and $\eta \to \infty$. We define, respectively, to each initial state the expected cost starting at $X(0)=x$, 

Fig. 6. Cost as a function of $a$ and $b$ for the values $d=1$, $M=2$, $\mu=\frac{1}{2}$, $\lambda=\frac{1}{4}$, $\delta=20$, $C^+=20$, $C^-=60$. 

by \( J_1(x), J_1^*(x) \) and \( J_0(x) \). The Hamilton–Bellman–Jacobi equations for this problem are

\[
0 = g(x) - d \frac{\partial J_0(x)}{\partial x} - (\mu + \alpha) J_0(x) + \mu J_1(x), \quad (36)
\]

\[
0 = g(x) + \min_{u=0,1} \left\{ u \left( - (\lambda + \alpha) J_1(x) + (M - d) \frac{\partial J_1(x)}{\partial x} + \lambda J_0(x) \right) + (1 - u) \left( - (\lambda + \alpha + \eta) J_1(x) + (M - d) \frac{\partial J_1(x)}{\partial x} + \lambda J_0(x) + \eta J_1^*(x) \right) \right\}, \quad (37)
\]

\[
0 = g(x) + \min_{u=0,1} \left\{ u \left( - (\gamma + \alpha) J_1(x) + (M - d) \frac{\partial J_1(x)}{\partial x} + \gamma \delta \right) + (1 - u) \left( - (\lambda J_1(x) - d \frac{\partial J_1(x)}{\partial x} + \lambda J_0(x) \right) \right\}. \quad (38)
\]

Now, we will suppose the control known, which will divide the space into four regions (again we have to distinguish the case \( a > 0 \) from the case \( a \leq 0 \)). Here we will treat the case \( a < 0 \): region A is defined by \( x < a \), region B by \( a \leq x < 0 \), region C by \( 0 \leq x < b \) and region D by \( b \leq x \). We obtain for each of the regions, a system of partial differential equations, for example in the C region

\[
\frac{\partial}{\partial x} J_{c0}(x) + (\alpha + \mu) J_{c0}(x) - \mu J_{c1}(x) = C^+ x, \quad (39)
\]

\[
-(M - d) \frac{\partial}{\partial x} J_{c1}(x) + (\alpha + \lambda) J_{c1}(x) - \lambda J_{c0}(x) = C^+ x, \quad (40)
\]

\[
d \frac{\partial}{\partial x} J_{c1}(x) + \lambda J_{c1}(x) = C^+ x, \quad (41)
\]

where we have put a subscript \( c \) to indicate that \( x \) is in the region C. We solve these equations in each region of the space and we stick together the costs corresponding to the different regions:

\[
J_{a0}(a) = J_{b0}(a), \quad J_{b0}(0) = J_{c0}(0), \quad J_{c0}(b) = J_{d0}(b)
\]

for \( i = 1, 1^* \) and 0. With these equations, we can find the integration constants and we find the minimum of \( J_{d0}(0) \) for example, with respect to \( a \) and \( b \). Note here that, as the policy is optimal, the minimization of any, \( J_1(x) \) at any point \( x \) leads to the same values of \( a \) and \( b \).

For the case \( \alpha = \frac{1}{10}, d = 1, M = 2, \mu = \frac{1}{7}, \lambda = \frac{1}{4}, \delta = 20, C^+ = 20, C^- = 60 \), we find \( a = -0.107 \) and \( b = 1.553 \), which agree exactly with numerical results (dynamic programming).

### 7. Optimality of the \((s; S)\) policy

In this section, we will show that the \((s; S)\) policy is optimal. Only the case of the stochastic system with a discounted cost will be considered as the others are limiting cases.

Up to now, we have only found the optimal values of \( a \) and \( b \) for a \((s; S)\) policy and we have to show that these solutions give the minimum of the expected discounted cost over all acceptable policies. Unfortunately, the algebra seems to become too heavy to allow a direct proof in the general case. To circumvent this difficulty, we can compute the expected discounted cost for any given values of the parameters and then show that it does indeed satisfy the HJB equations \((36)-(38)\). To illustrate this procedure, we treat explicitly the case discussed in Section 6 and we show that the \((s; S)\) policy is optimal in this case.

By construction, the costs \( J_1(x), J_1^*(x) \) and \( J_0(x) \) are solutions of the Eqs. \((36)-(38)\) for a specific choice of \( u \). Therefore, we only have to check that the minima in these equations are obtained for our choice of \( u \). Eqs. \((37)\) and \((38)\) can be rewritten as

\[
0 = g(x) + C_1(x) + \min_{u=0,1} \left\{ (1 - u)(\eta J_1^*(x)) - J_1(x) \right\}, \quad (43)
\]

\[
0 = g(x) + C_2(x) + \min_{u=0,1} \left\{ u(\gamma J_1(x) + \delta - J_1^*(x)) \right\}, \quad (44)
\]

where

\[
C_1(x) = - (\lambda + \alpha) J_1(x) + (M - d) \frac{\partial J_1(x)}{\partial x} + \lambda J_0(x) \quad (45)
\]

\[
C_2(x) = -\lambda J_1^*(x) - d \frac{\partial J_1^*(x)}{\partial x}. \quad (46)
\]

Remark that for \( \gamma \to \infty \) and \( \eta \to \infty \), the minimum is obtained in region A \((x < a)\) and D \((x > b)\) by a
Fig. 7. Difference between $J_1(x)$ and $J_1^*(x)$ for the values $x = \frac{1}{10}, d = 1, M = 2, \mu = \frac{1}{2}, \lambda = \frac{1}{4}, \delta = 20, C^+ = 20$ and $C^- = 60$.

Fig. 8. Optimal costs $J_1(x)$, $J_1^*(x)$ and $J_0(x)$ for the values $x = \frac{1}{10}, d = 1, M = 2, \mu = \frac{1}{2}, \lambda = \frac{1}{4}, \delta = 20, C^+ = 20$ and $C^- = 60$.

(s; S) control as $J_1(x) = J_1^*(x) - \delta$ in region A and $J_1(x) = J_1(x)$ in region D. Between $a$ and $b$ where the policy does not switch from one state to the other, we have to show that

$$J_1(x) \leq J_1^*(x) \leq J_1(x) + \delta. \quad (47)$$

Fig. 7 shows that, when $a$ and $b$ are optimal, condition (47) is satisfied. In Fig. 8, we sketch the optimal costs $J_1(x), J_1^*(x)$ and $J_0(x)$.

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