Uniform bounds on the limiting and marginal derivatives of the analytic center solution over a set of normalized weights

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Abstract

While properties of the weighted central path for linear and non-linear programs have been studied for years, the effect that weighting the central path has on its limiting derivatives has not been fully explored. We show that the limiting derivatives of the central path for a linear program are uniformly bounded over a set of normalized weights. Consequently, this allows a similar bound for the marginal derivatives of the analytic center solution with respect to changes in the right-hand side.

Keywords: Analytic central path; Computational economics; Sensitivity analysis

1. Introduction

Consider the standard form primal linear programming problem

(LP) \[ \min \{ cx : Ax = b, \ x \geq 0 \} \]

and its associated dual:

(LD) \[ \max \{ yb : yA + s = c, \ s \geq 0 \}, \]

where \( A \in \mathbb{R}^{m \times n}, \ m \leq n, \ rank(A) = m, \ b \in \mathbb{R}^m, \ c \in \mathbb{R}^n, \ x \in \mathbb{R}^n, \ s \in \mathbb{R}^n, \ y \in \mathbb{R}^m, \) and \( c, y, \) and \( s \) are taken to be row vectors. Since linear changes in the right-hand side, \( b, \) are considered, the primal and dual feasible regions are denoted by \( \mathcal{P}_b, \) and \( \mathcal{D}, \) respectively. Similarly, the primal and dual optimal faces are \( \mathcal{P}_b^\star, \) and \( \mathcal{D}_b^\star, \) respectively.

Both the primal and dual feasibility regions are assumed to satisfy Slater’s constraint qualification. That is, we assume the existence of an \( x \in \mathcal{P}_b \) and a pair \( (y, s) \in \mathcal{D}, \) such that \( x > 0 \) and \( s > 0. \) The right-hand side vector, \( b, \) is admissible if the corresponding LP satisfies Slater’s constraint qualification.

The assumption that \( b \) is admissible implies the existence of the omega central path [15],

\[ \{(x(\mu, b, \omega), (y(\mu, b, \omega), s(\mu, b, \omega))): \mu > 0\} \]

where \( \omega \in \mathbb{R}^n_+ \) and the elements are the unique solution to

\[ Ax(\mu, b, \omega) = b, \]

\[ y(\mu, b, \omega)A + s(\mu, b, \omega) = c, \]

\[ S(\mu, b, \omega)x(\mu, b, \omega) = \mu \omega. \]
In the last equality, $S(\mu, b, \omega)$ is the diagonal matrix formed by $s(\mu, b, \omega)$. These equations are the necessary and sufficient Lagrange conditions for
\[
\min \left\{ cx - \mu \sum_{i=1}^{n} \omega_i \ln(x_i): Ax = b, \ x > 0 \right\}
\]
and
\[
\max \left\{ yb - \mu \sum_{i=1}^{n} \omega_i \ln(s_i): yA + s = c, \ s > 0 \right\}.
\]
Furthermore,
\[
\lim_{\mu \to 0^+} \left( x(\mu, b, \omega), (y(\mu, b, \omega), s(\mu, b, \omega)) \right) = (x^*(b, \omega), (y^*(b, \omega), s^*(b, \omega)))
\]
is a strictly complementary solution to the linear programs, called the omega analytic center solution. This solution induces the optimal partition:
\[
B(b) = \{ i: x_i^*(b, \omega) > 0 \}
\]
and
\[
N(b) = \{ 1, 2, 3, \ldots, n \} \setminus B(b).
\]
These sets do not depend on any specific choice of $\omega \in \mathbb{R}^n_{++}$. If no confusion results, the argument $(b)$ is dropped. The sets $B$ and $N$ are used as subscripts to indicate the components of a vector, or columns of a matrix, that correspond to elements in $B$ and $N$, respectively.

Sonnevend [16–18] showed the mathematical programming community that analytic centers were important because of their connection to interior point algorithms. In fact, a polynomial-time algorithm for linear programming problems seems to require a centering component in its search direction [8]. Because of this, properties of the omega central path are important and have been investigated by many researchers [3,4,6,10–12,14,21,22].

In this paper, the limiting derivatives of the omega central path are of interest. For any $\bar{\mu} > 0$, the derivative of $x(\bar{\mu}, b, \omega)$ is denoted by
\[
D_{\mu}x(\bar{\mu}, b, \omega) = \lim_{\mu \to \bar{\mu}} \frac{x(\mu, b, \omega) - x(\bar{\mu}, b, \omega)}{\mu - \bar{\mu}}.
\]
Furthermore, we use $D_{\mu^*}x^*(b, \omega)$ to represent $\lim_{\mu \to 0^+} D_{\mu}x(\mu, b, \omega)$. The existence of the first-order limiting derivatives was shown by Witzgall et al. [20] and Adler and Monteiro [1]. Güler [13] established the existence of the higher-order limiting derivatives. This result was used in [9] to show that along any given direction $\delta b$, the analytic center solution is one-sided infinitely, continuously differentiable.

In Section 2, the vector of right-sided derivatives of $x^*(b, \omega)$ with respect to $\mu$, is shown to be uniformly bounded over $\{ \omega \in \mathbb{R}^n_{++}: \| \omega \| = 1 \}$. In Section 3, a similar result is presented for the right-sided marginal derivatives, denoted $D_{\mu^*}x^*(b, \omega)$, where $b_\mu = b + p\delta b$. Notice that for any admissible $b$ and any $\delta b$, $b_\mu$ is admissible for sufficiently small $p$.

2. The limiting derivatives of the omega central path

The components of $D_{\mu^*}x^*(b, \omega)$ are partitioned into $D_{\mu^*}x_N^*(b, \omega)$ and $D_{\mu^*}x_S^*(b, \omega)$. The next lemma shows that $D_{\mu^*}x_N^*(b, \omega)$ is uniformly bounded over a set of normalized weights.

Lemma 1. For any admissible $b$, there exists a constant $\mathcal{M} > 0$ such that
\[
\sup \{ \| D_{\mu^*}x_N^*(b, \omega) \| : \omega \in \mathbb{R}^n_{++}, \| \omega \| = 1 \} \leq \mathcal{M} < \infty.
\]
Proof In all of [1,13,20], it is shown that
\[
D_{\mu^*}x_N^*(b, \omega) = (S_N^*(b, \omega))^{-1} \omega_N.
\] (1)
Consider,
\[
\max \left\{ \sum_{i \in N} \omega_i \ln(s_i): yA_N = c_N, \ yA_N + s_N = c_N, \ s_N > 0 \right\},
\]
with solution $(y^*(b, \omega), s^*_N(b, \omega))$. The necessary and sufficient optimality conditions for this program are the existence of $u^*(b, \omega)$ such that
\[
u^*(b, \omega) > 0,
\] (2)
\[
u^*(b, \omega)A = c_N,
\] (3)
\[
u^*(b, \omega)A_N = c_N,
\] (4)
\[
u^*(b, \omega)A_N + s^*_N(b, \omega) = c_N.
\] (5)
\( S_N^*(b, \omega) u_N^*(b, \omega) = \omega_N, \quad (6) \)
\( s_N^*(b, \omega) > 0. \quad (7) \)

Let \((\hat{u}, (\hat{y}, \hat{s}_N)) = (u^*(b, e), (y^*(b, e), s_N^*(b, e)))\), where \(e\) is the vector of ones. Then,
\( \hat{u}_N > 0, \quad (8) \)
\( A\hat{u} = 0, \quad (9) \)
\( \hat{y}A_N = c_N, \quad (10) \)
\( \hat{y}s_N = c_N, \quad (11) \)
\( \hat{s}_N > 0. \quad (12) \)

From (2), (7), (8), and (12) it follows that for all \( i \in N \),
\( \hat{s}_i u_i^*(b, \omega) \leq \hat{s}_n u_n^*(b, \omega) + s_N^*(b, \omega) \hat{u}_N \). \quad (13)

From (3) and (9), \( u^*(b, \omega) - \hat{u} \in \text{null}(A) \), and from (5) and (11), \((0, s^*_N(b, \omega)) = (0, \hat{s}_N) \in \text{row}(A) \). Since \( \text{null}(A)^\perp = \text{row}(A) \),
\( \hat{s}_N u_N^*(b, \omega) + s_N^*(b, \omega) \hat{u}_N = s_N^*(b, \omega) u_N^*(b, \omega) + \hat{s}_N \hat{u}_N. \quad (14) \)

Combining (6), (13), and (14) with the fact that \( \omega \) is normalized, gives
\( \hat{s}_i u_i^*(b, \omega) \leq \frac{\omega_i}{s_i^*(b, \omega)} \omega_N + \hat{s}_N \hat{u}_N \leq n + \hat{s}_N \hat{u}_N. \quad (15) \)

The following completes the result and holds from (6), (7) and (15),
\( \frac{\omega_i}{s_i^*(b, \omega)} = u_i^*(b, \omega) \leq \frac{n + \hat{s}_N \hat{u}_N}{\hat{s}_i}. \quad (16) \)

What is left to show is that \( D_{\mu \omega} X_N^*(b, \omega) \) is bounded. From Güler [13, Theorem 2.1] and Eq. (1),
\( D_{\mu \omega} X_N^*(b, \omega) = -\Omega_N^{-1/2} X_N^*(b, \omega) (A_B \Omega_N^{-1/2} X_N^*(b, \omega))^\dagger \times A_N(S_N (b, \omega))^{-1} \omega_N \)
\( = -\Omega_B^{-1/2} X_B^*(b, \omega) (A_B \Omega_B^{-1/2} X_B^*(b, \omega))^\dagger \times A_N(D_{\mu \omega} X_N^*(b, \omega), (17) \)

where the superscript “\(+\)" denotes the Moore–Penrose generalized inverse. Hence, Lemma 1 implies that to bound \( D_{\mu \omega} X_N^*(b, \omega) \), only
\( \Omega_N^{-1/2} X_N^*(b, \omega) (A_B \Omega_N^{-1/2} X_N^*(b, \omega))^\dagger \)
needs to be bounded. Lemma 3 provides the needed bound, and depends on Lemma 2 and its subsequent corollary. The absence of a rank requirement makes the statements of Lemma 2 and Corollary 1 slight generalizations of Theorem 1 in [19] due to Stewart. Stewart’s proof, without modification, continues to work for the relaxed statement found in the next lemma. However, the proof of the corollary does not follow from the proof found in [19] without an additional property of the Moore–Penrose generalized inverse. Define \( D_{++} \) to be the set of positive diagonal matrices. For any \( D \in D_{++} \), set \( M_D = D^{1/2} M \) and \( P_D = M M_D^{1/2} \). The following notation is used in the proof of Lemma 2: \( \text{col}(M) = \{Mx: x \in \mathbb{R}^n\} \), \( \text{leftnull}(M) = \{v: vM = 0\} \), and \( \tilde{K} \) is the closure of the set \( K \).

Lemma 2 (Stewart [19]). Let \( M \in \mathbb{R}^{p \times q} \). Then, there is a number \( \mathcal{M} > 0 \), such that
\[ \sup_{D \in D_{++}} \|P_D\| \leq \mathcal{M}. \]

Proof Let
\[ C = \{u \in \text{col}(M): \|u\| = 1\} \]
and
\[ K = \bigcup_{D \in D_{++}} \text{leftnull}(DM). \]

We show that \( \tilde{K} \cap C = \emptyset \), so that
\[ \inf_{(u,w) \in C \times K} \|v - u\| > 0, \]
and then use this bound to complete the result. Suppose for the sake of attaining a contradiction that \( \tilde{K} \cap C \neq \emptyset \). Then, there exists a sequence \( \{v^k \in K\} \rightarrow u \in C \). For every \( k \) there exists \( D^k \in D_{++} \) such that \( v^k \in \text{leftnull}(D^k M) \) and \( u \in \text{col}(M) \). So, \( (D^k)^{1/2} v^k \in \text{leftnull}((D^k)^{1/2} M) \) and \( (D^k)^{1/2} u \in \text{col}((D^k)^{1/2} M) \). Hence,
\[ 0 = v^k D^k u = \sum_{u_i \neq 0} u_i d_i^k v_i. \]

Since \( \|u\| = 1 \), some \( u_i \neq 0 \). However, for sufficiently large \( k, u_i d_i^k > 0 \) when \( u_i \neq 0 \), which implies the contradiction that the right-hand side of the last equality is positive. Hence, \( \tilde{K} \cap C = \emptyset \). Define
\[ \rho \equiv \inf_{(u,w) \in C \times K} \|v - u\| > 0. \]

For any fixed \( D \in D_{++}, \|P_D\| = \max_{\|w\| = 1} \|P_D w\| \).
Let \( \|w\| = 1 \) and \( w = u + v \) where \( u \in \text{col}(M) \) and
v ∈ \text{leftnull}(DM). Since \(D^{1/2}P_Dx\) is the projection of \(D^{1/2}x\) onto \(\text{col}(D^{1/2}M)\), we have
\[
D^{1/2}P_Dw = D^{1/2}P_Du + D^{1/2}P_Dv = D^{1/2}P_Du.
\]
Furthermore, because \(D^{1/2}u ∈ \text{col}(D^{1/2}M), D^{1/2}P_Du = D^{1/2}u\). Hence \(P_Dw = u\), and to show that \(\|P_D\|\) is bounded we may show that \(\|u\|\) is bounded. Without loss of generality we assume that \(u ≠ 0\). Since \((u/\|u\|) \in C\) and \((v/\|u\|) \in \text{leftnull}(DM)\),
\[
1 = \frac{\|w\|}{\|u\|} = \frac{\|u + v\|}{\|u\|} = \frac{\|u\| + \|v\|}{\|u\|} \leq \frac{1}{\rho}.
\]
So, \(\|P_D\| ≤ 1/\rho\), and since \(\rho\) is independent of \(D\), we have that
\[
\sup_{D ∈ D_+} \|P_D\| ≤ \frac{1}{\rho}.
\]

\textbf{Corollary 1.} There exists \(\mathcal{M}\) such that
\[
\sup_{D ∈ D_+} \|(D^{1/2}M)^+D^{1/2}\| ≤ \|M^+\|\mathcal{M}.
\]

\textbf{Proof} From Corollary 1.4.1 in [5],
\[(D^{1/2}M)^+ = M^+(D^{1/2}MM^+)^+.
\]
Hence,
\[
M^+P_D = M^+M(D^{1/2}M)^+D^{1/2} = M^+MM^+(D^{1/2}MM^+)^+D^{1/2}
\]
\[
= M^+(D^{1/2}MM^+)^+D^{1/2} = (D^{1/2}M)^+D^{1/2}.
\]
So,
\[
\sup_{D ∈ D_+} \|(D^{1/2}M)^+D^{1/2}\| = \sup_{D ∈ D_+} \|M^+P_D\|
\]
\[
\leq \|M^+\| \sup_{D ∈ D_+} \|P_D\|
\]
\[
\leq \|M^+\|\mathcal{M}.
\]
which completes the result. \(\square\)

\textbf{Lemma 3.} The matrix in (17) is uniformly bounded over \(\{ω ∈ \mathbb{R}^m_+\}\).

\textbf{Proof} Let \(D = \Omega_B^{-1/2}X_B^*(b, ω)\) and \(M^T = A_B\). Then,
\[
\|\Omega_B^{-1/2}X_B^*(b, ω)(A_B\Omega_B^{-1/2}X_B^*(b, ω))^+\|
\]
\[
= \|D(M^T)^+\| = \|(DM)^+D\|.
\]
From Corollary 1 there exists a constant \(\mathcal{M} > 0\) such that for all \(ω ∈ \mathbb{R}_+^m,\)
\[
\|\Omega_B^{-1/2}X_B^*(b, ω)(A_B\Omega_B^{-1/2}X_B^*(b, ω))^+\|
\]
\[
≤ \|A_B\|\mathcal{M} < \infty. \quad \square
\]

The following theorem is the main result of this section, and it shows that the limiting derivatives of the omega analytic center solution are uniformly bounded over a set of normalized weights.

\textbf{Theorem 1.} For any admissible right-hand side \(b\), there exists a constant \(\mathcal{M} > 0\) such that
\[
sup \{\|D_μx^*(b, ω)\| : \|ω\| = 1, \|ω\| > 0\} ≤ \mathcal{M} < \infty.
\]

\textbf{Proof} Partition the vector of derivatives to obtain
\[
\|D_μx^*(b, ω)\| ≤ \|D_μx_1^*(b, ω)\| + \|D_μx_2^*(b, ω)\|.
\]
Lemma 1, Eq. (16), and Lemma 3, imply the existence of an \(\mathcal{M}_1\) and \(\mathcal{M}_2\) such that
\[
sup \{\|D_μx^*(b, ω)\| : \|ω\| ∈ \mathbb{R}^m_+^\prime, \|ω\| = 1\}
\]
\[
≤ \|A_B\|\mathcal{M}_1\|Ax\|\mathcal{M}_2 + \mathcal{M}_2
\]
\[
≤ \mathcal{M} < \infty. \quad \square
\]

3. The marginal derivatives of the omega central path

In this section, the boundedness of the first-order limiting derivatives is shown to imply that the first-order marginal derivatives are also bounded. This follows because \(D_μx^*(b, ω)\) is expressible as a limiting derivative of an omega central path. The following subset relationships are needed.

\textbf{Lemma 4} (Adler and Monteiro [2]). For a given \(db\) and sufficiently small \(ρ,\)
\(B(b) ⊆ B(b_ρ)\) and \(N(b) ⊇ N(b_ρ)\).

Lemma 4 shows that if \(db\) is a direction of change that forces an immediate change in the optimal partition, then the cardinality of \(B\) increases. To accommodate such a situation, define for sufficiently small
\( \rho, \Delta B = B(b_\rho)/B(b) \). From [7] we know that \( B(b_\rho) \) is invariant on \((0, \hat{\mu})\), for \( \hat{\rho} \) sufficiently small. The establishment of the fact that the marginal derivatives of \( x^*(b, \omega) \) are the limiting derivatives of an omega central path is found in Lemma 5. The main result follows.

**Lemma 5** (Holder et al. [9]). For a given \( \delta b \), let
\[
\{ (z_\mu(\mu), z_{\Delta B}(\mu), \rho(\mu)) : \mu \geq 0 \}
\]
be the \( \omega \) central path for
\[
\min \{ \rho : A_B z_B + A_{\Delta B} z_{\Delta B} - \rho \delta b = b, \quad z_B \geq 0, \quad z_{\Delta B} \geq 0, \quad \rho \geq 0 \}.
\]
Then,
\[
D_{\rho'} x_B^*(b_0, \omega) = D_{\mu'} z_B^*
\]
and
\[
D_{\rho'} x_{\Delta B}^*(b_0, \omega) = D_{\mu'} z_{\Delta B}^*.
\]

**Theorem 2.** For any \( \delta b \), there exists a constant \( \mathcal{M} \) with
\[
\sup \{ \| D_{\rho'} x^*(b_0, \omega) \| : \| \omega \| = 1, \quad \omega > 0 \} \leq \mathcal{M} < \infty.
\]

**Proof** Let \( v \) be the dual slack vector associated with the omega central path in (18). Since the optimal partition for (18) is \( (B | \Delta B \cup \{ \rho \}) \), Eq. (16) and Lemma 4 imply that
\[
D_{\rho'} x_B^*(b_0, \omega) = D_{\mu'} z_B^*(b, \omega)
\]
\[
= - \Omega_B^{-1/2} Z_B^*(b, \omega) A_B \Omega_B^{-1/2} Z_B^* \times (b, \omega)^T A_N D_{\mu'} \left( \frac{x_{\Delta B}^*(b, \omega)}{\rho^*} \right).
\]
Since,
\[
D_{\mu'} \left( \frac{x_{\Delta B}^*(b, \omega)}{\rho^*} \right) = (V_{\Delta B \cup \{ \rho \}}^*)^{-1} \omega_{\Delta B \cup \{ \rho \}},
\]
the result that \( D_{\rho'} x^*_{\{\rho \} \cup \{ \Delta B \}}(b_0, \omega) \) is uniformly bounded over \( \{ \omega > 0 : \| \omega \| = 1 \} \) is analogous to Theorem 1. Furthermore, since \( x_{\Delta B}^*(b_\rho, \omega) = 0 \) for sufficiently small \( \rho \), \( D_{\rho'} x_{\Delta B}^*(b_\rho, \omega) = 0 \), and the proof is complete. \( \square \)

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