Asymptotics for renewal-reward processes with retrospective reward structure

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Abstract

Let \( \{X_i, Y_i; i = \ldots, -1, 0, 1, \ldots \} \) be a doubly infinite renewal-reward process, where \( \{X_i; i = \ldots -1, 0, 1, \ldots \} \) is an i.i.d. sequence of renewal cycle lengths and \( Y_i = g(X_{i-q}, X_{i-q+1}, \ldots, X_i) \) is the lump reward earned at the end of the \( i \)th renewal cycle, with some function \( g: \mathbb{R}^{q+1} \rightarrow \mathbb{R} \). Starting with the first renewal cycle (of duration \( X_1 \)) at the time origin, let \( C(t) \) denote the expected cumulative reward earned in \((0, t]\). In this paper, an asymptotic representation for \( C \) of the form

\[
C(t) = t + o(1) \quad t \to \infty
\]

is derived. An application of this result in single item replacement modelling is discussed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Motivating example and main result

Consider the following example from replacement modelling. A customer needs a certain item of equipment continuously. He buys the equipment from the same supplier and operates a failure replacement policy with instantaneous replacement upon failure. On delivery, the usual price payable is \( K_1 \). However, if the total of the lifetimes of the last \( q \) + 1 items (\( q \geq 1 \)) received is below some threshold \( t_0 \), the supplier is willing to charge a discounted price \( K_2 < K_1 \) (as a sign of goodwill and so as to compensate for the inconvenience of what the customer may perceive as a premature failure). We are interested in the long-term expected price per unit time this policy costs the customer.

The above problem is identical to that of a usual failure replacement policy except that the replacement price payable is dependent not only on the duration of the most recent lifetime, but also on that of the previous \( q \) periods. While the usual failure replacement problem is modelled by a renewal-reward process [5], here, the following novel modelling framework is appropriate. The lifetime variables \( X_1, X_2, \ldots \) are i.i.d. and onto them a sequence of random rewards \( Y_1, Y_2, \ldots \) is imposed where \( Y_i \) is interpreted as the reward earned (or cost payable) at the time of the end of the \( i \)th renewal cycle. As opposed to the usual assumption, where it is assumed that \( (X_1, Y_1), (X_2, Y_2), \ldots \) is an independent sequence of random vectors, we now require that \( Y_i \)
be some function of the lengths of the last \(q+1\) cycles, i.e.

\[
Y_i = g(X_{i-q}, X_{i-q+1}, \ldots, X_i)
\]

with some function \(g: \mathbb{R}^{q+1} \to \mathbb{R}\). In our specific example,

\[
g(x_0, \ldots, x_q) = \begin{cases} 
K_1 & \text{if } x_0 + \cdots + x_q > t_0, \\
K_2 & \text{if } x_0 + \cdots + x_q \leq t_0.
\end{cases}
\]

We are interested in the asymptotic behaviour of \(C(t)\), the total expected cost during the time interval \((0, t]\), assuming that at \(t = 0\) a new renewal interval is started. To state our result in precise terms, we want first to describe in more detail the framework used. For mathematical reasons we assume that we are given a doubly infinite sequence of pairs of random variables \(\ldots, (X_{-1}, Y_{-1}), (X_0, Y_0), (X_1, Y_1), \ldots\) where \(\ldots X_{-1}, X_0, X_1, \ldots\) is a doubly infinite i.i.d. sequence of non-negative, non-lattice random variables (the lengths of the renewal cycles) and the \(Y_i\) are given by (1) with a Borel-measurable \(g: \mathbb{R}^{q+1} \to \mathbb{R}, \ g \geq 0\). \(Y_i\) is interpreted as the reward received at the end of the \(i\)th renewal cycle. The time origin is placed at the beginning of the first renewal cycle, and, as a consequence of this, the reward \(Y_i\) is received at time \(t = X_1 + \cdots + X_i\) if \(i \geq 0\) and at \(t = -X_{i+1} - \cdots - X_0\) if \(i \leq -1\). (The empty sum is by definition zero.)

The assumption of a doubly infinite sequence of renewal lengths is a mathematical device allowing all the costs \(Y_i\) to be written in the same form (1); as a consequence, (1) holds for all renewal intervals, especially also those in the start-up phase. Of course, since we have a steady-state analysis, its result will not be affected by using this device.

For \(t > 0\), \(N(t)\) denotes, as usual, the number of renewals in \((0, t]\), i.e.

\[
N(t) = \sum_{i=1}^{\infty} I_{\{X_i + \cdots + X_{i+1} < t\}}
\]

where \(I_{\{\cdot\}}\) stands for the indicator function of the event \(\{\ldots\}\). Notice that the ‘initial renewal’ at \(t = 0\) is not counted in this definition of \(N\). The total reward received during \((0, t]\) is then

\[
Y(t) = \sum_{i=1}^{N(t)} Y_i,
\]

and its expected value is

\[
C(t) = E(Y(t)) = E \left( \sum_{i=1}^{N(t)} Y_i \right).
\]

Our main result, as stated in the next Theorem, is an asymptotic representation of \(C(t)\) of the form

\[
C(t) = \zeta t + \eta + o(1), \quad t \to \infty,
\]

where \(o(1)\) stands for a term which tends to zero as \(t\) is taken to infinity.

**Theorem.** Under the above assumptions, and assuming that all quantities in the formulae below are finite, the asymptotic form of \(C(t)\) is given by (3) with

\[
\zeta = E(g(X_0, X_1, \ldots, X_q))/E(X_0),
\]

\[
\eta = \left\{ q + 1 + \frac{\text{Var}(X_0) - (E(X_0))^2)}{2(E(X_0))^2} \right\} E(g(X_0, X_1, \ldots, X_q)) - \sum_{j=0}^{q} \frac{E(X_j g(X_0, X_1, \ldots, X_q))}{E(X_0)}.
\]
The proof of the Theorem will be given in Section 2. There, beginning as in the standard case (see e.g. [4]), first the total random reward accumulated in $(0, t]$ is written as a random sum of random variables plus a remainder term such that the upper limit in the summation is a stopping time with respect to the summands involved. Then, Wald’s Equation is invoked to obtain the expectation of the random sum and some additional work is performed to deal with the expectation of the remainder term. Dealing with this latter point will turn out to be the most intricate and non-standard part of the proof. The corresponding result is summarised as a Lemma in Section 2 and its proof is given in the appendix. A discussion in Section 3 closes the paper.

The notation used here is standard in OR except perhaps that for integrals: at times, $\int f(x)P(X \in dx)$ is used to denote the integral over $f$ with respect to the probability measure generated by the random variable $X$. This notation is used for example in [2] which is referenced here also for its general measure-theoretic viewpoint underlying this paper.

Let us conclude this section with the introductory example from replacement modelling. To represent the quantities involved, we shall need $F(t) = P(X_0 \leq t)$, the cumulative distribution function of the equipment lifetime, and $F^{(n)}(t) = P(X_1 + \cdots + X_n \leq t)$, for which it is easily shown that

$$F^{(1)}(t) = F(t), \quad F^{(n)}(t) = \int_0^t F^{(n-1)}(t-x) dF(x), \quad n \geq 2.$$  

Furthermore, it will prove useful to indicate the number of arguments of the $g$-function and its dependence on $t_0$ by appropriate subscripting. With this notation then, the following holds.

$$E(g_{t_0, q+1}(X_0, \ldots, X_q)) = K_1 + (K_2 - K_1)F^{(q+1)}(t_0),$$

and

$$E(X_j g_{t_0, q+1}(X_0, \ldots, X_q))$$

$$= \int E(x g_{t_0, q+1}(x, X_1, \ldots, X_q) | X_0 = x) P(X_0 \in dx)$$

$$= \int x E(g_{t_0-x, q}(X_1, \ldots, X_q)) P(X_0 \in dx)$$

$$= \int x (K_1 + (K_2 - K_1)F^{(q)}(t_0 - x)) P(X_0 \in dx)$$

$$= K_1 \int_0^\infty x dF(x) + (K_2 - K_1) \int_0^{t_0} x F^{(q)}(t_0 - x) dF(x).$$

For $t_0 = 0$, we have the usual failure replacement policy and the above result becomes

$$E(g_{0, q+1}(X_0, \ldots, X_q)) = K_1$$

and

$$E(X_j g_{0, q+1}(X_0, \ldots, X_q)) = K_1 \int_0^\infty x dF(x),$$

implying the familiar formulae (e.g. [4])

$$\zeta = K_1/E(X_0), \quad \eta = \frac{\text{Var}(X_0) - (E(X_0))^2}{2(E(X_0))^2} K_1.$$
2. Proof of Theorem

Let us first turn to the required representation of $Y(t)$. Obviously, (2) can be written as

$$Y(t) = \sum_{i=1}^{M_q(t)} Y_i - \sum_{j=1}^{g+1} Y_{N(t)+j},$$

with $M_q(t) = N(t) + 1 + q$. The advantage of this representation is that $M_q(t)$ is a stopping time for the sequence $Y_1, Y_2, \ldots$, i.e. the event $\{M_q(t) = m\}$ is independent of $Y_{m+1}, Y_{m+2}, \ldots$ for all $m \geq 1$. To see this, we note that because of

$$N(t) + 1 = n \iff N(t) = n - 1 \iff X_1 + \cdots + X_{n-1} \leq t < X_1 + \cdots + X_n,$$

the event $\{N(t) + 1 = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$, from which it follows that $\{N(t) + 1 = n\}$ is independent of $g(X_{n+1}, X_{n+2}, \ldots, X_{n+q+1})$, $g(X_{n+2}, X_{n+3}, \ldots, X_{n+q+2}), \ldots$, i.e. $\{N(t) + 1 = n\}$ is independent of $Y_{n+1+q}, Y_{n+2+q}, \ldots$. The last event can be written as $\{M_q(t) = m\}$ with $m = n + q$ from which it is finally seen that $\{M_q(t) = m\}$ is independent of $Y_{m+1}, Y_{m+2}, \ldots$. For the sake of completeness, we state Wald’s Equation in the form needed.

**Proposition (Wald’s Equation).** Let $Y = (Y_1, Y_2, \ldots)$ be a sequence of random variables with the same finite mean $E(Y_1) = E(Y_2) = \ldots$ and let $v \geq 1$ be an integer-valued random variable with $E(v) < \infty$. If $v$ is a stopping time for the sequence of random variables $Y$ (i.e. if $\{v = i\}$ is independent of $Y_i, Y_{i+1}, \ldots$ for all $i$) then the random variable $Z = Y_1 + \cdots + Y_v$ has finite mean and it is $E(Z) = E(Y_1)E(v)$.

**Proof.** The proof as outlined in [5, pp. 349–350] for the case of an i.i.d. sequence $Y$ carries over unchanged.

By Wald’s Equation we get from (4) that

$$E(Y(t)) = E(M_q(t))E(Y_1) - \sum_{j=1}^{q+1} E(Y_{N(t)+j}). \tag{5}$$

The first term on the r.h.s. of (5) is dealt with in the usual manner by using the well-known expansion of the renewal function

$$E(M_q(t)) = E(N(t)) + q + 1 = t E(X_0) + \frac{\text{Var}(X_0) - (E(X_0))^2}{2(E(X_0))^2} + q + o(1).$$

The second term on the r.h.s. of (5) is addressed in the Lemma below and its proof can be found in the appendix.

**Lemma.** For $q \geq 0$ let $g: \mathbb{R}^{q+1} \to \mathbb{R}$ be a bounded, Borel-measurable function. For $j \in \{1, \ldots, q+1\}$ and $t \in [0, \infty)$ let

$$\gamma(t) = E(Y_{N(t)+j}) = E(g(X_{N(t)+j}, \ldots, X_{N(t)+j})).$$

Then, for $t \to \infty$ it is the case that

$$\gamma(t) = \frac{E(X_{q+1-j}g(X_0, X_1, \ldots, X_q))}{E(X_0)} + o(1).$$

The proof idea of the Lemma involves establishing first a general integral equation for a certain conditional expectation of $Y_{N(t)+j}$ which then in its partly integrated form gives rise to a renewal equation to which the
Key Renewal Theorem can be applied. The result of the Lemma specialises to a known formula for $q = 0$ (see [7]).

3. Discussion and future work

Our results are not an exhaustive treatment of all aspects of renewal reward theory with a retrospective reward structure. Naturally, all that has been known for the standard case may also be asked here. For example, one might also consider continuously accruing rewards (rather than just lump rewards), refined asymptotics for the variance of the accumulated reward [3]) and extended renewal theory [1] just to name a few possibilities. The reward structure can also be made more complicated by allowing $Y_i$ to depend (in addition to $X_i − q, X_i − q + 1, \ldots, X_i$) on some other random variable $Z_i$, itself independent of the cycle durations. It is hoped that the paper will arouse interest in, and show an approach to, dealing with the retrospective reward structure and that the method presented here should be applicable in all other related but more complex situations. We also note in passing that some tools from measure theory (the Monotone Class Theorem, [6]) seem indispensable in our method (see the appendix), emphasising the need for a broad set of tools in probability models in OR.

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Appendix A. Proof of Lemma

Throughout this appendix, all results will be displayed in their most general form but, to keep the notational complexity manageable and for the sake of clarity, the ideas used will be demonstrated by way of selected special cases. These special cases will convey the main essence of the method used and they will also allow the general pattern of the resulting formulae to be recognised.

A.1. The basic integral equation

Proposition. For $q \geq 0$, $j = 1, \ldots, q + 1$ and $t \in [0, \infty)$ put

$$ G(t; x_{q+1}, x_{q+2}, \ldots, x_0) = E(Y_{N(t)+j} | X_{q+1} = x_{q+1}, X_{q+2} = x_{q+2}, \ldots, X_0 = x_0). \quad (A.1) $$

Then, the following integral equation holds

$$ G(t; x_{q+1}, x_{q+2}, \ldots, x_0) = \int_{[0,t]} G(t - x_1; x_{q+2}, x_{q+3}, \ldots, x_1) P(X_1 \in dx_1) \right. $$

$$ \left. + \int_{(t, \infty)} E(g(x_{q+j}, \ldots, x_0, x_1, x_2, \ldots, x_j)) P(X_1 \in dx_1). \quad (A.2) $$

Proof. The proof relies on the process’s regenerative structure. For the case $q = 1$ and $j = 1$, we have an example for when in the second integrand on the r.h.s. of (A.2) none of the random variables occurs.
\[ G(t; x_0) = E(Y_{N(t) + 1} | X_0 = x_0) \]
\[ = \int_{[0, t]} g(Y_{N(t)} + 1, X_0 = x_0, X_1 = x_1) P(X_1 \in d x_1) + \int_{(t, \infty)} g(Y_{N(t)}, X_0 = x_0, X_1 = x_1) P(X_1 \in d x_1) \]
\[ = \int_{[0, t]} E(Y_{N(t) + 1} | X_0 = x_1) P(X_1 \in d x_1) + \int_{(t, \infty)} E(Y_{N(t)}, X_0 = x_0, X_1 = x_1) P(X_1 \in d x_1) \]
\[ = \int_{[0, t]} g(t - x_1; x_1) P(X_1 \in d x_1) + \int_{(t, \infty)} g(x_0, x_1) P(X_1 \in d x_1). \] (A.3)

To see the above, observe that \( X_1 = x_1 \leq t \Rightarrow N(t) + 1 \geq 2 \) and that then the value of \( X_0 \) will have no bearing on the cost in the second and in any of the subsequent periods. Also, we may move the time origin to the right by \( x_1 \) units and there restart a probabilistically identical process. The second integral in (A.3) arises by observing that \( X_1 = x_1 > t \Rightarrow N(t) + 1 = 1. \)

The case \( q = 1 \) and \( j = 2 \) gives rise to a r.h.s. in (A.2) where the random variable \( X_2 \) truly occurs. The corresponding equation now reads as
\[ G(t; x_0) = E(Y_{N(t) + 2} | X_0 = x_0) \]
\[ = \int_{[0, t]} E(Y_{N(t) + 2} | X_0 = x_0, X_1 = x_1) P(X_1 \in d x_1) + \int_{(t, \infty)} E(Y_{N(t) + 2}, X_0 = x_0, X_1 = x_1) P(X_1 \in d x_1) \]
\[ = \int_{[0, t]} E(Y_{N(t) + 2} | X_0 = x_1) P(X_1 \in d x_1) + \int_{(t, \infty)} E(Y_{N(t) + 2}, X_0 = x_0, X_1 = x_1) P(X_1 \in d x_1) \]
\[ = \int_{[0, t]} g(t - x_1; x_1) P(X_1 \in d x_1) + \int_{(t, \infty)} g(x_1, X_2) P(X_1 \in d x_1). \] (A.3)

A.2. Proof of Lemma

The proof of the Lemma will be conducted by using the Key Renewal Theorem on a renewal equation which will be obtained from the integral equation (A.2) by integrating both of its sides with respect to \( P(X_{-q+1} \in d x_{-q+1}, X_{-q+2} \in d x_{-q+2}, \ldots, X_0 \in d x_0) \). As a preliminary step, we are going to examine first \( g \)-functions of the following specific form:
\[ g(x_0, x_1, \ldots, x_q) = I_{B_0 \times B_1 \times \ldots \times B_q} (x_0, x_1, \ldots, x_q). \] (A.4)

These are indicator functions of Cartesian products of subsets of the real line (also called 'cylinder sets'). We also assume that \( B_1, \ldots, B_q \) are measurable. To start with, in the next proposition an explicit form is given of the conditional expectation in (A.1).

**Proposition.** For functions of the form (A.4), it is for \( j = 1, \ldots, q + 1, \)
\[ E(I_{B_0 \times \ldots \times B_q} (X_{N(t) - q + j}, \ldots, X_{N(t) + j}) | X_{-q+1} = x_{-q+1}, \ldots, X_0 = x_0) \]
\[ = E(I_{B_0 \times \ldots \times B_q} (X_{N(t) - q + j}, \ldots, X_{N(t) + j})) \]
\begin{align}
& + \sum_{k=0}^{q-j} \{ E(I_{B_0} \cdots \times I_{B_q}(X_{-k}, X_{-k+1}, \ldots, X_0, X_1, \ldots, X_{-k+q}) I_{(X_1, \ldots, X_{q-j-k} \leq t < X_1, \ldots, X_{q-j-k+1})}) \\ 
& - E(I_{B_0} \cdots \times I_{B_q}(X_{-k}, X_{-k+1}, \ldots, X_{-k+q}) I_{(X_1, \ldots, X_{q-j-k} \leq t < X_1, \ldots, X_{q-j-k+1})}) \}. 
\end{align}

(The empty sum is zero by definition.)

\textbf{Proof.} We shall reason on the case \( q = 2, \ j = 1 \). By the definition of the conditional expectation (as is usually defined in measure theory; see, e.g., [2]) we may concentrate on events of the form

\[ D = \{ X_{-1} \in A_{-1}, X_0 \in A_0, X_{N(t)-1} \in B_0, X_{N(t)} \in B_1, X_{N(t)+1} \in B_2 \}. \]

By the equivalences \( N(t) \geq 2 \iff X_1 + X_2 \leq t, \ N(t) = 1 \iff X_1 \leq t < X_1 + X_2 \) and \( N(t) = 0 \iff X_1 > t \) it is easily seen that

\[
P(D) = P(X_{-1} \in A_{-1}, X_0 \in A_0) P(X_{N(t)-1} \in B_0, X_{N(t)} \in B_1, X_{N(t)+1} \in B_2, N(t) \geq 2) \\
+ P(X_{-1} \in A_{-1}, X_0 \in A_0 \cap B_0, X_1 \in B_1, X_2 \in B_2, X_1 \leq t < X_1 + X_2) \\
+ P(X_{-1} \in A_{-1} \cap B_0, X_0 \in A_0 \cap B_1, X_1 \in B_2, X_1 > t) \\
= P(X_{-1} \in A_{-1}, X_0 \in A_0)\{ P(X_{N(t)-1} \in B_0, X_{N(t)} \in B_1, X_{N(t)+1} \in B_2) \\
- P(X_{N(t)-1} \in B_0, X_{N(t)} \in B_1, X_{N(t)+1} \in B_2, N(t) \leq 1) \} \\
+ P(X_{-1} \in A_{-1}) P(X_0 \in A_0 \cap B_0) P(X_1 \in B_1, X_2 \in B_2, X_1 \leq t < X_1 + X_2) \\
+ P(X_{-1} \in A_{-1} \cap B_0) P(X_0 \in A_0 \cap B_1) P(X_1 \in B_2, X_1 > t) \\
= P(X_{-1} \in A_{-1}, X_0 \in A_0)\{ P(X_{N(t)-1} \in B_0, X_{N(t)} \in B_1, X_{N(t)+1} \in B_2) \\
- P(X_0 \in B_0, X_1 \in B_1, X_2 \in B_2, X_1 \leq t < X_1 + X_2) \\
- P(X_{-1} \in B_0, X_0 \in B_1, X_1 \in B_2, X_1 > t) \} \\
+ P(X_{-1} \in A_{-1}) P(X_0 \in A_0 \cap B_0) P(X_1 \in B_1, X_2 \in B_2, X_1 \leq t < X_1 + X_2) \\
+ P(X_{-1} \in A_{-1} \cap B_0) P(X_0 \in A_0 \cap B_1) P(X_1 \in B_2, X_1 > t). \tag{A.6}
\]

The rules of the conditional probability [2] allow us to infer from (A.6) that

\[
P(X_{N(t)-1} \in B_0, X_{N(t)} \in B_1, X_{N(t)+1} \in B_2 | X_{-1} = x_{-1}, X_0 = x_0) \\
= P(X_{N(t)-1} \in B_0, X_{N(t)} \in B_1, X_{N(t)+1} \in B_2) \\
+ I_{B_0}(x_0) P(X_1 \in B_1, X_2 \in B_2, X_1 \leq t < X_1 + X_2) - P(X_0 \in B_0, X_1 \in B_1, X_2 \in B_2, X_1 \leq t < X_1 + X_2) \\
+ I_{B_0 \times B_1}(x_{-1}, x_0) P(X_1 \in B_2, X_1 > t) - P(X_{-1} \in B_0, X_0 \in B_1, X_1 \in B_2, X_1 > t), \tag{A.7}
\]

which is (A.5) for \( q = 2, \ j = 1 \). \qed

We are now ready to conduct the proof proper of the Lemma. Again, we shall reason on the case \( q=2, \ j=1 \). The renewal equation to be derived will first be established for \( g \)-functions of the form (A.4). From (A.7) we know that in this case \( G \) can be represented in the form

\[
G(t; x_{-1}, x_0) = G_1(t) + a_0(x_0)b_1(t) + a_0(x_{-1})a_1(x_0)b_2(t) - c_1(t) - c_2(t), \tag{A.8}
\]
where
\[ G_1(t) = P(X_{N(t)} \in B_1, X_{N(t)} \in B_1, X_{N(t)+1} \in B_2), \]
\[ a_0(x) = I_{B_0}(x), \quad a_1(x) = I_{B_1}(x), \]
\[ b_1(t) = P(X_1 \in B_1, X_2 \in B_2, X_1 \leq t < X_1 + X_2), \quad b_2(t) = P(X_1 \in B_2, X_1 > t_2), \]
\[ c_1(t) = P(X_0 \in B_0, X_1 \in B_1, X_2 \in B_2, X_1 \leq t < X_1 + X_2), \]
\[ c_2(t) = P(X_{-1} \in B_0, X_0 \in B_1, X_1 \in B_2, X_1 > t). \]

We know that the function \( G \) in (A.8) satisfies the integral equation (A.2); thus, we have
\[
G(t; x_{-1}, x_0) = \int_{[0,t]} G(t - x_1; x_{0}, x_1) P(X_1 \in d x_1) + \int_{(t,\infty)} E(g(x_{-1}, x_0, x_1)) P(X_1 \in d x_1)
\]
\[ = \int_{[0,t]} G_1(t - x_1) P(X_1 \in d x_1) \]
\[ + \int_{[0,t]} a_0(x_1) b_1(t - x_1) P(X_1 \in d x_1) + a_0(x_0) \int_{[0,t]} a_1(x_1) b_2(t - x_1) P(X_1 \in d x_1) \]
\[ - \int_{[0,t]} (c_1(t - x_1) + c_2(t - x_1)) P(X_1 \in d x_1) + a_0(x_{-1}) a_1(x_0) P(X_1 \in B_2, X_1 > t). \quad (A.9) \]

Integrate now both sides of (A.9) with respect to \( P(X_{-1} \in d x_1, X_0 \in d x_0) \) to get
\[
G_1(t) = \int_{[0,t]} G_1(t - x_1) P(X_1 \in d x_1) + h(t), \quad (A.10)
\]
where
\[
h(t) = \int_{[0,t]} a_0(x_1) b_1(t - x_1) P(X_1 \in d x_1) + P(X_0 \in B_0) \int_{[0,t]} a_1(x_1) b_2(t - x_1) P(X_1 \in d x_1) \]
\[ - \int_{[0,t]} (c_1(t - x_1) + c_2(t - x_1)) P(X_1 \in d x_1) + P(X_{-1} \in B_0) P(X_0 \in B_1) P(X_1 \in B_2, X_1 > t). \]

It is now easily verified that
\[
\int_{[0,t]} a_0(x_1) b_1(t - x_1) P(X_1 \in d x_1) = E(g(X_0, X_1, X_2) I_{\{X_0 + X_1 \leq t < X_0 + X_1 + X_2\}},
\]
\[
\int_{[0,t]} a_1(x_1) b_2(t - x_1) P(X_1 \in d x_1) = P(X_1 \in B_1, X_2 \in B_2, X_1 \leq t < X_1 + X_2),
\]
\[
\int_{[0,t]} c_1(t - x_1) P(X_1 \in d x_1) = E(g(X_0, X_1, X_2) I_{\{X_1 + X_2 \leq t < X_1 + X_2 + X_3\}}),
\]
\[
\int_{[0,t]} c_2(t - x_1) P(X_1 \in d x_1) = E(g(X_{-1}, X_0, X_1) I_{\{X_2 \leq t < X_1 + X_2\}}),
\]
from which it is seen that
\[
h(t) = E(g(X_0, X_1, X_2) I_{\{X_0 + X_1 \leq t < X_0 + X_1 + X_2\}}) + E(g(X_0, X_1, X_2) I_{\{X_1 \leq t < X_1 + X_2\}})
\]
\[-E(g(X_0,X_1,X_2)I_{\{X_1+X_3 \leq t < X_1+X_2+X_3\}}) - E(g(X_1,X_0,X_1)I_{\{X_1 \leq t < X_1+X_2\}})\]
\[+ E(g(X_1,X_0,X_1)I_{\{X_1 > t\}}).\]  
(A.11)

Because of \(G_1(t) = E(g(X_{N(t)}-1,X_{N(t)},X_{N(t)+1}))\), the relationship as defined by (A.10) and (A.11) can be viewed as an equation in \(g\). We know that it is satisfied by all \(g\) which are indicator functions of (Borel-measurable) cylinder sets in the \((q+1)\)-dimensional Cartesian space. The equation is linear in \(g\), the system of cylinder sets is closed under the operation “\(\cap\)” and it generates the Borel \(\sigma\)-algebra. By the Monotone Class Theorem [6, p. 37], (A.10) therefore holds for any bounded measurable function \(g\). By integrating (A.11), it is seen (by Fubini) that
\[
\int_0^\infty h(t) \, dt = E(X_2g(X_0,X_1,X_2)),
\]
from which by the Key Renewal Theorem we infer that as \(t \to \infty\),
\[
E(Y_{N(t)+1}) = E(g(X_{N(t)}-1,X_{N(t)},X_{N(t)+1})) = G_1(t) = E(X_2g(X_0,X_1,X_2))/E(X_1) + o(1).
\]
(From (A.11) it is easily seen that \(h\) is a finite sum of monotone integrable functions and thus the Key Renewal Theorem is applicable; see [5].) \(\square\)

References