Inventory systems for goods with censored random lifetimes

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Abstract

We consider a Poisson inventory model for perishable goods in which the items have random lifetimes and are scrapped either when reaching the end of their lifetime or a fixed constant expiration age. The crucial process to describe this system is the virtual death process \((W(t))_{t \geq 0}\), where \(W(t)\) is the residual waiting time after time \(t\) until the next ‘death’ of an item if there were no demand arrivals after \(t\). We derive its stationary law in closed form and determine the distribution of the number of items in the system (also in steady state). © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

This note generalizes the perishable inventory system (PIS) introduced in [9] by adding the new feature that the items in the system can lose their value already before their expiration date. It is assumed that the arrival times of the items and those of the demands form independent Poisson processes of rates \(\lambda\) and \(\mu\), respectively. The items are stored on an infinite shelf waiting for demands. If there are items on the shelf upon the arrival of a demand, the oldest one is taken away; demands that enter an empty system immediately leave unsatisfied. Any item has a random lifetime (with general lifetime distribution function \(H(\cdot)\)) which is independent of those of the other items and of the arrival times. It is scrapped either when reaching a fixed expiration age, which we set equal to one time unit, or when reaching the end of its lifetime, whichever is smaller. PIS models in which the input is governed by an \((s, S)\) ordering policy were investigated by Liu [7], Kalpakam and Sapna [4–6], Ravichandran [13], and Perry and Posner [10] usually with Poisson demands and exponential lead-times. A PIS with demands that are willing to wait for a certain time and with state-dependent item and demand arrival rates was studied in [11]. For diffusion models of related PISs we refer to Bar-Ilan [1,12]. Earlier work on PISs was reviewed by Nahmias [8].

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The crucial process that describes this system is the \textit{virtual death process} \( W = (W(t))_{t \geq 0} \), where \( 1 - W(t) \) is the age of the oldest item present at time \( t \), if \( W(t) \leq 1 \), and \( W(t) - 1 \) is the residual waiting time until the arrival of the next item, if \( W(t) > 1 \). In other words, \( W(t) \) is the residual time until the next outdating after \( t \), provided that no demands occur after \( t \).

A typical path of \( W(\cdot) \) is shown in Fig. 1. \( W(0) = w_0 > 1 \) means that the first item arrives at time \( w_0 - 1 \). The jump at time \( t_i, i = 1, 2, 3, 5 \), can be due to a demand arrival or to a random ‘death’ of the oldest item present at time \( t_i \). For example, after the jump at \( t_2 \) the item that arrived at time \( s_1 \) becomes the oldest item. At time \( t_4 \) the oldest item reaches age 1 and is therefore outdated and thus removed, leaving an empty shelf behind. During \( (s_2, t_5) \) neither new items nor demands arrive; when the item that arrived at \( s_2 \) is taken away at \( t_5 \) either by demand or by random death, it is still the only item in the system.

We will determine the steady-state law of \( W \) in closed form and apply our result to several common lifetime distributions. From this key result many important characteristics of the model concerning the unsatisfied demands, the outdatings and the number of items in the system can be derived. To illustrate this point, let \( W_e \) be a random variable having the steady-state law of \( W \). Then, clearly the probability of an arbitrary demand not to be satisfied is \( P(W_e > 1) \), so that \( \mu P(W_e > 1) \) is the rate of unsatisfied demands. Furthermore, we will see below that the density \( f(x) \) of \( W_e \) is equal to the long-run average number of downcrossings per unit time of level \( x \). Since an outdating occurs if and only if \( W(\cdot) \) hits zero, the rate of the outdating process is equal to \( f(0) \). Finally, if there are items in the system, one of them is the oldest one, while the others have arrived during its shelf lifetime and have not yet perished. This idea will be used in the next section to express the steady-state distribution of the number of items in the system in terms of that of \( W_e \).

Let \( H(x) \) be the distribution function of an item’s lifetime. The density of \( H \) is assumed to exist, and we denote the corresponding failure rate by \( r(x) \). For simplicity we suppose that \( r \) is continuous on \([0, 1)\).

In practice, the underlying lifetime distribution is sometimes of a simple discrete type. For example, in a blood bank the donated blood portions that have not been used within a certain time period of prespecified length (usually 21 days) are inspected; if found to be in proper condition (which happens with probability \( p \in (0, 1) \)), they are kept in the system for potential demands for an extra time of given length. In our model this procedure leads to a two-point distribution \( H \) with atoms at some \( a \in (0, 1) \) and at 1 with probabilities \( p \) and \( 1 - p \), respectively. It will be seen that our result on \( W \) also applies to this situation by simply replacing \( r(x) \) by \( dH(x)/(1 - H(x)) \).

\section{The steady-state distribution}

\( W \) is a regenerative process. By means of level-crossing theory (LCT) it can be shown that \( W \) has an absolutely continuous stationary distribution; we denote its density by \( f \). This powerful technique is expounded
Fig. 2. A jump from $W(t)=w$ to $W(t+)>x$ occurs if the oldest item is removed at time $t$ and no item having arrived in $(t-(1-w),t-(1-x)]$ is still present at time $t$.

e.g. by Cohen [2] and Doshi [3]; it is based on the fact that the long-run rates at which the sample paths $W(\cdot)$ downcross or upcross specific levels are equal and that the long-run downcrossing rate of a level $x$ is equal to $f(x)$. In our situation we obtain for $f$ the following Pollaczek–Khintchine-type equation:

$$f(x) = \int_0^x (\mu + r(1-w)) \exp \left\{ -\lambda \int_w^x (1-H(1-y)) \, dy \right\} f(w) \, dw$$

$$+ f(0) \exp \left\{ -\lambda \int_0^x (1-H(1-y)) \, dy \right\}$$

if $0 < x \leq 1$  \hspace{1cm} (2.1)

and

$$f(x) = ce^{-\lambda(x-1)}$$

if $x > 1$ \hspace{1cm} (2.2)

for some constant $c$. Moreover, $f$ satisfies the continuity condition $f(1-)=f(1+)$ and the normalizing condition $\int_0^\infty f(x) \, dx = 1$.

To derive (2.1), suppose that the virtual death process is at level $w \in (0,1]$ at some time $t$, i.e. the oldest item in the system is of age $1-w$. Reflecting on Fig. 2, it is easily seen that there is an instantaneous upcrossing of level $x$ at that time if and only if the following two events occur:

(a) The oldest item is removed either by an arriving demand or by death. The rate of this to happen is $\mu + r(1-w)$.

(b) None of the items that have entered the system during the time interval $(t-(x-w)-(1-x),t-(1-x)]$ is still present at time $t$ (as otherwise $W(t)$ would still be less than $x$ after the jump) (see Fig. 2). Conditioning on the number of arrivals in an interval, the arrival times are i.i.d. and uniform so that the probability that all these items are gone at time $t$ is equal to

$$\sum_{n=0}^\infty e^{-\lambda(x-w)} \frac{(\lambda(x-w))^n}{n!} \left( \frac{1}{x-w} \int_0^{x-w} H(1-w-u) \, du \right)^n = \exp \left\{ -\lambda \int_w^x (1-H(1-y)) \, dy \right\}.$$
where \( \phi(x) = \exp\{ -\lambda \int_0^x (1 - H(1 - y)) \, dy \} \) and \( K(x, w) = (\mu + r(1 - w)) \phi(x)/\phi(w) \). Assume that the constant \( f(0) \) is known. Then the unique solution of (2.1) is given by the Neumann series

\[
    f(x) = f(0) \phi(x) + \sum_{n=1}^{\infty} \int_0^x K^{(n)}(x, w) f(0) \phi(w) \, dw, \quad 0 \leq x \leq 1,
\]

which converges for all \( x \in [0, 1] \); here \( K^{(n)} \) is defined recursively by \( K^{(1)} = K, K^{(n+1)}(x, w) = \int_0^w K(x, y) K^{(n)}(y, w) \, dy, n \geq 1 \). The constants \( c \) and \( f(0) \) can then be determined from the two equations

\[
    c = f(1+) = f(1-) \quad \text{and} \quad \int_0^\infty f(x) \, dx = 1.
\]

However, in our situation there is a simpler way to solve for \( f \). Let \( g(x) = f(x)/\phi(x) \). Then it follows from (2.1) that

\[
    g(x) = f(0) + \int_0^x (\mu + r(1 - w)) g(w) \, dw, \quad 0 < x \leq 1.
\]

Thus, \( g \) satisfies \( g'(x) = (\mu + r(1 - x)) g(x) \) on \( (0, 1] \), so that

\[
    g(x) = C \exp\{ \mu x + R(x) \}, \quad 0 < x \leq 1, \quad (2.3)
\]

where \( R(x) = \int r(1 - x) \, dx \) is an indefinite integral of \( r(1 - x) \) and \( C \) is some constant. From (2.3) we obtain

\[
    f(x) = C \exp\{ \mu x + R(x) \} \phi(x)
    = C \exp\left\{ \mu x + R(x) - \lambda \int_0^x (1 - H(1 - w)) \, dw \right\}, \quad 0 < x \leq 1. \quad (2.4)
\]

As \( f(1+) = f(1-) \), (2.2) and (2.4) yield

\[
    f(x) = C \exp\left\{ \lambda \mu + R(1) - \lambda \int_0^1 (1 - H(1 - w)) \, dw \right\} e^{-\lambda x}, \quad x \geq 1. \quad (2.5)
\]

Finally, the constant \( C \) can be determined from the normalizing condition \( \int_0^\infty f(x) \, dx = 1 \). We get

\[
    f(0) = C = \left( \lambda^{-1} \exp\left\{ \mu + R(1) - \lambda \int_0^1 (1 - H(1 - w)) \, dw \right\} \right)^{-1}
    + \int_0^1 \exp\left\{ \mu x + R(x) - \lambda \int_0^x (1 - H(1 - w)) \, dw \right\} \, dx. \quad (2.6)
\]

Eqs. (2.4)–(2.6) provide a closed-form solution for \( f \) if \( H \) is absolutely continuous. In the discrete case let \( 0 < a_1 < \cdots < a_k \leq 1 \) be the atoms of \( H \) in \( (0, 1] \) and \( q(a_i) = H(a_i) - H(a_{i-1}) \). Then it is easily seen (either by properly modifying the above derivation or by approximating \( H \) by a sequence of absolutely continuous distributions) that the density of the (absolutely continuous) distribution of \( W \) is

\[
    f(x) = C \exp\left\{ \mu x - \sum_{i: \ 1 < i < k} \left[ \lambda (1 - H(a_i)) - \frac{q(a_i)}{1 - H(a_i)} \right] \right\}, \quad 0 \leq x \leq 1
\]

and

\[
    f(x) = C \exp\left\{ \lambda \mu - \sum_{i: \ 0 < a_i \leq 1} \left[ \lambda (1 - H(a_i)) - \frac{q(a_i)}{1 - H(a_i)} \right] \right\} e^{-\lambda x}, \quad x \geq 1,
\]

where the constant \( C \) again has to be chosen so as to make \( f \) a probability density.
Once we know \( f \), we can also compute the steady-state distribution of \( K \), the number of items in the system. To this end, note that

\[
\{ K = 0 \} = \{ \text{no items in the system} \} = \{ W_e > 1 \}
\]

and, for \( k \geq 1 \),

\[
\{ K = k \} = \{ \text{there is an oldest item and exactly } k - 1 \text{ other items} \}
\]

arrived during its shelf life and have not yet perished}. To compute \( P(K = k) \) for \( k \geq 1 \), we consider the conditional probability \( P(K = k \mid W_e = w) \), \( k \geq 1 \), \( w \leq 1 \), which has then to be integrated using the stationary distribution \( f(w) \) \( dw \). By stationarity, we may assume that the oldest item (of age \( 1 - w \)) arrived at time 0. Then \( K = k \) means that for exactly \( k - 1 \) items that have arrived in \((0, 1 - w)\) their lifetimes are so long that they are still present at time \( 1 - w \). For an item arriving at time \( y \in (0, 1 - w) \) this occurs with probability \( 1 - H(1 - w - y) \). Since the sequence of lifetimes and the item arrival times are stochastically independent, the arrivals during \((0, 1 - w)\) of items surviving \( 1 - w \) form a nonhomogeneous Poisson process with rate function \( \lambda [1 - H(1 - w - y)] \), \( y \in (0, 1 - w) \). It follows that the generating function \( E(z^K) \), \( |z| \leq 1 \), of \( K \) is given by

\[
E(z^K) = P(W_e > 1) + \sum_{k=1}^{\infty} z^k \int_0^1 P(K = k \mid W_e = w) f(w) \, dw \\
= P(W_e > 1) + \sum_{k=1}^{\infty} z^k \int_0^1 \exp \left\{ - \int_0^{1-w} \lambda (1 - H(1 - w - y)) \, dy \right\} \\
\times \left( \int_0^{1-w} \lambda (1 - H(1 - w - y)) \, dy \right)^{k-1} f(w) \, dw \\
= \int_1^\infty f(w) \, dw + z \int_0^1 f(w) \exp \left\{ -\lambda (1 - z) \int_0^{1-w} (1 - H(y)) \, dy \right\} \, dw.
\]

3. Examples

1. **Uniform distribution**: Let \( H(x) = x/a \), \( 0 \leq x < a \), for some \( a \geq 1 \). Then \( r(x) = (a - x)^{-1} \), \( 0 \leq x < a \). If \( a > 1 \), then by (2.4)–(2.6),

\[
f(x) = C \exp \left\{ \mu x + \int_0^x \frac{dw}{a - 1 + w} - \int_0^x \lambda \left( 1 - \frac{1-w}{a} \right) \, dw \right\} \\
= C (x + a - 1) \exp \left\{ - \frac{1}{a} \left[ \frac{\lambda x^2}{2} + \left( (\lambda - \mu) a - \lambda \right) x \right] \right\}, \quad 0 \leq x < 1
\]

and

\[
f(x) = Ca \exp \left\{ \mu + \frac{\lambda}{2a} - \lambda x \right\}, \quad x \geq 1.
\]

The constant \( C \) is given by

\[
C = \left( \int_0^1 (x + a - 1) \exp \left\{ - \frac{1}{a} \left[ \frac{\lambda x^2}{2} + \left( (\lambda - \mu) a - \lambda \right) x \right] \right\} \, dx \\
+ \frac{a}{\lambda} \exp \left\{ \mu + \frac{\lambda}{2a} - \lambda \right\} \right)^{-1}
\]
It is easily seen that (3.1) and (3.2) also hold for \( a = 1 \), i.e. if \( H \) is the uniform distribution on \([0,1)\); in this case

\[
f(x) = \begin{cases} 
  C x \exp \left\{ \mu x - \frac{\lambda}{2} x^2 \right\} & \text{if } 0 \leq x < 1, \\
  C \exp \left\{ \frac{\lambda}{2} + \mu - \frac{\lambda}{x} \right\} & \text{if } x > 1,
\end{cases}
\]

where

\[
C = \left( \int_0^1 x \exp \left\{ -\frac{\lambda x^2}{2} + \mu x \right\} \, dx + \frac{1}{\lambda} \exp \left\{ -\frac{\lambda}{2} \right\} \right)^{-1}.
\]

Note that \( f(0) > 0 \) if (and only if) \( a > 1 \). Indeed, \( f(0) \) is the rate of the outdating process, and if \( H \) is the uniform distribution on \([0,1)\), there are no outdatings. The same phenomenon occurs in the next example.

2) Triangular density: Let \( H(x) = 1 - (1-x)^2 \), \( 0 \leq x < 1 \). Then \( r(1-w) = 2/w \), \( 0 < w \leq 1 \), and by (2.4),

\[
f(x) = C \exp \left\{ \mu x + R(x) - \int_0^x \frac{\lambda}{2} (1-w)^2 \, dw \right\}
= C x^2 e^{-\lambda x^2 + (x/3) + \mu x}, \quad 0 \leq x < 1.
\]

By (2.5) and (2.6),

\[
f(x) = C e^{(2/3) + \mu - \lambda x}, \quad x \geq 1
\]

and

\[
C = \left( \lambda^{-1} e^{-(\lambda/3) + \mu} + \int_0^1 x^2 e^{-\lambda (x-x^2+(x/3)) + \mu x} \, dx \right)^{-1}.
\]

3) Exponential distribution: Let \( H(x) = 1 - e^{-\eta x} \), \( x \geq 0 \), for some \( \eta > 0 \). A short calculation yields

\[
f(x) = \begin{cases} 
  C \exp \left\{ (\mu + \eta)x - \frac{\lambda}{\eta} e^{-\eta(1-x)} \right\} & \text{if } 0 \leq x < 1, \\
  C \exp \left\{ \frac{\lambda}{\eta} + \mu + \eta - \frac{\lambda}{\eta} - \lambda x \right\} & \text{if } x \geq 1,
\end{cases}
\]

where

\[
C = \left( \lambda^{-1} \exp \left\{ \mu + \eta - \frac{\lambda}{\eta} \right\} + \int_0^1 \exp \left\{ (\mu + \eta) x - \frac{\lambda}{\eta} e^{-\eta(1-x)} \right\} \, dx \right)^{-1}.
\]

4) Pareto distribution: Let \( H(x) = x/(1+x) \), \( x \geq 0 \). Then \( r(x) = 1/(1+x) \), and as above we find that

\[
f(x) = \begin{cases} 
  (2-x)^{\lambda-1} C e^{\mu x} & \text{if } 0 \leq x < 1, \\
  C e^{\lambda x - \lambda x} & \text{if } x \geq 1,
\end{cases}
\]

where again \( C \) has to be chosen so as to make \( f \) a probability density, i.e.

\[
C = \left( \int_{1}^{\infty} e^{\lambda x - \lambda x} \, dx + \int_0^1 (2-x)^{\lambda-1} e^{\mu x} \, dx \right)^{-1}
= \left( \lambda^{-1} e^{\mu} + \int_0^1 (2-x)^{\lambda-1} e^{\mu x} \, dx \right)^{-1}.
\]
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References