A Sokhotski–Plemelj problem related to a robot-safety device system

E.J. Vanderperre*

Department of Quantitative Management, University of South Africa

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Abstract

We introduce a robot-safety device system attended by two different repairmen. The twin system is characterized by the natural feature of cold standby and by an admissible “risky” state. In order to analyse the random behaviour of the entire system (robot, safety device, repair facility), we employ a stochastic process endowed with probability measures satisfying general steady-state differential equations. The solution procedure is based on the theory of sectionally holomorphic functions. An application of the Sokhotski–Plemelj formulae determines the long-run availability of the robot-safety device. Finally, we consider the particular but important case of deterministic repair. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Innovations in the field of microelectronics and micromechanics have enhanced the involvement of “smart” robots in all kind of advanced technical systems [2].

Unfortunately, no robot is completely reliable. Therefore, up-to-date robots are often connected with a safety device [3–5]. Such a device prevents possible damage, caused by a robot failure, in the robot’s neighbouring environment. However, the random behaviour of the entire system (robot, safety device, repair facility) could jeopardize some prescribed safety requirements. For instance, if we allow the robot to operate during the repair time of the failed safety device. Such a “risky” state is called admissible, if the associate event: “The robot is operative but the safety device is under repair”, constitutes a rare event. Therefore, an appropriate statistical analysis of robot-safety device systems is quite indispensable to support the system designer in problems of risk acceptance and safety assessments.

In order to avoid undesirable delays in repairing failed units, we introduce a robot-safety device attended by two different repairmen (henceforth called a $T$-system). The $T$-system satisfies the usual conditions, i.e. independent identically distributed random variables and perfect repair [6].

Each repairman has his own particular task. Repairman $S$ is skilled in repairing the safety unit, whereas...
repairman $\mathcal{R}$ is an expert in repairing robots. Both repairmen are jointly busy if, and only if, both units (robot + safety device) are down. In the other case, at least one repairman is idle. In any case, the safety device always waits (in cold standby [1]) until the repair of the robot has been completed.

In order to analyse the random behaviour of the $T$-system, we introduce a stochastic process endowed with probability measures satisfying general steady-state differential equations. The solution procedure is based on the theory of sectionally holomorphic functions [7]. An application of the Sokhotski–Plemelj formulæ determines the long-run availability of the robot-safety device.

Finally, we consider the particular but important case of deterministic repair.

### 2. Formulation

Consider a $T$-system satisfying the usual conditions.

The robot has a constant failure rate $\lambda > 0$ and a general repair time distribution $R(\bullet), R(0) = 0$ with mean $\mu$ and variance $\sigma^2$.

The operative safety device has a constant failure rate $\lambda_n > 0$ but a zero failure rate in standby (the so-called cold standby state) and a general repair time distribution $R_s(\bullet), R_s(0) = 0$ with mean $\mu_s$ and variance $\sigma^2_s$. The corresponding repair times are denoted by $r$ and $r_s$.

Characteristic functions (and their duals) are formulated in terms of a complex transform variable. For instance,

$$Ee^{i\omega x} = \int_{0}^{\infty} e^{i\omega x} dR(x), \quad \text{Im} \, \omega \geq 0.$$

Note that

$$Ee^{-i\omega x} = \int_{0}^{\infty} e^{-i\omega x} dR(x)$$

$$= \int_{-\infty}^{0} e^{i\omega x} d(1 - R((-x)-)), \quad \text{Im} \, \omega \leq 0.$$

The corresponding Fourier–Stieltjes transforms are called dual transforms.

Without loss of generality (see our forthcoming remarks), we may assume that both repair time distributions have bounded densities (in the Radon–Nikodym sense) defined on $[0, \infty)$.

In order to analyse the random behaviour of the $T$-system, we introduce a stochastic process $\{N_t, t \geq 0\}$ with arbitrary discrete state space $\{A, B, C, D\} \subset [0, \infty)$, characterized by the following events:

- $\{N_t = A\}$: “Both units are operating in parallel at time $t$”.
- $\{N_t = B\}$: “The robot is operative but the safety device is under repair at time $t$”. State $B$ is the so-called risky state.
- $\{N_t = C\}$: “The safety device is in cold standby and the robot is under repair at time $t$”.
- $\{N_t = D\}$: “Both units are simultaneously down at time $t$”.

A Markov characterization of the process $\{N_t\}$ is piecewise and conditionally defined by:

$\{N_t\}$, if $N_t = A$ (i.e. if the event $\{N_t = A\}$ occurs).

- $\{N_t, X_t\}$, if $N_t = B$, where $X_t$ denotes the remaining repair time of the safety device in progressive repair at time $t$.
- $\{N_t, Y_t\}$, if $N_t = C$, where $Y_t$ denotes the remaining repair time of the robot in progressive repair at time $t$.
- $\{(N_t, X_t, Y_t)\}$, if $N_t = D$.

The state space of the underlying Markov process is given by

$$\{A\} \cup \{(B, x); x \geq 0\} \cup \{(C, y); y \geq 0\}$$

$$\cup \{(D, x, y); x \geq 0, y \geq 0\}.$$

Next, we consider the $T$-system in stationary state (the so-called ergodic state) with invariant measure $\{p_K; K = A, B, C, D\}$, $\sum_K p_K = 1$, where

$$p_K := P\{N = K\} := \lim_{t \to \infty} P\{N_t = K\} = A.$$ 

Finally, we introduce the measures

$$\phi_B(x) \, dx := P\{N = B, X \in dx\}$$

$$:= \lim_{t \to \infty} P\{N_t = B, X_t \in dx\} = A,$$

$$\phi_C(y) \, dy := P\{N = C, Y \in dy\}$$

$$:= \lim_{t \to \infty} P\{N_t = C, Y_t \in dy\} = A.$$
\[ \varphi_D(x, y) \, dx \, dy := P\{N = D, X \in dx, Y \in dy\} \]
\[ := \lim_{t \to \infty} P\{N_t = D, X_t \in dx, Y_t \in dy|N_0 = A\}. \]

**Notations.** The indicator of an event \( \mathcal{E} \) is denoted by
\[ \mathbb{1}(\mathcal{E}) := \begin{cases} 1 & \text{if } \mathcal{E} \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases} \]

Note that, for instance,
\[ E\{e^{ix} \mathbb{1}(N = D)\} \]
\[ = \int_0^{\infty} \int_0^{\infty} e^{ix} e^{iy} \varphi_D(x, y) \, dx \, dy, \]
\[ \text{Im } \omega \geq 0, \text{ Im } \eta \geq 0 \]
so that
\[ p_D = \int_0^{\infty} \int_0^{\infty} p_D(x, y) \, dx \, dy. \]

The real line and the complex plane are denoted by \( \mathbb{R} \) and \( \mathbb{C} \) with obvious superscript notations \( \mathbb{C}^+, \mathbb{C}^- \), \( \mathbb{C}^+ \cup \mathbb{R}, \mathbb{C}^- \cup \mathbb{R} \). For instance, \( \mathbb{C}^+: = \{\omega \in \mathbb{C} : \text{Im } \omega > 0\} \).

The robot and the safety device are only jointly available (operative) in state \( A \).

Therefore, the long-run availability of the robot-safety device, denoted by \( \mathcal{A} \), is given by \( p_A \).

Finally, we propose the following risk-criterion: State \( B \) is admissible if \( p_B \) satisfies the relation \( p_B < \delta \ll 1 \) for some \( \delta > 0 \), called the security level.

**3. Differential equations**

In order to determine the \( \varphi \)-functions, we first construct a system of steady-state differential equations based on a time independent version of Hokstad’s supplementary variable technique (see e.g. Ref. [8, p. 526]). For \( x > 0, y > 0 \), we obtain
\[ (\lambda s + \lambda) p_A = \varphi_B(0) + \varphi_C(0), \]
\[ (\hat{\lambda} - \frac{d}{dx}) \varphi_B(x) = \varphi_D(x, 0) + \check{\lambda}_s p_A \frac{d}{dx} \hat{R}_s(x), \]
\[ -\frac{d}{dy} \varphi_C(y) = \varphi_D(0, y) + \check{\lambda} p_A \frac{d}{dy} \hat{R}(y), \]
\[ \left( -\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \varphi_D(x, y) = \lambda \varphi_B(x) \frac{d}{dy} \hat{R}(y). \]

**4. Solution procedure**

Note that our equations are well adapted to an integral transformation. The integrability of the \( \varphi \)-functions and their corresponding derivatives implies that each \( \varphi \)-function vanishes at infinity irrespective of the asymptotic behaviour of the underlying repair time densities! Applying a routine Fourier transform technique to the equations and invoking the boundary condition \((\check{\lambda}_s + \check{\lambda}) p_A = \varphi_B(0) + \varphi_C(0)\), reveals that
\[ i(\omega + \eta) E\{e^{ix} e^{iy} \mathbb{1}(N = D)\} \]
\[ + (\check{\lambda}(1 - E e^{iy}) + i\omega) E\{e^{ix} \mathbb{1}(N = B)\} \]
\[ + i\eta E\{e^{iy} \mathbb{1}(N = C)\} + \check{\lambda} p_A (1 - E e^{iy}) \]
\[ + \check{\lambda}_s p_A (1 - E e^{iy}) = 0. \quad (1) \]

Observe that Eq. (1) holds for any pair \((\omega, \eta) \in \mathbb{C} \times \mathbb{C} \): \( \text{Im } \omega > 0, \text{ Im } \eta > 0 \). Therefore, substituting \( \omega = t, \eta = -it (t \in \mathbb{R}) \) into Eq. (1), yields the functional equation
\[ \psi^+(t) - \psi^-(t) = \varphi(t), \quad (2) \]
where
\[ \psi^+(t) := p_A^{-1} E\{e^{it} \mathbb{1}(N = B)\}, \]
\[ \psi^-(t) := \frac{1}{1 + \check{\lambda} \rho \varphi^-(t)} \]
\[ \check{\lambda} \rho \varphi^-(t) \]
\[ \frac{1}{1 + \check{\lambda} \rho \varphi^-(t)}, \]
\[ \varphi^+(t) := E e^{it\rho} - 1 \]
\[ \check{\lambda}_s p_A \varphi^{+(t)} \]
\[ \frac{1}{1 + \check{\lambda} \rho \varphi^-(t)}. \]

Eq. (2) constitutes a boundary value problem on the real line which can be solved by the theory of sectionally holomorphic functions. An application of Rouché’s (see the appendix) theorem reveals that the function \( 1 + \check{\lambda} \rho \varphi^-(t) \), \( \text{Im } \omega \leq 0 \), has no zeros in \( \mathbb{C}^- \cup \mathbb{R} \). Consequently, the function \( \psi^-(t) \).
Im \( \omega \leq 0 \), is analytic in \( C^- \), bounded and continuous on \( C^- \cup \mathbb{R} \), whereas \( \psi^+(\omega) \), Im \( \omega \geq 0 \), is analytic in \( C^+ \), bounded and continuous on \( C^+ \cup \mathbb{R} \) and

\[
\lim_{|\omega| \to \infty} \psi^- (\omega) = \lim_{x \in \arg \omega \in 2\pi \atop \omega \in C^+} \psi^+(\omega) = 0.
\]

Moreover, \( \phi \) is (uniformly) Lipschitz continuous on \( \mathbb{R} \). (Indeed, \( |\phi(t)| \) is bounded on \( \mathbb{R} \). Therefore, our assertion follows from the mean value theorem). Finally, \( \phi \) is Hölder continuous at infinity, i.e. \( |\phi(t)| = O(|t|^{-1}) \), if \( |t| \to \infty \).

Consequently, the Cauchy-type integral

\[
\frac{1}{2\pi j} \int_{\Gamma} \phi(\tau) \frac{dT}{\tau - \omega},
\]

(see the appendix) exists for all \( \omega \in C \) (real or complex) and defines a sectionally holomorphic function vanishing at infinity. A straightforward application of the Cauchy formulae for the regions \( C^+ \) and \( C^- \), entails that

\[
E\{e^{inX}1(N = B)\} = \lim_{\omega \to -\infty} \int_{\Gamma} \phi(\tau) \frac{dT}{\tau - \omega}, \quad \omega \in C^+,
\]

\[
E\{e^{-inY}1(N = C)\}
\]

\[
= \lim_{\omega \to -\infty} \int_{\Gamma} \phi(\tau) \frac{dT}{\tau - \omega}
\]

\[
+ \frac{1}{2\pi} \int_{\Gamma} \phi(\tau) \frac{dT}{\tau - \omega} \quad \omega \in C^-.
\]

In particular, \( p_B = p_A \psi^+(0) \), where

\[
\psi^+(0) = \lim_{\omega \to -\infty} \frac{1}{2\pi} \int_{\Gamma} \phi(\tau) \frac{dT}{\tau - \omega}
\]

Note that by the Sokhotski–Plemelj formulæ,

\[
\psi^+(0) = \frac{1}{2} \phi(0) + \frac{1}{2\pi} \int_{\Gamma} \phi(\tau) \frac{dT}{\tau}.
\]

Moreover,

\[
p_D = \lim_{\eta \to 0} \lim_{\omega \to -\infty} E\{e^{inX}e^{it\tau}1(N = D)\}.
\]

Applying the limit procedure to Eq. (1) and invoking the condition \( p_A + p_B + p_C + p_D = 1 \), yields the additional relations

\[
p_A + p_B = \frac{1}{1 + \lambda \rho}, \quad p_C + p_D = \frac{\lambda \rho}{1 + \lambda \rho},
\]

\[
p_D + p_B = \lambda \rho p_A.
\]

Substituting \( p_B = p_A \psi^+(0) \) into the first relation yields \( p_A = [(1 + \lambda \rho)(1 + \psi^+(0))]^{-1} \). Observe that we have completely determined the invariant measure simply and solely depending upon \( \psi^+(0) \). We state the following

**Theorem.**

\[
\sigma = \frac{1}{(1 + \lambda \rho)(1 + \psi^+(0))},
\]

where

\[
\psi^+(0) = \frac{1}{2} \phi(0) + \frac{1}{2\pi} \int_{\Gamma} \phi(\tau) \frac{dT}{\tau}.
\]

**Remarks.** Note that the kernel \( \phi(t), t \in \mathbb{R} \), preserves all the relevant properties to ensure the existence of the Cauchy integral for arbitrary repair time distributions with finite mean and variance. First of all, the order relation \( |\phi(t)| = O(|t|^{-1}) \), \( |t| \to \infty \), also holds for arbitrary characteristic functions. Moreover, the H-continuity of \( \phi \) on \( \mathbb{R} \) does not depend on the canonical structure (decomposition) of \( R \) or \( R_c \). For instance, the Hölder inequality

\[
|Ee^{it\tau} - Ee^{it\tau'}| \leq \rho |t_2 - t_1|, \quad (t_1, t_2 \in \mathbb{R}),
\]

always holds for any \( \tau \) with mean \( \rho \).

The requirement of finite variances is extremely mild since the current probability distributions employed to model repair times [1] even have moments of all orders!

Consequently, our initial assumptions concerning the existence of repair time densities are totally superfluous to ensure the existence of an invariant measure.

As an example, we consider the case of deterministic repair, i.e., \( t_0 > 0 \) and \( \theta < 1 \), let

\[
R(x) = \begin{cases} 
1 & \text{if } x > t_0, \\
0 & \text{if } x < t_0,
\end{cases}
\]

\[
R_s(x) = \begin{cases} 
1 & \text{if } x \geq \theta t_0, \\
0 & \text{if } x < \theta t_0.
\end{cases}
\]

Clearly, \( Ee^{it\tau} = e^{it\theta}, \rho = t_0, \sigma^2 = 0 \) and \( Ee^{-it\tau} = e^{-it\theta}, \rho_s = \theta t_0, \sigma^2_s = 0 \).
From the identity
\[
\frac{1}{2\pi i} \int_G \frac{\varphi(t)}{t - \omega} \, dt = \begin{cases} 
\frac{i\lambda_s(e^{-i\lambda t} - e^{i\omega t})}{\omega - i\lambda} & \text{if } \Im \omega > 0, \omega \neq i\lambda, \\
i\lambda e^{-i\lambda t} & \text{if } \omega = i\lambda,
\end{cases}
\]
we obtain \( \psi^+(0) = \lambda^{-1} \lambda_s(1 - e^{-i\lambda t}) \). Hence,
\[
\mathcal{A} = \frac{1}{(1 + \lambda^{-1} \lambda_s(1 - e^{-i\lambda t}))(1 + \lambda t_0)}.
\]

Appendix

**Definition.** Let \( f(t), t \in \mathbb{R} \) be a bounded and continuous function. \( \varphi \) is called \( \Gamma \)-integrable, if
\[
\lim_{T \to \infty} \int_{L,T} f(t) \frac{dt}{t - u}, \quad u \in \mathbb{R},
\]
events, where \( L,T := (-T, u - \varepsilon) \cup [u + \varepsilon, T) \). The corresponding integral, denoted by
\[
\frac{1}{2\pi i} \int_G f(t) \frac{dt}{t - u},
\]
is called a Cauchy principal value in double sense.

**Lemma.** The function \( 1 + \lambda \rho \varphi^-(\omega), \Im \omega \leq 0 \), has no zeros in \( \mathbb{C}^- \cup \mathbb{R} \).

**Proof.** Clearly, \( 1 + \lambda \rho \varphi^-(0) = 1 + \lambda \rho > 0 \). Hence, \( \omega = 0 \) is not a zero. Consider region D with boundary \( \mathcal{D} \) as depicted in Fig. 1.

For \( \omega \neq 0 \) we have
\[
1 + \lambda \rho \varphi^-(\omega) = \frac{\lambda + i\omega}{i\omega}(1 - L^- (\omega)),
\]
where
\[
L^- (\omega) := \frac{\lambda}{\lambda + i\omega} \mathbb{E} e^{-i\omega r}, \quad \Im \omega \leq 0.
\]

Note that \( L^- (t) \) is the characteristic function of a non-lattice random variable. Consequently, \( L^- (t) \neq 1 \) on \( (-T, -\varepsilon] \cup [\varepsilon, T) \). Therefore, \( 1 + \lambda \rho \varphi^-(t) \) has no zeros in \( \mathbb{R} \). Furthermore, \( \|L^- (\omega)\| < 1, \forall \omega \in \mathcal{D} \). Moreover, \( L^- (\omega) \) is analytic in D and continuous on \( D \cup \mathcal{D} \). Applying Rouche’s theorem to the functions 1 and \(-L^- (\omega)\) reveals that \( 1 + \lambda \rho \varphi^-(\omega) \) has no zeros in \( \mathbb{C}^- \cup \mathbb{R} \). In addition, note that
\[
\lim_{|\omega| \to \infty} (1 + \lambda \rho \varphi^-(\omega)) = 1.
\]

**References**