Relatively monotone variational inequalities over product sets

Igor V. Konnov

Department of Applied Mathematics, Kazan University, UL Lenina, Kazan 420008, Russia

Received 1 February 2000; received in revised form 1 September 2000

Abstract

New concepts of monotonicity which extend the usual ones are introduced for variational inequality problems over product sets. They enable us to strengthen existence results and extend the field of applications of the solution methods.

Keywords: Variational inequalities; Product sets; Relative monotonicity; Existence results

1. Introduction

Variational inequalities (VIs) proved to be a very useful tool for investigation and solutions of various equilibrium-type problems arising in Operations Research, Economics, Mathematical Physics and other fields. It is well known that most of such problems arising in game theory, transportation and network economics have a decomposable structure, i.e., they can be formulated as VIs over Cartesian product sets; see e.g. Nagurney [11] and Ferris and Pang [4]. At the same time, most existence results for such VIs were established under either compactness of the feasible set in the norm topology or monotonicity-type assumptions regardless of the decomposable structure of the VI; see e.g. [1,6]. In fact, Bianchi [2] considered the corresponding extensions of concepts of P-mappings and noticed that they are not sufficient to derive existence results with the help of Fan’s Lemma.

The main goal of this paper is to present other monotonicity-type concepts which are suitable for decomposable VIs and to obtain new existence theorems for such VIs by employing Fan’s Lemma. In addition, we show that iterative methods which are convergent for usual (pseudo) monotone VIs can be, in fact, applied to solve much more general problems, thus extending the field of their applications.

In our notation, $\mathbb{R}^m_+$ denotes the set of all positive vectors in $\mathbb{R}^m$, i.e., $\mathbb{R}^m_+ = \{ a \in \mathbb{R}^m | a_i > 0 \ i = 1, \ldots, m \}$, $e$ denotes the unit vector in $\mathbb{R}^m$, i.e., $e = (1, \ldots, 1)$. We use superscripts to denote different vectors. Subscripts are used to denote different scalars or components of vectors.

Let $E$ be a real Banach space and $E^*$ its dual. Then $\langle x, f \rangle$ denotes the duality pairing between $x \in E$ and $f \in E^*$, $\| x \|$ denotes the norm of $x$. Also, for each set $B \subseteq E$, we denote by $B^w$ the weak closure of $B$.

2. Basic definitions

Let $M$ be the set of indices $\{1, \ldots, m\}$. For each $s \in M$, let $X_s$ be a real Banach space and $X^*_s$ its dual. Set $X = \prod_{s \in M} X_s$, so that for each $x \in X$, we have...
Define\( x = (x_s | s \in M) \) where \( x_s \in X_s \). Analogously, for each \( s \in M \), let \( K_s \) be a nonempty, convex and closed set in \( X_s \) and let

\[
K = \prod_{s \in M} K_s.
\]

Next, for each \( s \in M \), let \( G_s : K \to X_s^* \) be a mapping. Then we can define the following VI: Find \( x^* \in K \)

\[
\sum_{s \in M} \langle G_s(x^*), x_s - x_s^* \rangle \geq 0 \quad \forall x_s \in K_s, \quad s \in M.
\]

Of course, if we define the mapping \( G : K \to X^* \) as follows:

\[
G(x) = (G_s(x) | s \in M),
\]

then (2) can be equivalently rewritten as the usual VI: Find \( x^* \in K \)

\[
\langle G(x^*), x - x^* \rangle \geq 0 \quad \forall x \in K.
\]

We denote by \( K^\gamma \) the solution set of VI (2). Now, we recall the known monotonicity properties for \( G \).

**Definition 1.** The mapping \( G : K \to X^* \) is said to be

(a) monotone, if for all \( x, y \in K \), we have

\[
\langle G(x) - G(y), x - y \rangle \geq 0,
\]

and strictly monotone, if the inequality is strict for all \( x \neq y \);

(b) pseudomonotone, if for all \( x, y \in K \), we have

\[
\langle G(y), x - y \rangle \geq 0 \Rightarrow \langle G(x), y - x \rangle \geq 0,
\]

and strictly pseudomonotone, if the second inequality is strict for all \( x \neq y \).

It is clear that (strict) monotonicity implies (strict) pseudomonotonicity and that the reverse assertions are not true. We now give other concepts of monotonicity which extend those in Definition 1.

**Definition 2.** We say that the mapping \( G : K \to X^* \), defined by (3), is

(a) relatively monotone, if there exist vectors \( x, \beta \in R^n \), such that for all \( x, y \in K \), we have

\[
\sum_{s \in M} (\beta_s G_s(x) - \beta_s G_s(y), x_s - y_s) \geq 0,
\]

and relatively strictly monotone, if the inequality is strict for all \( x \neq y \); (b) relatively pseudomonotone, if there exist vectors \( x, \beta \in R^n \), such that for all \( x, y \in K \), we have

\[
\sum_{s \in M} \beta_s (G_s(y), x_s - y_s) \geq 0
\]

\[
\Rightarrow \sum_{s \in M} \alpha_s (G_s(x), x_s - y_s) \geq 0,
\]

and relatively strictly pseudomonotone, if the second inequality is strict for all \( x \neq y \).

It what follows, we reserve the symbols \( x \) and \( \beta \) for parameters associated to relative (pseudo) monotonicity of \( G \). From the definitions, it follows that relative (strict) monotonicity implies relative (strict) pseudomonotonicity, but the reverse assertions are not true in general. Moreover, relative monotonicity type properties obviously extend the usual monotonicity ones.

We also need the following continuity assumption.

**Definition 3.** The mapping \( G : K \to X^* \) is said to be hemicontractive, if for any \( x \in K \), \( y \in K \) and \( \lambda \in [0, 1] \), the mapping \( \lambda \to (G(x + \lambda z), z) \) with \( z = y - x \) is continuous.

3. **Existence results**

Fix an element \( \gamma \in R^m_\geq \). Then we can consider two problems associated with \( \gamma \). The first is to find a point \( x^* \in K \) such that

\[
\sum_{s \in M} \gamma_s (G_s(x^*), y_s - x_s^*) \geq 0 \quad \forall y_s \in K_s, \quad s \in M.
\]

The second is to find an element \( x^* \in K \) such that

\[
\sum_{s \in M} \gamma_s (G_s(y), y_s - x_s^*) \geq 0 \quad \forall y_s \in K_s, \quad s \in M.
\]

It is clear that (4) coincides with VI (2) when \( \gamma = e \) and that (5) represents the so-called dual VI. We denote by \( K^\gamma (\gamma) \) and \( K^\gamma (e) \) the solution sets of problems (4) and (5), respectively.

The following lemma justifies the coincidence of solution sets for all VIs of form (4).

**Lemma 1.** (i) For each \( \gamma \in R^m_\geq \), VI (4) is equivalent to the problem of finding an element \( x^* \in K \) such that

\[
(G_s(x^*), y_s - x_s^*) \geq 0 \quad \forall y_s \in K_s, \quad s \in M.
\]

(ii) For each \( \gamma \in R^m_\geq \), \( K^\gamma (\gamma) = K^\gamma \).
Proof. Obviously, (6) implies (4). The reverse assertion is true due to (1). Thus, (i) holds and (ii) follows immediately from (i). □

We also recall the known relationship between (4) and (5).

Lemma 2. If $G$ is hemicontinuous, then $K^d(\gamma) \subseteq K^\gamma(\gamma)$ for all $\gamma \in R^*_s$.

Proof. Indeed, if $G$ is hemicontinuous, so is $G^{(\gamma)}(x, s) = (\gamma, G_s | s \in M)$ and the result follows from the Minty Lemma, see e.g. Kinderlehrer and Stampacchia [7, Chapter 3, Lemma 1.5]. □

From Lemmas 1 and 2 we obtain the following immediately.

Lemma 3. If $G$ is hemicontinuous and relatively pseudomonotone, then $K^\gamma = K^\gamma(\beta) = K^d(\alpha) = K^\gamma(\alpha)$.

We are now ready to establish an existence result for relatively pseudomonotone VIs.

Theorem 1. Suppose that $G$ is hemicontinous and relatively pseudomonotone and that $K$ is weakly compact. Then there exists a solution to VI (2).

Proof. Define set-valued mappings $A, B : K \rightarrow 2^K$ by

$$B(y) = \left\{ x \in K \mid \sum_{s \in M} \beta_s(G_s(x), y_s - x_s) \geq 0 \right\}$$

and

$$A(y) = \left\{ x \in K \mid \sum_{s \in M} \alpha_s(G_s(y), y_s - x_s) \geq 0 \right\}.$$

We divide the proof into the following three steps.

(i) $\bigcap_{y \in K} \overline{B(y)}^w \neq \emptyset$. Let $z$ be in the convex hull of any finite subset $\{y^1, \ldots, y^n\}$ of $K$. Then $z = \sum_{j=1}^n \mu_j y^j$ for some $\mu_j \geq 0$, $j = 1, \ldots, n$; $\sum_{j=1}^n \mu_j = 1$. If $z \notin \bigcup_{y \in K} B(y)$, then

$$\sum_{s \in M} \beta_s(G_s(z), y^j_s - z_s) < 0 \quad \forall j = 1, \ldots, n.$$

It follows that

$$0 = \sum_{s \in M} \beta_s(G_s(z), z - z) < 0,$$

a contradiction. Since $\overline{B(y)}^w$ is weakly compact, we conclude that $\bigcap_{y \in K} \overline{B(y)}^w \neq \emptyset$ due to Fan’s Lemma [3, Lemma 10].

(ii) $\bigcap_{y \in K} A(y) \neq \emptyset$. From relative pseudomonotonicity of $G$ it follows that $B(y) \subseteq A(y)$, but $A(y)$ is obviously weakly closed. Therefore, (i) now implies (ii).

(iii) $K^\gamma \neq \emptyset$. On account of (ii), we have $K^d(\alpha) \neq \emptyset$. The result follows now from Lemma 3.

Thus, VI (2) is solvable. □

Corollary 1. Suppose that $G$ is hemicontinuous and relatively strictly pseudomonotone and that $K$ is weakly compact. Then VI (2) has a unique solution.

Proof. Due to Theorem 1, if suffices to show that VI (2) has at most one solution. Assume, for contradiction, that $x', x'' \in K^\gamma$ and $x' \neq x''$. Then, due to Lemma 3, we have $x' \in K^\gamma(\beta)$ and

$$\sum_{s \in M} \beta_s(G_s(x', y''_s - x'_s)) \geq 0.$$

By relative strict pseudomonotonicity, we obtain

$$\sum_{s \in M} \alpha_s(G_s(x''_s), y'_s - x'_s) < 0,$$

i.e., $x'' \notin K^\gamma(x) = K^\gamma$, a contradiction. □

In order to obtain existence results on unbounded sets we need certain coercivity conditions.

Corollary 2. Suppose that $G$ is hemicontinuous and relatively pseudomonotone and that there exist a weakly compact subset $C$ of $X$ and a point $\hat{y} \in C \cap K$ such that

$$\sum_{s \in M} \beta_s(G_s(x), \hat{y}_s - x_s) < 0 \quad \text{for all } x \in K \setminus C.$$

Then VI (2) is solvable.

Proof. In this case it suffices to follow the proof of Theorem 1 and observe that $B(\hat{y}) \subseteq C$ under the above assumptions. Indeed, it follows that $\overline{B(\hat{y})}^w$ is weakly compact, hence the assertion of Step (i) will be true due to Fan’s Lemma as well. □
The new concepts enable us to strengthen the previous existence results, as the following example illustrates.

**Example 1.** Let \( Q : K \to X^* \) be a hemicontinuous monotone mapping of the form (3) and let \( K \) be weakly compact. Denote by \( F(Q) \) the family of all “weighted” mappings \( Q^{(s)} \) of the form (3), where

\[
Q^{(s)}(x) = \delta_s Q(x), \quad \delta_s > 0 \quad \text{for all} \ s \in M.
\]

Clearly, each element of \( F(Q) \) is relatively monotone. Hence, due to Theorem 1, we now conclude that VI of form (2) with any cost mapping belonging to \( F(Q) \) will be solvable, whereas the general theory guarantees the existence of solutions only for VI with the cost mapping \( Q \). The difference between these results is appreciable even in the simplest case where \( m = 2 \) and \( G \) is an affine monotone mapping. For instance, set \( Q(x) = Ax \), where

\[
A = \begin{pmatrix} 6.2 & 3.1 \\ 3.1 & 2 \end{pmatrix}.
\]

Since \( A \) is positive definite, \( Q \) is monotone. Next, set \( \delta = (10/31, 1) \) and choose \( Q^{(0)} \in F(Q) \). Clearly, \( Q^{(0)}(x) = A^{(0)}x \), where

\[
A^{(0)} = \begin{pmatrix} 2 & 1 \\ 3.1 & 2 \end{pmatrix}.
\]

It is easy to see that the symmetric part of \( A^{(0)} \) is not positive semidefinite, hence \( Q^{(0)} \) is not monotone. Moreover, \( Q^{(0)} \) is not pseudomonotone either, since taking \( x = (1.2, -1.1) \) and \( y = (-1.1, 1.2) \) yields \( \langle Q^{(0)}(y), x - y \rangle = 0.023 > 0 \) and \( \langle Q^{(0)}(x), x - y \rangle = -0.506 < 0 \). At the same time, \( G = Q^{(0)} \) is relatively monotone with \( \alpha = \beta = (3.1, 1) \).

In a reflexive Banach space setting, we can use strong monotonicity-type assumptions.

**Definition 4.** We say that the mapping \( G \), defined by (3), is

(a) relatively strongly monotone with constant \( \tau > 0 \), if there exist vectors \( \alpha, \beta \in \mathbb{R}^m_+ \) such that for all \( x, y \in K \), we have

\[
\sum_{s \in M} \beta_s \langle G_s(y), x_s - y_s \rangle \geq \tau \|x - y\|^2;
\]

(b) relatively strongly pseudomonotone with constant \( \tau > 0 \), if there exist vectors \( \alpha, \beta \in \mathbb{R}^m_+ \) such that for all \( x, y \in K \), we have

\[
\sum_{s \in M} \beta_s \langle G_s(y), x_s - y_s \rangle \geq \tau \|x - y\|^2.
\]

It is clear that relative strong monotonicity implies relative strong pseudomonotonicity and that relative strong (pseudo) monotonicity implies relative strict (pseudo) monotonicity.

**Theorem 2.** Let \( X_s, s \in M \) be real reflexive Banach spaces and let \( G \) be hemicontinuous and relatively strongly pseudomonotone. Then VI (2) has a unique solution.

**Proof.** Due to Theorem 1 and Corollary 1 it suffices to show that VI (2) is solvable. Let \( B_s(r) \) denote the closed ball with center at 0 and radius \( r > 0 \) in \( X_s \) for each \( s \in M \). If \( K_s \cap B_s(r) \neq \emptyset \) for all \( s \in M \), there exists the unique solution \( x' \) of the following VI: Find \( x'_s \in K_s \cap B_s(r) \), \( s \in M \) such that

\[
\sum_{s \in M} \beta_s \langle G_s(x'_s), y_s - x'_s \rangle \geq 0
\]

\[
\forall y_s \in B_s(r) \cap K_s, \quad s \in M.
\]

From the relative strong pseudomonotonicity of \( G \) follows that the set \( \{x' \mid r > 0 \} \) is bounded. In fact, otherwise we fix \( y \in \bigcap_{s \in M}(B_s(r) \cap K_s) \) for some \( r > 0 \) and obtain

\[
\sum_{s \in M} \alpha_s \langle G_s(y), y_s - x'_s \rangle \geq \tau \|x' - y\|^2
\]

for \( r \to +\infty \), i.e., for \( \|x'\| \to +\infty \), which is a contradiction. Hence, there exists an \( r' > 0 \) such that \( \|x'_s\| < r' \) for all \( s \in M \). Take any \( y \in K \). Then we can choose \( \varepsilon > 0 \) small enough such that \( x'_s + \varepsilon(y_s - x'_s) \in K_s \cap B_s(r') \) for \( s \in M \). If we now suppose that

\[
\sum_{s \in M} \beta_s \langle G_s(x'_s), y_s - x'_s \rangle < 0,
\]

then

\[
\sum_{s \in M} \beta_s \langle G_s(x'_s), x'_s + \varepsilon(y_s - x'_s) - x'_s \rangle < 0,
\]

a contradiction. Thus, \( x' \in K \cap \beta \) and the result follows now from Lemma 1(ii). \( \Box \)
Remark 1. The approach to establish existence results in this paper can be viewed as an application of general approaches of Oettli [12] and Konnov and Schaible [10]. However, in this case, the cost mapping of the problem under consideration, in contrast to those in [12,10], need not be of pseudomonotone type with respect to any mapping.

4. Applications in convergence theory

Most algorithms for VIs of (2) and (1) are known to be convergent under (strict) monotonicity assumptions on $G$; see e.g. Patriksson [13]. Obviously, one can apply such algorithms to VI (2) in the case where $G$ is relatively (strictly) monotone with $\alpha = \beta$. It suffices to replace $G$ with the “weighted” mapping $G^{(s)}$, where

$$G^{(s)}(x) = \alpha_s G_s(x) \quad \text{for all } s \in M. \quad (7)$$

Recently, Konnov in [8,9] proposed several algorithms which converge to a solution of VI (2) if there exists a solution of the dual VI (5) with $\gamma = e$. Note that $K^d(\alpha) \neq \emptyset$ if $G$ is relatively pseudomonotone and $K^v(\beta) = K^v \neq \emptyset$. Therefore, replacing $G$ with $G^{(s)}$, defined by (7), we can apply these algorithms as well. On the other hand, algorithms [8,9] can be applied under the pseudo P-monotonicity assumption.

Definition 5 (M. Bianchi [2]). The mapping $G$, defined by (3), is said to be pseudo P-monotone, if for all $x, y \in K$, we have

$$\min_{s \in M} \langle G_s(y), x_s - y_s \rangle \geq 0 \Rightarrow \sum_{s \in M} \langle G_s(x), x_s - y_s \rangle \geq 0.$$ 

It was noticed in [2] that the pseudo P-monotonicity is not sufficient to derive existence results based on applying Fan’s Lemma. Nevertheless, if $G$ is pseudo P-monotone and $K^v \neq \emptyset$, then we see that the dual VI (5) with $\gamma = e$ is solvable; hence, we can apply algorithms [8,9] to find a solution of the initial VI (2). It is clear that relative pseudomonotonicity with $\alpha = e$ implies pseudo P-monotonicity, but the corresponding classes of mappings do not contain each other in general.

Let us now turn to the simplest case where $X_s = R$ for $s \in M$. Then VI (2) is called a box-constrained VI and involves, in particular, the well-known complementarity problem. Hence, one can verify relative (pseudo) monotonicity with $\alpha = \beta$ by considering positive definiteness of the symmetric part of $\nabla G^{(s)}(x)$. In particular, affine $M$-mappings (see [5,14]) satisfy this property. In fact, it is known that for every $M$-matrix there exists a diagonal matrix with positive diagonal entries such that the symmetric part of their product is positive definite; see [15]. Hence, each affine mapping $G$ of the form $G(x) = Ax + b$ with $A$ being an $M$-matrix is relatively monotone, but it need not be monotone in general.

It would be of particular interest to obtain similar rules to verify relative (pseudo) monotonicity in general case.

5. Conclusions

In this paper, we have suggested new concepts of (pseudo) monotonicity which are expected to be useful in investigating VIs over product sets. Namely, these properties allow us to strengthen existence results for such VIs and enlarge the field of applications of existing iterative algorithms.

Also, it would be worthwhile to extend the above results to multivalued [9] and vector [1] decomposable VIs. The author intends to consider these questions in forthcoming work.

Acknowledgements

This research is supported in part by RFBR Grant No. 98-01-00200 and by the R.T. Academy of Sciences. The author would like to thank an anonymous referee for his/her valuable comments.

References