An analysis of multiple-class vacation queues with individual thresholds

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Abstract

This study examines a multiple-class M/G/1 queue with multiple thresholds. Each class of customer arrives according to an independent Poisson process, and each class has its own threshold. The idle server is reactivated as soon as any one of the thresholds is reached. Both FCFS and non-preemptive priority cases are considered. The Laplace–Stieltjes transform of the waiting time distribution function and the mean waiting time for each class of customers are derived. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Most studies on N-policy use a single threshold. This has been the tendency for studies involving multiple class queues. In these models, the idle server is reactivated when the total number of customers (including all classes) reaches a threshold (for an example of this, see [12]). In this paper a queueing system is proposed in which each class has its own threshold and the idle server is reactivated as soon as any one of the thresholds is reached. This paper was motivated by a real production system in which a machine needed to process multiple types of products.

The first study on N-policy was carried out by Yadin and Naor [14]. They obtained the mean waiting time and the mean queue length, and then derived the optimal \(N^*\) that minimizes the overall average operating cost. Hofri [4] studied two N-policy queues attended by a single server. Lee et al. [9] considered a threshold policy in which the server is reactivated when the number of the start-up class reached its threshold, regardless

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of the number of customers in other classes. For more studies on $N$-policy and related works, readers are advised to see Takagi [12].

The queueing system related to $N$-policy can be thought of as a vacation system. Vacation queues have attracted much attention from numerous researchers since Levy and Yechiali [10] and Heyman [3]. Kella [5] studied the $M/G/1$ queue with $N$-policy and vacations. Batch arrival queues under $N$-policy, with and without multiple vacations, were first studied by Lee and Srinivasan [8]. Lee et al. [6,7] considered batch arrival queues with $N$-policy and vacations, and derived the queue length distribution in decomposed forms. The decomposition property was first proved by Fuhrmann and Cooper [2] and then by Shanthikumar [11]. For a comprehensive survey on vacation queues, readers are advised to see Doshi [1] and Takagi [12].

2. The system and notation

A system with the following specifications is considered in this study:

1. There are $r$ classes.
2. Customers of class $p$, ($p = 1, 2, \ldots, r$) arrive according to a Poisson process with rate $\lambda_p$ independently of other classes.
3. Service times within a class are identically and independently distributed. Different classes have different service time distributions, but they are independent of each other.
4. There is a single server and the queue capacity is unlimited.
5. If no priorities are applied, FCFS is assumed regardless of the classes. If priorities are applied, non-preemption is assumed. Class-$i$ has higher priority over class-$j$, for $i < j$. FCFS is applied within a class.
6. Class-$p$ has a threshold of $N_p$.
7. The idle server is reactivated as soon as any one of the thresholds is reached.

Let us define the following notations. For the purpose of convenience, the test customer is assumed to belong to class-$p$. In most cases, the notations used by Takagi [12] are adopted, but some minor modifications are made.

HC: high-priority customer, i.e., a customer who belongs to one of the classes $\{1, 2, \ldots, p-1\}$.

EHC: equal- or high-priority customer, i.e., a customer who belongs to one of the classes $\{1, 2, \ldots, p\}$.

EC: equal-priority customer, i.e., a customer belonging to class-$p$.

LC: low-priority customer, i.e., a customer who belongs to one of the classes $\{p + 1, p + 2, \ldots, r\}$.

BPPIP: busy period initiation point.

$W^\ast_{q(p)}(\theta)$: the Laplace–Stieltjes transform (LST) of the distribution function (DF) of the waiting time of a class-$p$ test customer.

$E(W_{q(p)})$: mean waiting time of a class-$p$ test customer.

$\lambda$: total arrival rate.

$\lambda_p$: arrival rate of class-$p$ customers.

$\lambda^+_{p}$: arrival rate of EHCs.

$\lambda^-_{p}$: arrival rate of LCs.

$S_p$: service time of a class-$p$ customer (random variable).

$B_p(x)$: $= \Pr(S_p \leq x)$, DF of $S_p$.

$B^\ast_p(\theta)$: the LST of $B_p(x)$.

$B^\ast(\theta)$: $= \sum_{k=1}^p(\lambda_k/\lambda)B^\ast_k(\theta)$, the LST of the DF of the service time of a randomly selected customer from all classes.
$B_p^*(\theta) = \sum_{j=1}^{r_p} \left( \frac{\lambda_j}{\lambda_p} \right) B_j^*(\theta)$, the LST of the DF of the service time of a randomly selected EHC
$B_p^{**}(\theta) = \sum_{k=p+1}^{N} \left( \frac{\lambda_k}{\lambda_p} \right) B_k^*(\theta)$, the LST of the DF of the service time of a randomly selected LC

$b_p = E(S_p)$, the mean service time of a class-$p$ customer.
$b = \sum_{k=1}^{r_p} \left( \frac{\lambda_k}{\lambda_p} \right) b_k$, the mean service time of a randomly selected customer from all classes
$b_p^{(2)} = E(S_p^2)$, the second moment of the service time of a class-$p$ customer

$\rho_p = \hat{\lambda}_p b_p$ steady-state probability that the server is serving a class-$p$ customer
$\rho = \sum_{p=1}^{r} \rho_p$ steady-state probability that the server is busy
$\rho^*_p = \sum_{k=1}^{p} \rho_k$ steady-state probability that the server is serving an EHC
$\rho^*_p = \sum_{k=p+1}^{r} \rho_k$ steady-state probability that the server is serving an LC

$\Theta^*_p(\theta) = B_p^* [\theta + \lambda_p^* - \lambda_p^* \Theta^*_p(\theta)]$, the delay cycle generated by the service of an EHC and its EHC-offsprings

$(i_1, i_2, \ldots, i_r)$ a system state during the idle period in which there are $i_k$ class-$k$ customers, $(0 \leq i_k \leq N_k - 1, k = 1, 2, \ldots, r)$

$R_{(i_1, i_2, \ldots, i_r)}$ the remaining idle period from an arbitrary time point in the state $(i_1, i_2, \ldots, i_r)$. From PASTA (Wolff [13]), this is equal to the remaining idle period under a condition where an arriving customer comes upon the state $(i_1, i_2, \ldots, i_r)$.

### 3. Analysis of the idle period

Before we derive the LST of the waiting time distribution of a class-$p$ test customer, it is necessary to analyze the idle period.

First, the probability that the idle period visits state $(i_1, i_2, \ldots, i_r)$ is given by

$$\pi_{(i_1, i_2, \ldots, i_r)} = \frac{(i_1 + i_2 + \cdots + i_r)!}{i_1! i_2! \cdots i_r!} \left( \frac{\hat{\lambda}_1}{\lambda} \right)^{i_1} \left( \frac{\hat{\lambda}_2}{\lambda} \right)^{i_2} \cdots \left( \frac{\hat{\lambda}_r}{\lambda} \right)^{i_r}, \quad (0 \leq i_k \leq N_k - 1, k = 1, 2, \ldots, r).$$

(1)

The mean staying time for the state $(i_1, i_2, \ldots, i_r)$ is $1/\lambda$ for any combination of $i_1, i_2, \ldots, i_r$. Thus, the steady-state probability that the system is in the state $(i_1, i_2, \ldots, i_r)$ under the condition that the system is idle is given by

$$P_{(i_1, i_2, \ldots, i_r)} = \frac{\pi_{(i_1, i_2, \ldots, i_r)}}{\sum_{i_{1}=0}^{N_1-1} \sum_{i_{2}=0}^{N_2-1} \cdots \sum_{i_{r}=0}^{N_r-1} \pi_{(i_1, i_2, \ldots, i_r)}}.$$ (2)

From PASTA, it is also the probability that a test customer that arrives during the idle period encounters the state $(i_1, i_2, \ldots, i_r)$.

Now, let us obtain the LST of the DF of $R_{(i_1, i_2, \ldots, i_r)}$. Under the event $A_j$ that the first arrival during $(i_1, i_2, \ldots, i_r)$ is of class-$j$, we have

$$R_{(i_1, i_2, \ldots, i_r)} = T_j + R_{(i_1, \ldots, i_j+1, \ldots, i_r)},$$ (3)

where $T_j$ is the time until the first arrival under $A_j$. It is easily seen that $T_j$ follows the exponential distribution with mean $E(T_j) = 1/\sum_{j=1}^{r} \hat{\lambda}_j = 1/\lambda$. Also, we see that $T_j$ and $R_{(i_1, \ldots, i_j+1, \ldots, i_r)}$ are independent. Using $\Pr(A_j) = \hat{\lambda}_j/\lambda$,
the LST of the DF of $R_{(i_1, i_2, \ldots, i_r)}$ is given by

$$R^*_{(i_1, i_2, \ldots, i_r)}(\theta) = \sum_{j=1}^{r} \frac{\lambda_j}{\lambda} T^*_j(\theta) R^*_{(i_1, \ldots, i_j+1, \ldots, i_r)}(\theta),$$

(4a)

where

$$R^*_{(i_1, \ldots, N_k-1, \ldots, i_r)}(\theta) = \sum_{j=1}^{k-1} \frac{\lambda_j}{\lambda} T^*_j(\theta) R^*_{(i_1, \ldots, i_j+1, \ldots, N_k-1, \ldots, i_r)}(\theta) + \frac{\lambda_k}{\lambda} T^*_k(\theta) + \sum_{j=k+1}^{r} \frac{\lambda_j}{\lambda} T^*_j(\theta) R^*_{(i_1, \ldots, N_k-1, \ldots, i_j+1, \ldots, i_r)}(\theta),$$

(4b)

and

$$R^*_{(N_1-1, N_2-1, \ldots, N_r-1)}(\theta) = \sum_{j=1}^{r} \frac{\lambda_j}{\lambda} T^*_j(\theta).$$

(4c)

The mean can be obtained recursively from

$$E(R_{(i_1, i_2, \ldots, i_r)}) = \sum_{j=1}^{r} \frac{\lambda_j}{\lambda} \left[ \frac{1}{\lambda} + E(R_{(i_1, \ldots, i_j+1, \ldots, i_r)}) \right],$$

(5a)

where

$$E(R_{(i_1, \ldots, N_k-1, \ldots, i_r)}) = \sum_{j=1}^{k-1} \frac{\lambda_j}{\lambda} \left[ \frac{1}{\lambda} + E(R_{(i_1, \ldots, i_j+1, \ldots, N_k-1, \ldots, i_r)}) \right] + \left( \frac{\lambda_k}{\lambda} \right) \left( \frac{1}{\lambda} \right) + \sum_{j=k+1}^{r} \frac{\lambda_j}{\lambda} \left[ \frac{1}{\lambda} + E(R_{(i_1, \ldots, N_k-1, \ldots, i_j+1, \ldots, i_r)}) \right],$$

(5b)

and

$$E(R_{(N_1-1, N_2-1, \ldots, N_r-1)}) = \frac{1}{\lambda}.$$  

(5c)

4. Waiting times under FCFS

In this section, we derive the LST of the distribution function of the waiting time of a class-$p$ test customer under FCFS. The server is busy with a probability of $\rho = \sum_{k=1}^{r} \rho_k$. Thus, the LST $W^*_{q(p), FCFS}(\theta)$ of the DF of the waiting time of a class-$p$ test customer can be written as

$$W^*_{q(p), FCFS}(\theta) = \rho W^*_{q(p), FCFS}(\theta \mid \text{busy}) + (1 - \rho) W^*_{q(p), FCFS}(\theta \mid \text{idle}).$$

(6)

To obtain $W^*_{q(p), FCFS}(\theta \mid \text{busy})$ we need to find the probability distribution for the number of customers at the busy period initiation point (BPPIP). Let $K(z_1, z_2, \ldots, z_r)$ be the joint PGF (probability generating function) for the number of customers of each class at the BPPIP. After laborious manipulations, we get (a derivation
of (7) is provided in the appendix.

\[ K(z_1, z_2, \ldots, z_r) \]

\[ = \left( \frac{\lambda_1 z_1}{\lambda} \right)^{N_1} \times \sum_{i_0=0}^{N_1-1} \sum_{i_2=0}^{N_2-1} (i_2 + \cdots + i_r)! \left( \frac{N_1 + i_2 + \cdots + i_r - 1}{i_2 + \cdots + i_r} \right) \left( \frac{\lambda_2 z_2}{\lambda} \right)^{i_2} \cdots \left( \frac{\lambda_r z_r}{\lambda} \right)^{i_r} + \cdots \]

\[ + \left( \frac{\lambda_r z_r}{\lambda} \right)^{N_r} \times \sum_{i_{r-1}=0}^{N_{r-1}-1} \sum_{i_1=0}^{N_{r-1}-1} (i_1 + \cdots + i_{r-1})! \left( \frac{N_r + i_1 + \cdots + i_{r-1} - 1}{i_1 + \cdots + i_{r-1}} \right) \left( \frac{\lambda_{r-1} z_{r-1}}{\lambda} \right)^{i_1} \cdots \left( \frac{\lambda_1 z_1}{\lambda} \right)^{i_1}. \quad (7) \]

Let \( T_0 \) be the time that it takes to serve all existing customers at the BPIP. Let \( T^*_0(\theta) \) be the LST of the DF of \( T_0 \). Then \( T^*_0(\theta) \) can be obtained from

\[ T^*_0(\theta) = K(z_1, z_2, \ldots, z_r) \mid z_1 = \beta_1^*(\theta), z_2 = \beta_2^*(\theta), \ldots, z_r = \beta_r^*(\theta). \quad (8a) \]

The first moment becomes

\[ E(T_0) = -\frac{d}{d\theta} T^*_0(\theta) \mid \theta = 0 \]

\[ = \left( \frac{\lambda_1}{\lambda} \right)^{N_1} \times \sum_{i_0=0}^{N_1-1} \sum_{i_2=0}^{N_2-1} \left( \frac{i_2 + \cdots + i_r}{i_2! \cdots i_r!} \right) \left( \frac{N_1 + i_2 + \cdots + i_r - 1}{i_2 + \cdots + i_r} \right) \left( \frac{\lambda_2}{\lambda} \right)^{i_2} \cdots \left( \frac{\lambda_r}{\lambda} \right)^{i_r} \]

\[ \times \left( \frac{\lambda_r}{\lambda} \right)^{i_r} [N_1 b_1 + i_2 b_2 + \cdots + i_r b_r] + \cdots \]

\[ + \left( \frac{\lambda_r}{\lambda} \right)^{N_r} \times \sum_{i_{r-1}=0}^{N_{r-1}-1} \sum_{i_1=0}^{N_{r-1}-1} \left( \frac{i_1 + \cdots + i_{r-1}}{i_1! \cdots i_{r-1}!} \right) \left( \frac{N_r + i_1 + \cdots + i_{r-1} - 1}{i_1 + \cdots + i_{r-1}} \right) \left( \frac{\lambda_{r-1}}{\lambda} \right)^{i_1} \cdots \left( \frac{\lambda_1}{\lambda} \right)^{i_1} \]

\[ \times [N_r b_p + i_1 b_1 + \cdots + i_{p-1} b_{p-1} + i_p b_{p+1} + \cdots + i_r b_r] + \cdots \]

\[ + \left( \frac{\lambda_r}{\lambda} \right)^{N_r} \times \sum_{i_{r-1}=0}^{N_{r-1}-1} \sum_{i_1=0}^{N_{r-1}-1} \left( \frac{i_1 + \cdots + i_{r-1}}{i_1! \cdots i_{r-1}!} \right) \left( \frac{N_r + i_1 + \cdots + i_{r-1} - 1}{i_1 + \cdots + i_{r-1}} \right) \left( \frac{\lambda_{r-1} z_{r-1}}{\lambda} \right)^{i_1} \cdots \left( \frac{\lambda_1}{\lambda} \right)^{i_1} \]

\[ \times [N_r z_r + i_1 b_1 + \cdots + i_{r-1} b_{r-1}]. \quad (8b) \]

The second moment becomes

\[ E(T^*_0)^2 = \frac{d^2}{d\theta^2} T^*_0(\theta) \mid \theta = 0 \]
\begin{align*}
&\left(\frac{\lambda_1}{\lambda}\right)^{N_1} \times \sum_{i_0=0}^{N_2-1} \sum_{i_1=0}^{N_2-1} \frac{(i_2 + \cdots + i_r)!}{i_2! \cdots i_r!} \left(\frac{\lambda_2}{\lambda}\right)^{i_2} \cdots \left(\frac{\lambda_r}{\lambda}\right)^{i_r} \\
&\times \left\{ N_1(N_1 - 1)b_1^2 + N_1b_1^{(2)} + \sum_{a=2}^{r} [i_a(i_a - 1)b_a^2 + i_ab_a^{(2)}] + 2 \left[ \sum_{a=1}^{r-1} \sum_{c=a+1}^{r} i_ai_c b_ab_c - \sum_{c=2}^{r} i_i b_1 b_c \right] \\
&\quad + 2N_1 b_1 \sum_{a=2}^{r} i_ab_a \right\} \\
&+ \left(\frac{\lambda_1}{\lambda}\right)^{N_p} \sum_{i_1=0}^{N_p-1} \sum_{i_p=0}^{N_p-1} \sum_{i_r=0}^{N_r-1} \frac{(i_2 + \cdots + i_{p-1} + i_p + \cdots + i_r)!}{i_2! \cdots i_{p-1}! i_p! \cdots i_r!} \\
&\times \left\{ N_p(N_p - 1)b_p^2 + N_p b_p^{(2)} + \sum_{a=2}^{r} [i_a(i_a - 1)b_a^2 + i_ab_a^{(2)}] \\
&\quad + 2 \left[ \sum_{a=1}^{r-1} \sum_{c=a+1}^{r} i_ai_c b_ab_c - \sum_{c=2}^{r} i_i b_1 b_c \right] + 2N_1 b_1 \sum_{a=1}^{r} i_ab_a \right\} \\
&+ \left(\frac{\lambda_1}{\lambda}\right)^{N_r} \sum_{i_1=0}^{N_r-1} \sum_{i_r=0}^{N_r-1} \frac{(i_1 + \cdots + i_{r-1} - 1)!}{i_1! \cdots i_{r-1}!} \left(\frac{\lambda_r}{\lambda}\right)^{i_1} \left(\frac{\lambda_{r-1}}{\lambda}\right)^{i_{r-1}} \\
&\times \left\{ N_r(N_r - 1)b_r^2 + N_r b_r^{(2)} + \sum_{a=1}^{r-1} [i_a(i_a - 1)b_a^2 + i_ab_a^{(2)}] + 2 \left[ \sum_{a=1}^{r-1} \sum_{c=a+1}^{r} i_ai_c b_ab_c - \sum_{c=2}^{r} i_i b_1 b_r \right] \\
&\quad + 2N_r b_r \sum_{a=1}^{r-1} i_ab_a \right\}. \tag{8c}
\end{align*}

The busy period is a delay cycle with $T_0$ as the initial delay. The LST of the distribution function of the waiting time of an arbitrary customer that arrives during such a delay cycle under FCFS is given by (see Takagi [12])

\begin{equation}
W_{q(p),\text{FCFS}}(\theta|\text{busy}) = \frac{1 - T_0^*(\theta)}{\theta E(T_0)} \frac{\theta(1 - \rho)}{\theta - \lambda + \lambda B^*(\theta)} \tag{9}
\end{equation}
with mean

\[ E(W_{q,p,\text{FCFS}} | \text{busy}) = \frac{E(T_0^2)}{2E(T_0)} + \frac{\lambda b^2}{2(1 - \rho)}. \]  

(10)

Under the FCFS discipline, the DFs of the waiting times of the customers that arrive during the busy period do not differ among classes. Thus (9) and (10) can be applied to all classes.

Now let us obtain the LST of the DF of the waiting time of the class-\( p \) test customer that arrives during the idle period. Let \( E_{(i_1, i_2, \ldots, i_r)(p)} \) be the event that the class-\( p \) test customer encounters an idle state \((i_1, \ldots, i_r)\) when it arrives. Let \( w_{(i_1, i_2, \ldots, i_r)(p), q}^\ast(\theta) \) be the LST of the DF of the waiting time under \( E_{(i_1, i_2, \ldots, i_r)(p)} \). Then we get

\[ w_{(i_1, i_2, \ldots, i_r)(p), q}^\ast(\theta) = [B_1^\ast(\theta)]^{i_1} \cdots [B_r^\ast(\theta)]^{i_r} R_{(i_1, i_2, \ldots, i_r+1, \ldots, i_r)}^\ast(\theta). \]  

(11)

From PASTA, we have

\[ W_{q,p,\text{FCFS}}^\ast(\theta | \text{idle}) = \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_r=0}^{N_r-1} P_{(i_1, i_2, \ldots, i_r)} w_{(i_1, i_2, \ldots, i_r)(p), q}^\ast(\theta), \]  

(12)

where \( P_{(i_1, i_2, \ldots, i_r)} \) was obtained in (2). Unlike \( W_{q,p,\text{FCFS}}^\ast(\theta | \text{busy}) \), \( W_{q,p,\text{FCFS}}^\ast(\theta | \text{idle}) \) depends on \( p \). It can be shown that the mean becomes

\[ E(W_{q,p,\text{FCFS}} | \text{idle}) = \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_r=0}^{N_r-1} P_{(i_1, i_2, \ldots, i_r)} [i_1 b_1 + \cdots + i_r b_r + E(R_{(i_1, i_2, \ldots, i_r+1, \ldots, i_r)})]. \]  

(13)

Using (9) and (12) in (6), we get

\[ W_{q,p,\text{FCFS}}^\ast(\theta) = \rho \frac{1 - T_0^\ast(\theta)}{E(T_0)} \frac{\theta(1 - \rho)}{\theta - \lambda + \lambda B^\ast(\theta)} + (1 - \rho) \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_r=0}^{N_r-1} P_{(i_1, i_2, \ldots, i_r)} w_{(i_1, i_2, \ldots, i_r)(p), q}^\ast(\theta). \]  

(14)

The mean waiting time becomes

\[ E(W_{q,p,\text{FCFS}}) = \rho E(W_{q,p,\text{FCFS}} | \text{busy}) + (1 - \rho) E(W_{q,p,\text{FCFS}} | \text{idle}) \]

\[ = \rho \left[ \frac{E(T_0^2)}{2E(T_0)} + \frac{\lambda b^2}{2(1 - \rho)} \right] + (1 - \rho) \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_r=0}^{N_r-1} P_{(i_1, i_2, \ldots, i_r)} [i_1 b_1 + \cdots + i_r b_r + E(R_{(i_1, i_2, \ldots, i_r+1, \ldots, i_r)})]. \]  

(15)

The mean number of customers can be obtained from the Little’s law.

Figs. 1 and 2 show the behavior of the mean waiting time when one of the thresholds varies while the other two are fixed.
5. Waiting times under non-preemptive priority

In this section, we consider the waiting times in the case of non-preemptive priority (NP). We still have

\[
W_{q(p),NP}^*(\theta) = \rho W_{q(p),NP}^*(\theta \mid \text{busy}) + (1 - \rho) W_{q(p),NP}^*(\theta \mid \text{idle}).
\]  
(16)

First, we obtain \(X(z)\), the PGF of the number of EHCs at the BPIP. This can be obtained by using \(z\) in place of \(z_1, z_2, \ldots, z_p\) and by using 1 in \(z_{p+1}, z_{p+2}, \ldots, z_r\) in (7):

\[
X(z) = K(z_1, z_2, \ldots, z_r)\bigl| z_1 = z_2 = \ldots = z_p = z, \ z_{p+1} = z_{p+2} = \ldots = z_r = 1 \bigr. 
\]  
(17)

Let \(T_{0(p)}\) be the time that it takes to serve all EHCs existing at the BPIP and \(T_{0(p)}^*(\theta)\) be its LST. Then we get

\[
T_{0(p)}^*(\theta) = X[B_p^+(\theta)].
\]  
(18a)

The first moment becomes

\[
E(T_{0(p)}) = -\frac{d}{d\theta} T_{0(p)}^*(\theta)|_{\theta=0}
\]

\[
= \left( \frac{\tilde{\lambda}}{\lambda} \right)^{N_1} \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_r=0}^{N_r-1} \frac{(i_2 + \cdots + i_r)!}{i_2! \cdots i_r!} \left( \frac{N_1 + i_2 + \cdots + i_r - 1}{i_2 + \cdots + i_r} \right) \left( \frac{\tilde{\lambda}_2}{\lambda} \right)^{i_2} \cdots \left( \frac{\tilde{\lambda}_r}{\lambda} \right)^{i_r} 
\]

\[
+ \left( \frac{\tilde{\lambda}_p}{\lambda} \right)^{N_p} \sum_{i_0=0}^{N_p-1} \cdots \sum_{i_{p-1}=0}^{N_{p-1}-1} \sum_{i_p=0}^{N_p-1} \frac{(i_2 + \cdots + i_{p-1} + i_p + 1 + \cdots + i_r)!}{i_2! \cdots i_{p-1}! i_p! \cdots i_r!} 
\]

\[
\times \left( \frac{N_1 + i_1 + \cdots + i_p - 1}{i_1 + \cdots + i_p} \right) \left( \frac{N_p + i_1 + \cdots + i_p - 1}{i_1 + \cdots + i_p} \right) \cdots \left( \frac{\tilde{\lambda}_r}{\lambda} \right)^{i_r} (N_p + i_1 + \cdots + i_p - 1) b_p^+.
\]
\begin{align*}
&\expval{T_{\mathcal{O}(p)}^2} = \frac{d^2}{d\theta^2} T_{\mathcal{O}(p)}(\theta)|_{\theta = 0} \\
&= \left( \frac{\dot{\lambda}_1}{\lambda} \right)^{N_1} \times \sum_{i_0=0}^{N_1-1} \left( \frac{\dot{\lambda}_p}{\lambda} \right)^{i_0} \left( \frac{\tilde{\lambda}_{p+1}}{\lambda} \right)^{i_{p+1}} \cdot \left( \frac{\tilde{\lambda}_{p+2}}{\lambda} \right)^{i_{p+2}} \cdots \left( \frac{\tilde{\lambda}_r}{\lambda} \right)^{i_r} \times (i_1 + \cdots + i_p)b_p^+ + \cdots \\
&\times \left( \frac{\dot{\lambda}_1}{\lambda} \right)^{i_1} \left( \frac{\dot{\lambda}_p}{\lambda} \right)^{i_p} \left( \frac{\tilde{\lambda}_{p+1}}{\lambda} \right)^{i_{p+1}} \left( \frac{\tilde{\lambda}_{p+2}}{\lambda} \right)^{i_{p+2}} \cdots \left( \frac{\tilde{\lambda}_r}{\lambda} \right)^{i_r} (i_1 + \cdots + i_p)b_p^+. \tag{18b}
\end{align*}

The second moment becomes

\begin{align*}
E(T_{\mathcal{O}(p)}^2) &= \frac{d^2}{d\theta^2} T_{\mathcal{O}(p)}(\theta)|_{\theta = 0} \\
&= \left( \frac{\dot{\lambda}_1}{\lambda} \right)^{N_1} \times \sum_{i_0=0}^{N_1-1} \left( \frac{\dot{\lambda}_p}{\lambda} \right)^{i_0} \left( \frac{\tilde{\lambda}_{p+1}}{\lambda} \right)^{i_{p+1}} \cdot \left( \frac{\tilde{\lambda}_{p+2}}{\lambda} \right)^{i_{p+2}} \cdots \left( \frac{\tilde{\lambda}_r}{\lambda} \right)^{i_r} \times (i_1 + \cdots + i_p)b_p^+ + \cdots \\
&\times \left( \frac{\dot{\lambda}_1}{\lambda} \right)^{i_1} \left( \frac{\dot{\lambda}_p}{\lambda} \right)^{i_p} \left( \frac{\tilde{\lambda}_{p+1}}{\lambda} \right)^{i_{p+1}} \left( \frac{\tilde{\lambda}_{p+2}}{\lambda} \right)^{i_{p+2}} \cdots \left( \frac{\tilde{\lambda}_r}{\lambda} \right)^{i_r} (i_1 + \cdots + i_p)b_p^+. \tag{18b}
\end{align*}
The busy period is composed of two types of intervals:

(i) The busy period that has the first class-\( p \) test customer that arrives during the idle period consists of

\[
\begin{align*}
E(B^-_p\text{-cycle}) &= \left( \frac{b^-_p}{1 - \rho^+_p} \right), \\
\phi_{B^-_p} &= \lambda_p \left( \frac{b^-_p}{1 - \rho^+_p} \right) = \frac{\rho^-_p}{1 - \rho^-_p}. 
\end{align*}
\]

From \( \phi_{T_{0(p)}} + \phi_{B^-_p} = \rho \), we get

\[
\phi_{T_{0(p)}} = \frac{\rho^+_p (1 - \rho^-_p)}{1 - \rho^+_p}. 
\]

(ii) The delay cycle that starts with the service of an LC (there are no EHC s at this point) and is extended by its EHC-offsprings.

Let \( \phi_{T_{0(p)}} \) and \( \phi_{B^-_p} \) be the probabilities of the class-\( p \) test customer that arrives during the busy period and finds \( T_{0(p)}\)-cycle and \( B^-_p\)-cycle, respectively. Since every LC generates its own \( B^-_p\)-cycle, we get, after using

\[
\begin{align*}
E(W^*_{q(p),N\!p} | \text{busy}) &= \frac{(1 - \rho^-_p)(1 - T^*_p)\sigma^+_p}{\rho E(T_{0(p)})[\theta - \lambda_p + \lambda_p B^+_p \sigma^+_p]} + \frac{(1 - \rho^-_p)(1 - B^-_p \sigma^+_p)}{\rho E(B^-_p)[\theta - \lambda_p + \lambda_p B^+_p \sigma^+_p]}, \\
\end{align*}
\]

where \( \sigma^+_p = \theta + \lambda^+_p - \lambda^+_p \Theta^+_p \) and \( \Theta^+_p = B^+_p(\sigma^+_p) \). Note that \( \Theta^+_p(\theta) \) is a busy period that starts with the service of a HC and is extended by its HC-offsprings. The mean becomes

\[
\begin{align*}
E(W^*_{q(p),N\!p} | \text{busy}) &= \frac{(1 - \rho^-_p)\rho^+_p}{2\rho(1 - \rho^+_p)(1 - \rho^-_p)} \left( \frac{E(T_{0(p)}^2)}{E(T_{0(p)})} + \frac{\lambda_p^+_p b^+_p(2)}{1 - \rho^-_p} \right) \\
&+ \frac{\rho^-_p}{2\rho(1 - \rho^+_p)(1 - \rho^-_p)} \left( \frac{b^+_p(2)}{b^-_p} + \frac{\lambda_p^+_p b^+_p(2)}{1 - \rho^-_p} \right). 
\end{align*}
\]

The waiting time of a class-\( p \) test customer that arrives during the idle period consists of

(i) the remaining idle period,
(ii) the time it takes to service all HC s that arrive during the entire idle period and their HC-offsprings, and
(iii) the time it takes to serve all ECs that already exist at the arrival point of the test customer and their
EC-offsprings. Let \( \Phi_{(i_1, i_2, \ldots, i_r), p}(\theta) \) be the sum of (i) and (ii) under the condition that the arriving class-\( p \) test customer encounters
an idle state \((i_1, i_2, \ldots, i_r)\). Let \( \Phi_{(i_1, i_2, \ldots, i_r), p}^*(\theta) \) be the LST of the DF of \( \Phi_{(i_1, i_2, \ldots, i_r), p}(\theta) \). For \( p = 1 \), (ii) becomes
zero and \( \Phi_{(i_1, i_2, \ldots, i_r), 1}^*(\theta) \) is equal to \( R_{(i_1, i_2, \ldots, i_r)}^*(\theta) \). Conditioning on the class of the first arriving customer, we get

\[
\Phi_{(i_1, i_2, \ldots, i_r), p}^*(\theta) = \sum_{j=1}^{r} \frac{\hat{j}}{\lambda} T_j^*(\theta) \Phi_{(i_1, i_2, \ldots, i_j+1-\ldots, i_r, p)}^*(\theta),
\]

(23a)

where

\[
\Phi_{(i_1, \ldots, i_r), p}^*(\theta) = [B_p^*(\sigma_{p-1}^+)]^{i_1+\cdots+i_{p-1}}.
\]

(23b)

\( T_j^*(\theta) \) was defined and used in (4a). The mean becomes

\[
E(\Phi_{(i_1, i_2, \ldots, i_r), p}(\theta)) = \sum_{j=1}^{r} \frac{\hat{j}}{\lambda} \left[ \frac{1}{\lambda} + E(\Phi_{(i_1, \ldots, i_j, i_{j+1}, \ldots, i_r), p}(\theta)) \right],
\]

(24a)

where

\[
E(\Phi_{(i_1, \ldots, i_r), p}(\theta)) = \begin{cases} \frac{b_{p-1}^+}{1 - \rho_{p-1}^+} \left( N_j + \sum_{k=1}^{p-1} i_k \right), & (j = 1, 2, \ldots, p - 1), \\ \frac{b_{p-1}^+}{1 - \rho_{p-1}^+} \left( \sum_{k=1}^{p-1} i_k \right), & (j = p, p + 1, \ldots, r). \end{cases}
\]

(24b)

Let \( w_{(i_1, \ldots, i_r), p}^*(\theta) \) be the LST of the DF of the waiting time of the class-\( p \) test customer that arrives during
the idle period and encounters the state \((i_1, \ldots, i_r)\). \( w_{(i_1, \ldots, i_r), p}^*(\theta) \) is the LST of the DF of (i)+(ii)+(iii) and is given by

\[
w_{(i_1, \ldots, i_r), p}^*(\theta) = \Phi_{(i_1, \ldots, i_p+1-\ldots, i_r), p}^*(\theta)[B_p^*(\sigma_{p-1}^+)]^{i_p}.
\]

(25)

Then we get

\[
W_{q(p), NP}^*(\theta | idle) = \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_r=0}^{N_r-1} P_{(i_1, \ldots, i_r)} w_{(i_1, \ldots, i_r), p}^*(\theta).
\]

(26)

The mean becomes

\[
E(W_{q(p), NP} | idle) = \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_r=0}^{N_r-1} P_{(i_1, \ldots, i_r)} \left[ \frac{i_p b_p}{1 - \rho_{p-1}^+} + E(\Phi_{(i_1, \ldots, i_p+1-\ldots, i_r), p}(\theta)) \right].
\]

(27)

Finally, \( W_{q(p), NP}^*(\theta) \) can be obtained by using (21) and (26) in (16), and the mean waiting time becomes

\[
E(W_{q(p), NP}) = \rho E(W_{q(p), NP} | busy) + (1 - \rho) E(W_{q(p), NP} | idle)
\]

\[
= \rho \left[ \frac{(1 - \rho) \rho_{p-1}^+}{2 \rho (1 - \rho_{p-1}) (1 - \rho_{p}^+)} \left( \frac{E(T_{0(p)}^2)}{E(T_{0(p)})} + \frac{\lambda_p^+ b_p^+(2)}{1 - \rho_{p}^+} \right) + 2 \rho (1 - \rho_{p-1}^+) \frac{\rho_{p-1}^-}{b_p^-(2)} \left( \frac{\lambda_p^- b_p^+(2)}{b_p^-} + \frac{\lambda_p^+ b_p^+(2)}{1 - \rho_{p}^+} \right) \right]
\]

\[
+ \frac{\rho_{p-1}^-}{b_p^-(2)} \left( \frac{\lambda_p^- b_p^+(2)}{b_p^-} + \frac{\lambda_p^+ b_p^+(2)}{1 - \rho_{p}^+} \right)
\]
Fig. 3. Mean waiting time when \( N_1 \) varies (NP).

\[
\lambda_1 = \lambda_2 = \lambda_3 = 0.2 \\
E(S_1) = E(S_2) = E(S_3) = 1.0 \\
N_2 = 2, \ N_3 = 7
\]

\[
+(1 - \rho) \sum_{i_1 = 0}^{N_1 - 1} \cdots \sum_{i_r = 0}^{N_r - 1} P_{(i_1, i_2, \ldots, i_r)} \left[ \frac{i_r b_p}{1 - \rho_{p-1}} + E(\Phi_{(i_1, \ldots, i_r+1, \ldots, i_r)}(p)) \right]. \tag{28}
\]

The mean number of customers can be obtained from the Little’s law.

Figs. 3 and 4 show the behavior of the mean waiting time when one of the thresholds varies while the other two are fixed.

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Appendix Derivation of Eq. (7)

Let us consider the case for three classes. The \( r \)-class case is a simple extension. Let \( T_{N_p} \) be the time until \( N_p \) class-\( p \) customers arrive. \( T_{N_p} \) follows the Erlang distribution with pdf \( f_{T_{N_p}}(t) = \frac{N_p^{N_p}}{t^{N_p-1} e^{-\lambda t}}/N_p! \). Let \( F_{T_{N_p}}(t) \) be the distribution function of \( T_{N_p} \) and let \( F_{T_{N_p}}^*(\beta) \) be the LST of \( F_{T_{N_p}}(t) \). Then we have

\[
K(z_1, z_2, z_3) = \sum_{n=0}^{N_1} \sum_{m=0}^{N_2} \sum_{z_1}^{z_1} \sum_{z_2}^{z_2} \sum_{z_3}^{z_3} \int_0^\infty \frac{(\lambda_2 t)^m}{m!} \frac{(\lambda_3 t)^n}{n!} e^{-\lambda t} \, dF_{T_{N_p}}(t)
\]

\[
+ \sum_{n=0}^{N_1} \sum_{m=0}^{N_2} \sum_{z_1}^{z_1} \sum_{z_2}^{z_2} \sum_{z_3}^{z_3} \int_0^\infty \frac{(\lambda_1 t)^m}{m!} \frac{(\lambda_3 t)^n}{n!} e^{-\lambda t} \, dF_{T_{N_p}}(t)
\]

\[
+ \sum_{n=0}^{N_1} \sum_{m=0}^{N_2} \sum_{z_1}^{z_1} \sum_{z_2}^{z_2} \sum_{z_3}^{z_3} \int_0^\infty \frac{(\lambda_1 t)^m}{m!} \frac{(\lambda_2 t)^n}{n!} e^{-\lambda t} \, dF_{T_{N_p}}(t).
\] (A.1)

It suffices to see the first term on the right-hand side of (A.1). It becomes

\[
\sum_{n=0}^{N_1} \sum_{m=0}^{N_2} \sum_{z_1}^{z_1} \sum_{z_2}^{z_2} \sum_{z_3}^{z_3} \int_0^\infty \frac{(\lambda_2 t)^m}{m!} \frac{(\lambda_3 t)^n}{n!} e^{-\lambda t} \, dF_{T_{N_p}}(t)
\]

\[
= \sum_{n=0}^{N_1} \sum_{m=0}^{N_2} \sum_{z_1}^{z_1} \sum_{z_2}^{z_2} \sum_{z_3}^{z_3} \frac{1}{m! n!} \frac{1}{(\lambda_2)^m} (\frac{1}{(\lambda_3)^n}) \int_0^\infty t^{n+m} e^{-(\lambda_2+\lambda_3) t} \, dF_{T_{N_p}}(t).
\] (A.2)

To evaluate the right-hand side of (A.2), we consider the following identity:

\[
\int_0^\infty e^{-\beta t} t^k \, dF_{T_{N_p}}(t) = (-1)^k \frac{d^k}{d\beta^k} F_{T_{N_p}}^*(\beta) = (-1)^k \frac{d^k}{d\beta^k} \left( \frac{\lambda_1}{\lambda_1 + \beta} \right)^{N_1}
\]

\[
= \left( \frac{\lambda_1}{\lambda_1 + \beta} \right)^{N_1} \frac{1}{(\lambda_1 + \beta)^k}
\]

\[
\left[ N_1(N_1 - 1) \cdots (N_1 - k + 1) \right] + \frac{k!}{(k - 2)!} [N_1(N_1 - 1) \cdots (N_1 - k + 2)]
\]

\[
+ \frac{(k - 1)k!}{(k - 3)!} [N_1(N_1 - 1) \cdots (N_1 - k + 3)]
\]

\[
+ \cdots + \frac{(k - 1)(k - 2) \cdots 4k!}{2(k - 3)!} [N_1(N_1 - 1) - 1](N_1 - 2)] + \frac{(k - 1)(k - 2) \cdots 3k!}{1!(k - 2)!} [N_1(N_1 - 1)]
\]

\[
+ \frac{(k - 1)(k - 2) \cdots 2k!}{(k - 1)!} N_1
\]

(A.3)
By replacing \( \beta \) by \( \beta' + \beta_2 \) and \( k \) by \( n + m \) in (A.3), (A.2) becomes

\[
\sum_{n=0}^{N_1-1} \sum_{m=0}^{N_2-1} z_1^{N_1} z_2^{m} z_3^{n} \int_{0}^{\infty} \frac{e^{-\beta' t} \beta' t^m e^{-\beta_2 t} \beta_2 t^n n!}{m!} \text{d}F_{N_1}(t)
\]

\[
= \sum_{n=0}^{N_1-1} \sum_{m=0}^{N_2-1} z_1^{N_1} z_2^{m} z_3^{n} \frac{1}{m! n!} (\beta')^m (\beta_2)^n \left( \frac{\beta_1}{\beta} \right)^{N_1} \left( \frac{1}{\beta} \right)^{n+m} \times B,
\]

where

\[
B = \left\{ \begin{array}{l}
[N_1(N_1-1) \cdots (N_1-n-m+1)] + \frac{(m+n)!}{(m+n-2)!} [N_1(N_1-1) \cdots (N_1-m-n+2)] \\
+ \frac{(m+n-1)(m+n)!}{(m+n-3)!} [N_1(N_1-1) \cdots (N_1-m-n+3)] + \\
+ \frac{2!(m+n-3)!}{(m+n-1)(m+n-2)!} [N_1(N_1-1)(N_1-2)] \\
+ \frac{11!(m+n-2)!}{(m+n-1)(m+n-2)!} [N_1(N_1-1)] \\
- \frac{(m+n-1)(m+n)!}{(m+n-1)!} N_1
\end{array} \right.
\]

Now using the identity

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r},
\]

we get

\[
B = (m+n)! \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} \binom{N_1}{j+1} = (m+n)! \binom{N_1+m+n-1}{m+n}.
\]

Thus (A.2) becomes

\[
\sum_{n=0}^{N_1-1} \sum_{m=0}^{N_2-1} z_1^{N_1} z_2^{m} z_3^{n} \int_{0}^{1} \frac{e^{-\beta' t} \beta' t^m e^{-\beta_2 t} \beta_2 t^n n!}{m!} \text{d}F_{N_1}(t)
\]

\[
= \sum_{n=0}^{N_1-1} \sum_{m=0}^{N_2-1} z_1^{N_1} z_2^{m} z_3^{n} \frac{1}{m! n!} (\beta')^m (\beta_2)^n \left( \frac{\beta_1}{\beta} \right)^{N_1} \left( \frac{1}{\beta} \right)^{n+m} \times B
\]

\[
= \left( \frac{\beta_1}{\beta} \right)^{N_1} \sum_{n=0}^{N_1-1} \sum_{m=0}^{N_2-1} \binom{m+n}{m} \binom{N_1+m+n-1}{m+n} \left( \frac{\beta_2}{\beta} \right)^{n+m} \left( \frac{\beta_3}{\beta} \right)^{n}.
\]

Similar derivations can be applied to the second and third terms of (A.1) to get (7).

References