Delesse principle and statistical fractal sets: 2.
Unified Fractal Model for soil porosity

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Abstract

Self-similar fractals are useful models of the soil solid and pore sets along lines, across areas and inside volumes. The scaling properties of these sets, analysed by the divider method within the known scale range over which the fractal extends, were described by three parameters, the solid \( D_s \) and pore \( D_p \) linear box or capacity empirical fractal dimensions, and the total number of rulers that covered the line \( N_t \). The empirical fractal model for soil porosity was developed and compared with the more commonly used theoretical one, and the Unified Fractal Model (UFM) for soil porosity was proposed. This model extracts the soil linear (NL), areal (AP) and volumetric (VP) porosity from the solid and pore distributions along the lines. The accuracy of the empirical model was tested for NL, AP and VP, within the scale range from 0.008 to 3 mm, by using the real macro and micromorphological images of soils and sediments with contrasting genesis, available from field experiments and published by other researchers. It was shown that the proposed model offers a rapid and statistically coherent solution to porosity estimation. The UFM for soil porosity was derived starting from the theoretical and empirical ones. The UFM is useful to solve the relationship between NL, AP and VP, and to estimate the alternative space filling ability of solid and pore sets along a line, across an area and inside a volume. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Currently, image analysis is recognised as the most reliable method for measuring soil structure (Crawford et al., 1993; Anderson et al., 1998). However, the dimensional equivalents of spatial objects and their projected images in \( E \)-dimensional space (where \( E \) is the Euclidean dimension, i.e., \( E = 1, 2 \) or 3) are still poorly defined. These equivalents were studied in detail by stereology, and described in 1848 by Dellesse, (quoted by Weibel, 1989) and by Rosiwal (1898). There are two basic principles of stereology, known as Delesse's and Rosiwal's principles, that establish the next relation for each component \( a \) distributed randomly inside the object structure, and sectioned at random by testing lines:

\[
V_V = A_A = L_L, \tag{1}
\]
where \( V_{V_i}, A_{A_i} \) and \( L_{L_i} \) are the component density, corresponding to the fractions of \( a \)'s volume, area and length, respectively inside the object, traversed by test lines. The soil linear (NL), areal (AP) and volumetric (VP) porosity, estimated as the proportion of pores inside the tested \( E \)-section of soil, are similar in origin to length \( (L_S) \), surface \( (S_V) \) and volume \( (V_V) \) density, respectively, used as the basic structural descriptors in stereological methods (Dexter, 1976). Our previous research has shown that for statistical fractal solid and pore sets, the equality (1) between NL, AP and VP was never reached (Oleschko, 1998).

Starting from the Delesse and Rosiwal principles the dimensional equivalents for the object embedded in the Euclidean one- \((D_{NL})\), two- \((D_{AP})\) and, three- \((D_{VP})\) dimensional space \((D_{VP})\), can be established as follows:

\[
D_{NL} + 2 = D_{AP} + 1 = D_{VP}.
\]

However, for all studied soils and sediments of Mexico with contrasting genesis and marked differences in morphology, the fractal dimension across an area was close to double the value of the set dimension along the line (Oleschko, 1998). The correction for the Delesse and Rosiwal principles applied to statistical fractal sets, was derived from the empirical data, and it was supposed that this correction may be suitable to clarify the relation between the NL, AP and VP (Oleschko, 1998).

The main hypothesis of the present work, is that the set of the \( E \)-dimensional cross-sections of statistical fractal, is the self-similar fractal itself and then, the proportionality between the cross-section length, area and volume, established by Mandelbrot (1983) for fractals, does exist. The cited author has mentioned, that the linear and planar sections of a random fractal, are themselves fractal sets. Unfortunately, this result is hard to illustrate in the case of non-random fractals, which have axes of symmetry (Mandelbrot, 1983). However, Mandelbrot (1983) gave the examples of the Sierpinski carpet and the triadic Menger sponge sections, coming to the conclusion that these sets are Cantor dusts.

This study has three principal objectives:

1. To use the corrected Delesse principle, for the empirical solution of a more common \( E \)-dimensional theoretical fractal model of soil porosity.
2. To test the accuracy of the proposed model by using: (a) real macro- and micromorphological data, obtained by point- and box-counting procedures; (b) some published data; and (c) data from field experiments.
3. To derive the Unified Fractal Model for soil \( E \)-dimensional porosity.

2. Mathematical addendum

2.1. Theoretical fractal model

Rieu and Sposito (1991), Perrier (1995) and Rieu and Perrier (1998) considered lacunar models to be particularly suited to model soils, because they represent simultaneously the solid and the void phase. Rieu and Perrier (1998) have proposed two models, able to describe the spatial arrangement of both phases inside a fractal object, assuming an upper cut-off of scale \( (l_{\text{max}}) \) to the fractal set \( (F) \), and measuring \( (F) \) at increasing resolution, by using \( E \)-tiles of decreasing linear size \( (l_{\text{min}}) \) and \( E \)-volume \((l^E)\). In model 1 (A in the original work), \( F \) represents the solid phase and is bounded by two cut-off values of scale \( (l_{\text{max}} \text{ and} l_{\text{min}}) \), and the mass of the solid phase scales as a power-law of the object size. In this mass fractal model, pores are considered as a complement of \( F \) in the \( E \)-dimensional space. In model 2 (B in Rieu and Perrier, 1998), called the pore fractal, \( F \) represents the pore space and is also bounded by two cut-off values of scale (Rieu and Perrier, 1998). Models 1 and 2 have been extended to any continuous value of length-scale \( l_{\text{min}} \) and led to general equations expressing the volume \((VE)\) of the \( D \)-dimensional fractal object \( F \) of size \( L_i \), embedded in the Euclidean space of dimension \( E \), as power-laws of the length-scale:

\[
VE(L_i) = (l_{\text{min}})^{E-D}(L_i)^D.
\]

For a fractal object that presents both upper and lower cut-off values of scale, Rieu and Perrier (1998) arrived at compact and simple expressions, by using the dimensionless variable \( il_{\text{min}} \). Model 1 is able to estimate only the partial porosity \( \Phi \) (for pores with size >\( l \)):

\[
[\Phi > l] = 1 - (l/l_{\text{max}})^{E-D}.
\]
For $F$ described by model 2, the total porosity $\Phi$ is equal to:

$$\Phi = (l_{\text{min}}/l_{\text{max}})^{E-D}. \quad (5)$$

However, empirical testing of both models, is far from complete.

### 2.1.1. Point-counting procedure used for empirical fractal dimension estimation

Most algorithms, designed for soil fractal analysis, use the traditional box-counting technique, applying it to the dilated and eroded soil images (Anderson et al., 1996, 1998). However, by definition, each point of a fractal set should be covered by a measuring box (Falconer, 1990). In our previous research, the algorithms based on stereological principles were designed and used for the fractal dimension measurements (Oleschko et al., 1997, 1998; Oleschko, 1998). The basic test system applied to the soil sections, is a regular grid bounded by a frame, where every pair of perpendicular test lines determines a test point. This grid with varying box side ($r$, the ruler or divider length, or the scale of observation) is scratched on the surface of the thick and thin soil sections. The number of solid ($N_s$) and pore ($N_p$) intersections with the test lattice is described following the Dexter (1976) procedure, by using the binary system, with 1 representing the solid and 0 the void (Thompson et al., 1987; Oleschko et al., 1997). The model of $D$-dimensional fractal point sets on a straight line, is used for the soil dimension estimation.

A general property of the $D$-dimensional (where $D$ is a fractal and therefore fractional dimension) fractal point set on a straight line, is expressed as follows: any interval of length $L$ contains $N$ points of the fractal set, at resolution $r$ ($L/r \gg 1$) (Korvin, 1992), or:

$$N \propto (L/r)^D, \quad (6)$$

where $N$ was assumed to be equal to the number of test points coinciding with a solid ($N_s$) or pore ($N_p$) on each horizontal line of test grid. The ratio between ($L$) and ($r$) represents the total number ($N_i$) of test points, considered along each line of length $L$. Eq. (6) can be transformed in (7) for model 1:

$$N_s \propto N_i^D, \quad (7)$$

and in (8) for model 2:

$$N_p \propto N_i^D. \quad (8)$$

The similarity between the ratios ($L/r$) in (6), ($N_i$) in (7) and (8), and ($l_{\text{max}}/l_{\text{min}}$) in (4) and (5), should be noted.

From binary data, obtained for each test lattice, three logarithms have to be calculated at each scale of observation: $\ln(N_s)$, $\ln(N_p)$ and $\ln(N_i)$. Plotting of each pair of them ($\ln(N_s)/\ln(N_i)$) and ($\ln(N_p)/\ln(N_i)$) gives the set linear fractal dimension, determined as the slope of the corresponding graph. The first may be referred as solid ($D_s$) and the second as pore ($D_p$) mass fractal dimension. The soil linear fractal dimension, in one sense, is a measure of how much space a particular solid or pore set fills along a line (Brown, 1987; Falconer, 1990), and consequently it is the measure of the soil NL porosity.

### 2.2. Empirical fractal model

It has been mentioned that, the soil NL porosity is defined as the proportion of the length of each line ($N_i$ multiplied by $r$) occupied by pores ($N_p$ multiplied by $r$, where $r$ is the scale of observation). In model 2, it is estimated directly from the pore distribution, as:

$$NL = (N_p/N_i). \quad (9)$$

In the fractal model 1, NL is considered as the complement of the one-dimensional Euclidean space occupied by fractal solid set:

$$NL = 1 - (N_s/N_i). \quad (10)$$

It is easy to introduce the fractal dimensions $D_s$ and $D_p$ in Eqs. (9) and (10), by using the proportionality (7) and (8), and to derive analytical expressions for models 2 (11) and 1 (12), respectively:

$$NL \propto N_i^{D_p - 1}. \quad (11)$$

and,

$$NL \propto 1 - N_i^{D_s - 1}. \quad (12)$$

Two- and three-dimensional soil porosity can be derived similarly. By definition, the soil AP and VP porosity, are respectively the proportions of pore area across the total sample area, and pore volume inside the total volume (Jury et al., 1991).
It is easy to express the soil NL in terms of pore length:

\[ \text{NL} = \frac{N_p r}{N_t r} \]  

(13)

If the main hypothesis of the present work is correct, and the sections of a fractal are themselves a statistical fractal set, therefore the length–area–volume relations established by Mandelbrot (1983) for fractals, are applicable to the section sets. Consequently, to the AP and VP porosity modelling, the following proportion is useful:

\[ \text{AP}^{1/2} \propto \left( \frac{N_p r}{N_t r} \right) \]  

(14)

And, after transformation,

\[ \text{AP} \propto (\text{NL})^2 \propto N_t^{2(D_p - 1)} \]  

(15)

The latter equation however is correct only for model 2. For model 1 the area of the solid set should be estimated first. This area (SA) is proportional to the length of solid set (SL), and hence:

\[ \text{SA} \propto (\text{SL})^2 \propto N_t^{2(D_s - 1)} \]  

(16)

For model 1, porosity can be estimated as the complement of the fractal set of solids embedded in two-dimensional Euclidean space, as follows:

\[ \text{AP} \propto 1 - N_t^{2(D_s - 1)} \]  

(17)

The same simple mathematical solution is possible for VP, where it can be shown that:

\[ \text{VP} \propto 1 - N_t^{3(D_p - 1)} \]  

(18)

for model 1, and:

\[ \text{VP} \propto N_t^{3(D_p - 1)} \]  

(19)

for model 2.

In (18) and (19) it is possible to replace the variable \( N_t \) by \( L/r \), and both equations become exactly the same as (4) and (5) (Rieu and Perrier, 1998). Equivalencies between the terms \( 3D_s \) and \( 3D_p \) of (18) and (19), and \( D \) of (4) and (5), seem evident.

In Eqs. (15)–(19), the terms \( N_t^{2(D_p - 1)} \), \( N_t^{2(D_s - 1)} \), \( N_t^{3(D_p - 1)} \), and \( N_t^{3(D_s - 1)} \) have clear physical significance, and express how much space a fractal set of pores or solids fills across an area (AP) or inside a volume (VP).

3. Materials and methods

3.1. Model testing

For unbiased testing of the proposed empirical model in the \( E \)-dimensional space, three types of analysis were carried out: (I) In one-dimensional space, the soil linear porosity was modelled using the data obtained by point-counting technique; (II) In two-dimensional space, the box-counting procedure was applied to some published images, and estimated fractal parameters were used for the areal porosity modelling. The obtained results were compared with published data. (III) In three-dimensional space, the micromorphological images were used for the fractal dimension estimation, and modelled volumetric porosity was compared with field measured data. Any of the described procedures is useful for the \( E \)-dimensional porosity testing.

3.1.1. One-dimensional space

The binary experimental data obtained through Dexter’s technique (Dexter, 1976) during 12 years in previous research projects, related to the dynamics of some agricultural soils structure in Mexico, were used for the model testing in the one-dimensional space. These studies included the analysis of the undisturbed and disturbed samples from the topsoil (0–20 cm), of three representative agricultural soils of Mexico under different management systems. The results of fractal analysis of these soils along a line and across an area, have been published before (Oleschko et al., 1997, 1998). The Mollic Andosol (Michoacán) was studied for three tillage practices: (1) zero tillage (ZT); (2) reduced (or minimum) tillage (MT); and (3) traditional tillage (TT).

For the Eutric Vertisol (Veracruz), three agroecosystems were compared: (1) A Vertisol under secondary vegetation (NV), that had not been cultivated for several years. (2) An experimental lot that had been cropped since 1981 using minimum tillage (MT) system and where permanent beds (1.5 m wide) were established. In this field, two areas with different degree of compaction were sampled: the bed area (MT_b) and the furrows between the beds (MT_f), which drain the excess water during the rainy season. (3) The third experimental field had been tilled by conventional methods (TT) since 1980.
Luvic Acrisol (the savanna of Huimanguillo, Tabasco) was studied under three management systems: (1) cattle grazing (CG), practised for the last 15 years; (2) citrus orchard with “chapeo” (weeding with weed cutters) for weed control, used for 10 years (CHS); (3) citrus orchard with weed control by disc harrowing at a depth of 15 cm, used for the last 12 years (HAR). All mentioned systems were compared with Acrisol under secondary natural vegetation (NV), kept intact at least for the last 20 years. Undisturbed soil samples were collected with metal samplers (12 cm long, 14 cm wide and 25 cm high) at each experimental site of interest. All samples were taken at field moisture and carried in plastic bags without drying. In the laboratory, samples were dried by the acetone replacement (in liquid phase) method and then impregnated with a 1:1 mixture of polyester resin (HU-543) and acetone (Murphy, 1986). Soil cores were then re-impregnated with the same resin under vacuum conditions. After the resin had hardened sufficiently, samples were horizontally sectioned parallel to the soil surface in thick and thin sections. Three thin sections (2.4 cm, with an average thickness of 30 μm) were prepared from each thick section, and analysed under the petrographic microscope (Olympus, BH-2). Five horizontal lines distributed by probabilistic design were scratched on the thick and thin section surfaces. The point-counting procedure designed by Dexter (1976) and described above, was used for the solid and pore sets analysis along the test lines. The mean value of all parameters was obtained from 25 observed lines. The binary data were used for empirical fractal dimension estimation (see above).

### 3.1.2. Two-dimensional space

Two digital images of size 1000 × 1000 to 600 × 600 pixels, were obtained from each one of twelve vertical thin section images, published by Anderson et al. (1996). These images were submitted to the Linfrac and Fractal programs, designed by Parrot and Rico (1997) specially for the fractal analysis of solid and pore sets along the lines, and across the areas. Both mentioned programs were calibrated before the analysis, by using the images of the deterministic mass fractals (Mandelbrot, 1983). Calibration results were also published (Oleschko, 1998).

Four fractal dimensions were estimated for each image: the solid set mass fractal dimension along a line ($D_s$) and across an area ($D_{As}$), and the pore set mass fractal dimension along a line ($D_p$) and across an area ($D_{Ap}$). These values were used in Eq. (23) to model the soil AP. All modelled data were compared to the reported one.

### 3.1.3. Three-dimensional space

Three soils and sediments from Mexico with contrasting genesis and physical properties, were used for the model testing in the three-dimensional space. All profiles were sampled according to the genetic horizons, and described in detail by Oleschko (1998).

In the Melanic Andosol (Veracruz) under maize, three horizons with significant differences in bulk density (ρ_b) were sampled within the 130 cm profile. The topsoil was the most dense (ρ_b = 0.39 Mg m⁻³) and corresponded to the arable horizon. The densities of other horizons varied between 0.30 and 0.37 Mg m⁻³.

Eutric Vertisol from Bajio, Guanajuato State, was characterised as homogeneous from the morphological point of view. However, the bulk density of the soil increased from 1.07 to 1.35 Mg m⁻³ in the first 45 cm. The latter value coincided with the dense plough pan found at 30 cm. Three layers were sampled in this profile.

The 160 cm profile of lacustrine sediments was sampled on the bank of the dry Texcoco Lake, Mexico State. The profile is divided in two contrasting parts by the compacted basaltic volcanic ash horizon (ρ_b = 1.21 Mg m⁻³), presented at 39 cm depth. This horizon is the textural screen between the Mollic layer (ρ_b = 0.98 Mg m⁻³), developed in the upper part of the studied profile, and lacustrine clay sediments (from 69 to 160 cm). Within these sediments, the bulk density varied from 0.43 Mg m⁻³ in the upper part of the deposit to 0.28 Mg m⁻³ on the limit with underground water. Six horizons were sampled and used for the model testing.

One profile of cemented volcanic soils (tepetates) was sampled in Mexico State too. In this profile, various clay layers separate all cemented horizons (fragipans). Some of these layers are paleosols with vertic properties, and bulk densities range from 1.15 to 1.30 Mg m⁻³. Tepetates are characterised by specific pore space morphology: all pores are isolated and
occupied by cementing agents, and bulk densities vary from 1.25 to 1.68 Mg m\(^{-3}\). Nine horizons were described in this profile: the arable layer of recovered tepetate, three cemented horizons, and five paleosoils.

The sampling procedure for the micromorphological analysis and thin section preparation, was similar to that described above for the soils used in the one-dimensional model testing.

The VP of these soils and sediments, was estimated from the microscopic images, applying Eq. (26). The solid set fractal dimension inside the volume (three-dimensional space), was estimated from the set two-dimensional dimension, obtained on the micromorphological images. The set dimension across the area, calculated by the Fractal program (Parrot and Rico, 1997), was used for the set dimension estimation along the line. The latter value was obtained by using the corrected Delesse principle: \(D_s = D_{Asm}/2\). These values were always statistically similar to \(D_s\) estimated by the Linfrac program along a line. The set dimension inside the volume was obtained multiplying \(D_s\) by 3. Two sizes of the digitised images were used: 1000 \(\times\) 1000, and 600 \(\times\) 600 pixels.

Modelled volumetric porosity was compared with data, calculated from the soil bulk (\(\rho_b\)) and particle (\(\rho\)) densities, according to the traditional equation:

\[
P_{tot} = 1 - \rho_b/\rho.
\]  
(20)

The saran-coated undisturbed clod method, was used for bulk density measurements (Brasher et al., 1966).

Table 1

<table>
<thead>
<tr>
<th>Soil Type</th>
<th>Management practice</th>
<th>(N_t = (N_s + N_p)b)</th>
<th>(D_s^c)</th>
<th>NL(_{mod})</th>
<th>NL(_{mes})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mollic Andosol</td>
<td>Zero tillage</td>
<td>146(5.4)(^d) + 93(9.1)</td>
<td>0.91(1.5)</td>
<td>38.9(5.4)</td>
<td>38.9(4.7)</td>
</tr>
<tr>
<td></td>
<td>Minimum tillage</td>
<td>151(5.9) + 97(9.9)</td>
<td>0.91(1.2)</td>
<td>39.1(9.3)</td>
<td>39.1(9.0)</td>
</tr>
<tr>
<td></td>
<td>Traditional tillage</td>
<td>144(4.2) + 102(5.7)</td>
<td>0.90(0.8)</td>
<td>42.3(6.2)</td>
<td>41.5(5.2)</td>
</tr>
<tr>
<td>Luvic Acrisol</td>
<td>Natural vegetation</td>
<td>188(5.9) + 64(15.2)</td>
<td>0.95(1.0)</td>
<td>25.1(15.6)</td>
<td>25.4(15.6)</td>
</tr>
<tr>
<td></td>
<td>Cattle grazing</td>
<td>197(2.99) + 58(9.5)</td>
<td>0.95(0.5)</td>
<td>22.4(9.2)</td>
<td>22.7(9.2)</td>
</tr>
<tr>
<td></td>
<td>Clearing with weed cutter</td>
<td>199(4.0) + 53(12.0)</td>
<td>0.96(0.6)</td>
<td>21.2(11.7)</td>
<td>21.0(11.0)</td>
</tr>
<tr>
<td></td>
<td>Citrus orchards</td>
<td>211(4.2) + 43(21.1)</td>
<td>0.97(0.8)</td>
<td>17.0(21.0)</td>
<td>16.9(13.6)</td>
</tr>
<tr>
<td>Eutric Vertisol</td>
<td>Natural vegetation</td>
<td>185(2.8) + 61(8.5)</td>
<td>0.95(0.5)</td>
<td>25.3(8.6)</td>
<td>24.8(8.4)</td>
</tr>
<tr>
<td></td>
<td>Minimum tillage (bed)</td>
<td>206(3.9) + 42(16.7)</td>
<td>0.97(0.7)</td>
<td>17.1(17.0)</td>
<td>16.9(16.8)</td>
</tr>
<tr>
<td></td>
<td>Minimum tillage (furrows)</td>
<td>214(4.3) + 32(25.4)</td>
<td>0.98(0.7)</td>
<td>12.3(19.0)</td>
<td>13.0(19.0)</td>
</tr>
<tr>
<td></td>
<td>Traditional tillage</td>
<td>194(2.7) + 53(8.5)</td>
<td>0.96(0.4)</td>
<td>21.4(8.7)</td>
<td>21.5(7.9)</td>
</tr>
</tbody>
</table>

\(^a\) Only the first scale data are used (\(\varepsilon = 0.5\ mm\)), and the mean value of all parameters was obtained from 25 observed lines.

\(^b\) The total number of observed elements (\(N_t\)) is equal to the sum of solids (\(N_s\)) and pores (\(N_p\)).

\(^c\) The solid set fractal dimension along the lines.

\(^d\) The coefficient of variation is in parentheses.

The mean sample size (\(L\)) was equal to 6 \(\times\) 6 cm, and comparable to the soil section size, used for the fractal analysis.

### 3.2. Data variation

The data variation was analysed through classical, statistical procedures, and the variances were compared. Tukey’s test (\(P = 0.01\) and \(P = 0.05\)) was used to estimate the statistical significance of the differences between modelled and measured porosity.

### 3.3. Results

#### 3.3.1. Linear porosity: point-counting technique

The number of solids (\(N_s\)) and pores (\(N_p\)) along the lines of known length (\(L\)), was significantly different at all compared scales, for studied soils of contrasting genesis and under different management practices (Table 1). Consequently, the solid (\(D_s\)) and pore (\(D_p\)) mass fractal dimensions along the lines, and the measured one-dimensional soil linear porosity (NL\(_{mes}\)) were different. Notwithstanding, the plotted points, representing the quotients \(\ln(N_s)/\ln(N_t)\) and \(\ln(N_p)/\ln(N_t)\), are distributed near the straight line with very little scatter, that show the near “ideal” fractal nature of solid and pore sets (Oleschko et al., 1997). For the same soil, the solid (\(D_s\)) and pore (\(D_p\)) fractal dimensions along the lines, were different between them and depended on the management system used.
The observed differences were small, and in the former case ($D_s$ vs. $D_p$) were statistically negligible; however in the latter case (for management systems comparison), the differences were statistically significant (Oleschko et al., 1997). For instance, in the Eutric Vertisol, under native vegetation, the solid set linear fractal dimension was equal to 0.95, and under the more compacted condition (minimum tillage, furrows) it was 0.98. For the same sample, the pore set linear dimension was equal to 0.94, under both applied management systems.

The soil linear porosity ($NL_{mod}$), modelled by empirical model (Eq. (12)), was always close to that measured by the Dexter’s procedure ($NL_{mes}$) (Table 1). All detected differences were not statistically significant. This fact confirms that Eq. (12), is only the other fractal form of the relation classically defined as soil linear porosity (11).

3.3.2. Areal porosity: automatic image fractal analysis

At the next step, the two-dimensional images, published by Anderson et al. (1996), were used for the empirical model testing in two-dimensional space. All mentioned images are referred to soils with different level of compaction, including samples with three different porosities. For these soils, only pore set mass fractal dimensions ($D_m$) were estimated by the cited authors, because as a result of the dilated and eroded images used, inconsistent values for solid set fractal dimension were obtained. In this procedure a part of the fractal set remains unmeasured, and the value of solid mass fractal dimension becomes close to the upper topological limit for the fractal analysis in two-dimensional space, equal to 2 (Anderson et al., 1996, 1998).

The results of fractal analysis, carried out by Anderson et al. (1996), show a clear trend relating pore fractal dimension with compaction. As the porosity increases, the values of the pore mass fractal dimension generally increase as well (Anderson et al., 1996).

The “Fractal” program, used in the present work, was able to measure pore and solid set mass fractal dimensions. The pore fractal dimensions, reported by Anderson et al. (1996) and measured by the Fractal program were similar (Table 2, Fig. 1), and the small differences detected, may be related to differences in the nature of the applied measurement procedure (Oleschko et al., 1998). Soil pore and solid mass fractal dimensions were strongly dependent on soil porosity, and show the opposite tendency in their dynamics. As the image porosity is increased, the pore mass fractal dimension generally increases too. At the same time, a strong decrease in solid mass

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Table 2
Soil areal porosity modelled by Eq. (23) ($AP_{mod}$) compared with that reported by Anderson et al. (1996) ($AP_{rep}$)

<table>
<thead>
<tr>
<th>Soil</th>
<th>$N_i$</th>
<th>$D_{sm}$</th>
<th>$D_{pm}$</th>
<th>$D_{mrep}$</th>
<th>AP$_{mod}$ (%)</th>
<th>AP$_{rep}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Top</td>
<td>1000</td>
<td>1.9449</td>
<td>1.8061</td>
<td>1.779</td>
<td>31.7</td>
<td>27.0</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>1.9486</td>
<td>1.8067</td>
<td>–</td>
<td>27.9</td>
<td>27.0</td>
</tr>
<tr>
<td>1 Bottom</td>
<td>1000</td>
<td>1.9624</td>
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</table>

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a The first three images of Anderson et al. (1996), used for testing, are presented in this table.
b All images are specified in Anderson et al. (1996).
c The linear size of digital image in pixels.
d The solid set mass fractal dimension.
e The pore set mass fractal dimension.
f Pore mass fractal dimension, reported by Anderson et al. (1996).
fractal dimension was documented (Table 2). Anderson et al. (1996) concluded, that the solid set mass fractal dimension (Dsm) did not discriminate between the different soil structures. Nonetheless, the solid set mass fractal dimensions, measured by the Fractal program, show results always coherent and opposite to the pore dynamic. The AP, modelled by Eq. (23), was close to that reported by Anderson et al. (1996), for all analysed samples. The modelled porosity data, compared with the reported ones, fit the straight line with the determination coefficient of 0.9617 (Fig. 1). The modelled porosity was closer to the data reported by Anderson et al. (1996) for small images (600 x 600 pixels).

3.4. Volumetric porosity

The soils and sediments set with contrasting genesis, was involved in the model testing referred to the three-dimensional space. The VP, modelled by Eq. (26) (VPmod), was compared with VP estimated from bulk and particle densities, by using the reference Eq. (20) (VPmes). The high correlation between these variables, with determination coefficient of 0.9285, was obtained for Melanic Andosol, Eutric Vertisol and Tepetates profiles, all analysed together (Fig. 2). The porosity estimated from 600 x 600 pixel images was smaller then the value obtained from 1000 x 1000 pixel images, but the differences were not statistically significant (Table 3). In general, a small digital image size corresponds to a small solid and pore set fractal dimension estimated across an image area. Sometimes, a decrease in Nt was compensated by a Dsm drop, and the porosity value becomes quasi-stable, and independent on the image size.

For a better understanding of these results, the data obtained for the dry Texcoco Lake profile, consisting of contrasting materials, will be discussed in detail. The porosity modelling results of four more contrasting layers, are shown in Table 3. The images of thin sections were digitised, and two sizes of digital images were used for fractal analysis and modelling: 1000 x 1000, and 600 x 600 pixels.

Very similar values for VP, ranging from 0.49 to 0.59, were obtained by model 1 (Eq. (26)), for contrasting materials in digital images of different size. The modelled porosity was close to the values estimated from the field bulk densities for all horizons, except clay. The best agreement between the modelled and measured volumetric porosity, was observed in the volcanic ash horizon. Here, the modelled VPmod was close to 0.56 and the measured one close to 0.55. The sandy homogeneous texture of this horizon, may be responsible for such a good fit between the modelled and measured porosity. The real problem for the modelling was found in the clay horizon, where the model was able to extract the VP of 0.49–0.51, while the VP estimated from the bulk density data, was close to 0.86. These results confirm our early suggestion (Oleschko, 1998), that the sub-microscopic porosity, invisible on the microphotographs, has escaped from the fractal analysis, and therefore from the modelling by model 1. In all horizons with high clay content, an underestimation of pore set mass fractal dimension,
and also of the modelled porosity, should be expected. Anderson et al. (1998) came to the same conclusion, and have emphasised that the fractal analysis of the micromorphological images, takes into account only pores that are visible at the resolution of the thin section, or the photograph taken from the thin section. Nevertheless, in general the digital image size has clear influence on the pore and solid set fractal dimension, and also on the modelled porosity value. In general (except the image of the C horizon), the larger the image size, the larger the modelled porosity. Therefore, in all image fractal analysis, it is necessary to consider that the process of digitalisation has strong effect on fractal parameters. The calibration of used program and testing of all images, including the image size, the grey level etc., seems to be necessary. The known deterministic mass fractals are the best objects for this type of testing (Oleschko, 1998).

4. Discussion

4.1. Unified Fractal Model for soil porosity

The results of the empirical model testing have confirmed its usefulness for the E-dimensional porosity estimation. In order to unify the theoretical and empirical fractal models for soil porosity, discussed above, the solid and pore sets, can be referred to as the δ-parallel body $F_δ$, of a set $F$, embedded in the Euclidean $\mathbb{R}^E$ space. $F_δ$ is defined as the set of points within distance $δ$ of $F$; thus

$$F_δ = \{ x : |x-y| \leq δ \text{ for some } y \text{ in } F \},$$

where $x$ and $y$ are points in the Euclidean $\mathbb{R}^E$ space (Falconer, 1990).

Regarding soil, it is not clear at all which part of this δ-parallel body (solid or pore) is the real fractal, and which is the complement of $F$ in $\mathbb{R}^E$. However, if the ability to be fragmented is restricted only to solids, the solid set will have the fractal structure. In this case, the geometry of the complement (pore set) may be similar to $F$, but unprecise values should be expected for an empirically measured pore set fractal dimension. The continuous change of pore geometry, as the result of solid set instability and packing in $\mathbb{R}^E$, is responsible for these dynamics. Our experimental data show the statistically larger variance for pore fractal dimensions, when compared to solid fractal dimensions (Oleschko et al., 1997, 1998).

If the fractal dimension of a solid set is $D_i$, it is possible to propose the next general model for soil porosity $(\Phi)$, based on the theoretical and empirical models discussed above:

$$\Phi = 1 - (L/r)^{D_i - E},$$

(21)

where $L$ is the length of the E-dimensional section of the fractal set, and $r$ the size of the test grid unit or the divider size. This general model for solid fractal set, could be presented in three different variants, referring
to Euclidean one- (linear section, Eq. (22)), two-
(areal, 23), and three-dimensional (volume, 24) space,
and by using the solid set linear ($D_s$), areal ($D_{Asm}$) and
volumetric ($D_{Vsm}$) fractal dimensions:

\[ \Phi = 1 - (L/r)^{D_s^{-1}}, \quad (22) \]
\[ \Phi = 1 - (L/r)^{2(D_s^{-1})} = 1 - (L/r)^{D_{Asm}^{-2}}, \quad (23) \]
\[ \Phi = 1 - (L/r)^{3(D_s^{-1})} = 1 - (L/r)^{D_{Vsm}^{-3}}. \quad (24) \]

Mandelbrot (1983) has established a basic rule
regarding the planar fractal shape section by an interval “independent of the fractal”. He finds that if the
section is nonempty, it is “almost sure” that its
dimension is $D$-1, and the corresponding value in
space is $D$-2 (Mandelbrot, 1983, p. 135). The nature
of the exponent in Eqs. (22)–(24) may be derived from
this theoretical supposition.

Eqs. (23) and (24) are similar to:

\[ \Phi = 1 - (L/r)^{2D_s^{-2}}, \quad (25) \]
and
\[ \Phi = 1 - (L/r)^{3D_s^{-3}}, \quad (26) \]
respectively.

If (25) and (26) are correct, then it should be expected that the solid areal dimension ($D_{As}$) is equal
to or close to the double linear dimension ($2D_s$),
and the set’s three-dimensional dimension be equal or
similar to the three linear dimensions ($3D_s$). The
previous experimental data have confirmed this and
therefore, the classical Dellesse and Rosiwal princi-
pies (1 and 2) should be corrected for the statistical
fractal sets.

If the sign of proportionality in the Mandelbrot
length–area–volume relation, is to be respected, how-
ever, it seems to be correct to change the sign of
equality in Eqs. (22)–(26), for the sign of propor-
tionality. Then, the unified model for soil porosity will
appear as follows:

\[ \Phi \propto 1 - (L/r)^{D_s^{-E}}. \quad (27) \]

Only the solid set may be described by (27), and this
model is able to estimate the soil partial porosity (for
pores with size $\geq r$). Therefore, the porosity value,
estimated by empirical methods, should be strictly
dependent on the resolution of the method used for the
image collection.

It can be concluded therefore, that within the
known scale range over which the fractal extends or
within the fractal nature of the set was documented,
there are only two physical variables necessary for
the linear, areal, and volumetric porosity extraction:
the total number ($N_t$) of test points analysed along
the known sample length ($L$), using the test grid
with known length of box side ($r$), and the solid
($D_s$) or the pore ($D_p$) linear mass fractal dimension,
that expresses how much space one fractal set fills
along a line.

The main advantage of the proposed Unified Fractal
Model seems to be its ability to establish the simple
relation among the soil linear, areal and volumetric
porosity. Notwithstanding, assuming the validity of
both proposed models, the limitation of model 1 can
be related to the fact that it predicts only partial
porosity, whereas model 2 estimates the total soil
porosity.

5. Conclusions

The proposed Unified Fractal Model for soil $E$-
dimensional porosity, extracts the soil linear (NL),
areal (AP) and volumetric (VP) porosity from the solid
and pore distributions along the lines. Therefore,
within the scale range over which the fractal nature
of the set is documented, there are only two physical
variables necessary to model the fractal linear, areal
and volumetric porosity. These variables are: the total
number ($N_t$) of test points analysed along the known
sample length ($L$) by using the test grid of known box
side ($r$), and the solid ($D_s$) linear mass fractal dimen-
sion, that expresses how much space one fractal set
fills along a line. It was shown that the modelled
porosity in one-dimensional space, was close to the
measured by the point-counting technique; in two-
dimensional space, porosity was close to the values
reported by other authors, and in three-dimensional
space, the modelled porosity fits closely that measured
in the field. Therefore, all mentioned procedures are
useful for the proposed model testing in the $E$-dimen-
sional Euclidean space. These data confirm the main
hypothesis of the present research: the set of the $E$-
dimensional cross-sections of statistical fractal, is the
self-similar fractal itself and, the proportionality does
exist among the cross-section length, area and volume.
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