Non-degenerate conditionings of the exit measures of super Brownian motion

Thomas S. Salisbury\textsuperscript{a,\ast,1}, John Verzani\textsuperscript{b, 2}

\textsuperscript{a}Department of Mathematics and Statistics, York University, Toronto, Ont., Canada M3J 1P3
\textsuperscript{b}Department of Mathematics, CUNY – College of Staten Island, Staten Island, NY 10314, USA

Received 2 September 1998; received in revised form 8 November 1999

Abstract

We introduce several martingale changes of measure of the law of the exit measure of super Brownian motion. We represent these laws in terms of “immortal particle” branching processes with immigration of mass, and relate them to the study of solutions to \(Lu = cu^2\) in \(D\). The changes of measure include and generalize one arising by conditioning the support of the exit measure to hit a point \(z\) on the boundary of a 2-dimensional domain. In that case the branching process is the historical tree of the mass reaching \(z\), and our results provide an explicit description of the law of this tree. In dimension 2 this conditioning is non-degenerate. The representations therefore differ from the related representations studied in an earlier paper, which treated the degenerate conditionings that arise in higher dimensions. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 60G57; 60G42; 60F99

Keywords: Exit measure; Super Brownian motion; Martingale change of measure; Immortal particle description; Conditioning

1. Introduction

Many problems (for example, in biology) can be phrased in terms of recovering the law of a genealogical or historical tree, given information about a population at some time or place. We will study several conditionings of super Brownian motion that fit into this framework, in that the conditioned process can be broken into two pieces, one of which is such a historical tree (for a portion of the mass), while the other piece evolves as an unconditioned superprocess but with mass created along this tree.

What we actually investigate are conditionings of the exit measures of super Brownian motion in \(\mathbb{R}^d\). We can think of super Brownian motion as the limit of a particle
system, which can heuristically be described as follows (we give a precise formulation in Section 2). It consists of a cloud of particles, each diffusing as a Brownian motion and undergoing critical branching. A measure valued process is formed by assigning a small point mass to each particle’s position at a given time. The exit measure $X^D$ from a domain $D$ is then obtained by freezing this mass at the point the particle first exits from $D$. For an increasing sequence of subdomains, these measures can be defined on the same probability space, giving rise to a process indexed by the subdomains. In dimension 2, with positive probability, points on the boundary of a smooth enough domain will be hit by the support of the exit measure. In this paper, we study conditionings of the sequence of exit measures, analogous to the conditioning by this event. Unlike the case $d=2$, in higher dimensions the corresponding event has probability 0, and the analogous conditioning is a degenerate one. Such degenerate conditionings were treated in Salisbury and Verzani (1999), which we henceforth refer to as SV.

To be more specific, let $D$ be a bounded domain in dimension $d=2$, and let $D_k$ be an increasing sequence of subdomains. The domains $D_k$ give rise to a process of exit measures $X^k$, each defined on the boundary of $D_k$. We work under $\mathbb{N}_x$, the excursion measure under which Le Gall’s Brownian snake evolves, starting from location $x$. Let $\mathbb{M}_x$ be the law of super Brownian motion, conditioned on the exit measure hitting a fixed point $z$ on $\partial D$ (that is, conditioned on it charging all balls containing $z$). Let $\mathcal{F}_k$ be the $\sigma$-field generated by the particles before they exit $D_k$ and denote integration by $\langle \cdot \rangle$. Our first result is an explicit description of $\mathbb{M}_x$ on $\mathcal{F}_k$. Its densities with respect to $\mathbb{N}_x$ form a martingale (in $k$) which can be explicitly written in terms of the $X^k$.

More generally, the differential equation $Lu = 4u^2$ plays an important role in our discussion, and for the exit measures in general. In Lemma 3.1 it is shown that if $g \geq u \geq 0$ are both solutions in $D$ to $Lu = 4u^2$ then $\hat{M}_k = \exp -\langle X^k, u \rangle - \exp -\langle X^k, g \rangle$ is an $\mathcal{F}_k$ martingale. Letting $v = g - u$, we can define a general change of measure, using this martingale, to give a measure $\hat{\mathbb{M}}_x$ satisfying

$$\frac{d\hat{\mathbb{M}}_x}{d\mathbb{N}_x} \bigg|_{\mathcal{F}_k} = \frac{1}{v(x)} \hat{M}_k$$

for each $k$. (In the above example, $u = 0$ and $g = g_z = \mathbb{N}_x(z \in \mathcal{A}^D)$, where $\mathcal{A}^D$ denotes the range of the super Brownian “particles” before exiting $D$.)

Our second result is that the measures $\hat{\mathbb{M}}_x$ on $\mathcal{F}_k$ can be represented in terms of a branching process of “immortal particles” together with immigration of mass. Two such equivalent representations are given, in Theorems 3.2 and 3.4. The first involves a conditioned diffusion in which particles may die, but when this occurs two independent particles are born as replacements. The other uses a conservative conditional diffusion undergoing binary branching. The branching mechanism in both representations is homogeneous, unlike the representations of SV. (We use the terminology “immortal particle” to refer to a particle that is conditioned to exit $D$ through $\partial D$. The language comes from previous conditionings of the superprocess. See SV for references to earlier work.)
By using both descriptions, we can investigate the solutions to the equation \( Lu = 4u^2 \).

We see in two examples that the solutions given by

\[
g_z(x) = \mathbb{N}_x(z \in \mathbb{R}^D)
\]

for \( z \in \partial D \) and by

\[
g_f(x) = \mathbb{N}_x(1 - \exp(-\langle X^D, f \rangle))
\]

lead to quite different immortal particles pictures: the former having infinitely many branches and the latter just finitely many (for example, if \( f \) is bounded). We will show that the moderate functions (as studied in Le Gall (1995) and Dynkin and Kuznetsov (1998a)) are precisely those for which the mean number of branches is finite. In the first example, the immortal particle picture gives an explicit description of the historical tree of all mass reaching \( z \). A related genealogical interpretation can be given in the second example as well.

Finally, we draw an analogy between these conditionings and those treated in SV (see Section 3.5). In that paper we investigated transforms based on a different type of martingale than used here. That family of martingales generalized the ones arising from conditioning the exit measure to hit a given finite number of points on the boundary of \( D \), in the case that this conditioning was degenerate (that is, that the event conditioned on had probability 0). Because of this degeneracy, the results there had an asymptotic character, and required analytic estimates of small solutions to certain non-linear PDEs. Those conditionings also had immortal particle representations, though the particles in their backbones evolved in an inhomogeneous manner. In Section 4 of the current paper, we present a martingale change of measure combining features of the conditionings \( \tilde{M}_x \) described above, and those of SV. In Theorem 4.4 we derive an immortal particle representation for this general class of transforms.

2. Preliminaries

This paper is a sequel to Salisbury and Verzani (1999), which we refer to as SV, but for the convenience of the reader we restate some of the lemmas used therein. Any proofs appear in SV or the given references.

2.1. Notation

For a set \( A \), let \( |A| \) denote its cardinality, and let \( \mathcal{P}(A) \) denote the collection of partitions of \( A \). Choose some arbitrary linear order \( \prec \) on the set of finite subsets of the integers. For \( A \) such a finite subset, and \( \sigma \in \mathcal{P}(A) \), let \( \sigma(j) \) be the \( j \)th element of \( \sigma \) in this order. Thus for example,

\[
\prod_{C \in \sigma} \langle X^D, v^C \rangle = \prod_{j=1}^{|\sigma|} \langle X^D, v^{\sigma(j)} \rangle.
\]

We will switch between these notations according to which seems clearer.
2.2. Set facts

We make use of the following two simple lemmas. We will use the convention that a sum over an empty set is 0.

Lemma 2.1 (Lemma 2.1 of SV). Let \( A \subseteq B \subseteq C \) be subsets of \( \{1, 2, \ldots, n\} \). Then
\[
\sum_{A \subseteq B \subseteq C} (-1)^{|B| + 1} = (-1)^{|C|} 1_{A=C}.
\]

Lemma 2.2 (Lemma 2.2 of SV). Let \( A \) be finite, and let \( w_i \in \mathbb{R} \) for \( i \in A \). Then
\[
\prod_{i \in A} (1 - w_i) = 1 + \sum_{C \subseteq A, \emptyset \neq C} (-1)^{|C|} \left( \prod_{i \in C} w_i \right).
\]

2.3. Facts about conditioned diffusions

First we recall some formulae for conditioned Brownian motion. Let \( B \) be \( d \)-dimensional Brownian motion started from \( x \), under a probability measure \( P_x \). Write \( D = D(B) \) for the first exit time of \( B \) from \( D \).

Let \( g : D \to [0, 1) \) be bounded on compact subsets of \( D \), and set
\[
L_g = \frac{1}{2} A - g.
\]

Let \( \xi_t \) be a process which, under a probability law \( P_x^B \), has the law of a diffusion with generator \( L_g \) started at \( x \) and killed upon leaving \( D \). In other words, \( \xi \) is a Brownian motion on \( D \), killed at rate \( g \). Write \( \tau_D \) for the lifetime of \( \xi \). Then
\[
P_x^B(\xi_t \in A, \xi > t) = P_x \left( \exp - \int_0^t ds g(B_s), B_t \in A, \tau_D > t \right).
\]

Let \( U^g f(x) = \int_0^\infty P_x^B(f(\xi_t) 1_{\xi > t}) \, dt \) be the potential operator for \( L_g \). If \( g = 0 \) we write \( U \) for \( U^0 \). If \( 0 \leq u \) is \( L_g \)-superharmonic, then the law of the \( u \)-transform of \( \xi \) is determined by the formula
\[
P_x^B(\Phi(\xi)) 1_{\{\xi > t\}} = \frac{1}{u(x)} P_x^B(\Phi(\xi) u(\xi_t) 1_{\{\xi > t\}})
\]
for \( \Phi(\xi) \in \sigma\{\xi_s; s \leq t\} \). Assuming that \( 0 < u < \infty \) on \( D \), this defines a diffusion on \( D \). If \( u \) is \( L_g \)-harmonic, then it dies only upon reaching \( \partial D \). In fact, the generator of the \( u \)-transform is
\[
L_g u f = -\frac{1}{u} L_g (u f) = \frac{1}{2} A f + \frac{1}{u} \nabla u \cdot \nabla f.
\]

If \( u = U^g f \) for some \( f \geq 0 \) (that is, if \( u \) is a potential) then the \( u \)-transform dies in the interior of \( D \), and \( P_x^B u \) satisfies
\[
P_x^B(\Phi(\xi)) = \frac{1}{u(x)} \int_0^\infty P_x^B(\Phi(\xi_t) f(\xi_t) 1_{\{\xi > t\}}) \, dt.
\]
where \( \xi \) is the process \( \zeta \) killed at time \( t \). In addition, if \( u = h + v \), where \( h \geq 0 \) is \( L_0 \)-harmonic, and \( v = U^f f \) with \( f \geq 0 \), then

\[
P_x^{u,v} = \frac{1}{u(x)} (h(x)P_x^{h,v} + v(x)P_x^{v,v}).
\]

(2.3)

Suppose that \( \Phi \) vanishes on paths \( \xi \) that reach \( \partial D \). Applying (2.2) to (2.3), we see that

\[
P_x^{u,v}(\xi) = \frac{v(x)}{u(x)} P_x^{v,v}(\xi) = \frac{1}{u(x)} \int_0^\infty P_x^{v,v}(\Phi(\xi < t)f(\xi_t)1_{\xi > t}) \, dt.
\]

(2.4)

in this case as well.

2.4. Facts about the Brownian snake

Next we recall some useful facts about the Brownian snake. Readers are referred to Dawson (1993) or Dynkin (1991a) for a general introduction to superprocesses, and to Le Gall (1999) for the connection to random snakes.

The Brownian snake is a path-valued process, devised by Le Gall as a means to construct super Brownian motion without limiting procedures. The construction can be found in Le Gall (1999) or Le Gall (1994b).

We use the standard notation \((W(s),\xi(s))\) for the Brownian snake, and \(N_x\) for the excursion measure of the Brownian snake starting from the trivial path \((w,\zeta)\), \(\zeta = 0\), \(w(0) = x\). Note that \( W_t(\cdot) \) is constant on \([s,1]\), and \( \zeta \) has the distribution of a Brownian excursion under \( N_x \). We let \( \sigma > 0 \) denote the duration of this excursion.

Super Brownian motion \( X_t \) is defined as

\[
\langle X_t, \phi \rangle = \int \phi(W_t(s)) \, dL_t(s),
\]

where \( L_t \) is the local time of \( \zeta \) at level \( t \). Dynkin (1991b) introduced the exit measure \( X^D(\cdot) \) associated with \( X_t \), Le Gall’s snake-based approach to \( X^D \) (see Le Gall (1994b) or Le Gall (1999)) involves constructing a local time \( L^D(\cdot) \) for \( W_t(\xi(s)) \) on \( \partial D \), and then setting

\[
\langle X^D, \phi \rangle = \int \phi(W_t(\xi(s))) \, dL^D(s).
\]

We denote the range of the Brownian snake by \( R(W) = \{ W_t(s): 0 \leq s \leq \sigma, 0 \leq t \leq \xi(s) \} \) and the range inside \( D \) by \( R^D(W) = \{ W_t(s): 0 \leq s \leq \sigma, 0 \leq t \leq \tau_D(W) \wedge \xi(s) \} \). Recall that \( \tau_D(W) \) is the first exit time of \( W_t \) from \( D \). There is an obvious inclusion between the range inside \( D \) and the exit measures, given by

\[
\{(X^D,1_D) > 0\} \subseteq \{R^D(W) \cap A \neq \emptyset\}.
\]

We refer the reader to Le Gall (1999) for other facts about the Brownian snake, including the following result (Corollary V.8 of Le Gall (1999)).

**Lemma 2.3.** Let \( g \) be a solution to \( \Delta g = 4g^2 \) in \( D \), and let \( \{D_k\} \) be an increasing sequence of smooth subdomains of \( D \). Then for each \( k \) such that \( x \in D_k \),

\[
N_x(1 - \exp - \langle X^{D_k},g \rangle) = g(x).
\]
Let $\mathcal{F}_k = \mathcal{F}_{D_k}$ be the $\sigma$-field of events determined by the superprocess killed upon exiting $D_k$. See Dynkin (1991b) for a formal definition. Or refer to the final section of SV, which gives a definition in terms of the historical superprocess.

Dynkin (1991b) introduced a Markov property for the exit measures. In our context, it can be found in Le Gall (1995). The next result gives it in the form we will use it:

**Lemma 2.4.**

$$\mathbb{E}_x (\exp (-\langle X^D, \phi \rangle | \mathcal{F}_k)) = \exp (-\langle X^D, \mathbb{E}_x (1 - \exp (-\langle X^D, \phi \rangle)) \rangle).$$

We use the following notation, where $B_s$ denotes a path in $D$ whose definition will be clear from the context:

$e^D_\phi = \exp (-\langle X^D, \phi \rangle),$

$$\mathcal{N}_i (e^D_\phi) = \mathcal{N}_i (e^D_\phi, B) = \exp (-\int_0^t ds \mathbb{E}_B_s(1 - e^D_\phi)).$$

The Palm formula for the Brownian snake takes the form (see Proposition 4.1 of Le Gall (1994b)):

$$\mathbb{E}_x (\langle X^D, \phi \rangle e^D_\psi) = P_x (\phi (B_{\tau_D}), \mathcal{N}_{\tau_D} (e^D_\psi)).$$

Dawson and Perkins (1991) can be consulted for a general discussion of this type of result. We will make use of the following extension to the basic Palm formula:

**Lemma 2.5** (Lemma 2.6 of SV). Let $D$ be a domain and let $B$ be a Brownian motion in $D$ with exit time $\tau$. Let $\{\psi_i\}$ be a family of measurable functions. Then

$$\mathbb{E}_x \left( e^D_\phi \prod_{i \in N} \langle X^D, \psi_i \rangle \right) = \frac{1}{2} \sum_{M \subseteq N \cap \emptyset, N \neq M} P_x \left( \int_0^t dt \mathcal{N}_i (e^D_\psi) \mathbb{E}_B_t \left( e^D_\psi \prod_{i \in M} \langle X^D, \psi_i \rangle \right) \right) \times \mathbb{E}_B \left( e^D_\psi \prod_{i \in N \setminus M} \langle X^D, \psi_i \rangle \right).$$

Using the extended Palm formula one may show an exponential bound on the moments of the exit measure.

**Lemma 2.6** (Lemma 2.7 of SV). Let $D$ be a domain in $\mathbb{R}^d$ satisfying the condition $\sup_{x \in D} P_x (\tau_D) < \infty$, where $\tau_D$ is the exit time from $D$ for Brownian motion. Then there exists $\lambda > 0$ such that

$$\sup_{x \in D} \mathbb{E}_x (\exp \lambda \langle X^D, 1 \rangle - 1) < \infty.$$ 

**Remark 2.7.** A bounded domain $D$ in $\mathbb{R}^d$ will satisfy $\sup_{x \in D} P_x (\tau_D) < \infty$. 
3. Exponential transforms

In this section we investigate a type of $h$-transform of the exit measures, given by a martingale change of measure. This transform is then interpreted in terms of a branching system of particles, as in SV, but unlike the situation there, the branching system is now homogeneous. We present several examples of such transforms, and study whether or not the associated branching systems have finitely or infinitely many branches.

3.1. The martingale $\tilde{M}_k$

Suppose that $D$ is a bounded domain in $\mathbb{R}^d$ and that $D_k$ are smooth domains satisfying $D_k \uparrow D$. We will shorten our notation and will write $X^k$ for the exit measure $X^{D_k}$, and $e^{\xi}$ for $e^{\phi} = e^{-\langle X^k, \phi \rangle}$.

Suppose that $g > 0$ satisfies $\frac{1}{2}Ag = 2g^2$ in $D$. Let $u \geq g$ be a second solution to this equation, with $u \leq g$. Set $v = g - u$, and let $\tilde{M}_k = \exp - \langle X^k, u \rangle - \exp - \langle X^k, g \rangle$.

Lemma 3.1. $\tilde{M}_k$ is an $\mathcal{F}_k$ martingale.

Proof. Let $j < k$. Then by Lemmas 2.3 and 2.4,

$$\mathbb{E}_x(\tilde{M}_k \mid \mathcal{F}_j) = \mathbb{E}_x(\exp - \langle X^k, u \rangle - \exp - \langle X^k, g \rangle \mid \mathcal{F}_j)$$

$$= \exp - \langle X^j, \mathbb{E}_x(1 - \exp - \langle X^k, u \rangle) \rangle - \exp - \langle X^j, \mathbb{E}_x(1 - \exp - \langle X^k, g \rangle) \rangle$$

$$= \exp - \langle X^j, u \rangle - \exp - \langle X^j, g \rangle = \tilde{M}_j. \quad \square$$

As a consequence, we can define a transformed process via a martingale change of measure. If $\Phi_k$ is an $\mathcal{F}_k$-measurable function, set

$$\tilde{M}_x(\Phi_k) = \frac{1}{v(x)} \mathbb{E}_x(\Phi_k \tilde{M}_k) = \frac{1}{v(x)} \mathbb{E}_x(\Phi_k e^{\xi \langle X^k, v \rangle} - 1)$$

$$= \frac{1}{v(x)} \mathbb{E}_x \left( \Phi_k e^{\xi \sum_{n=1}^{\infty} \frac{1}{n!} \langle X^k, v \rangle^n} \right). \quad (3.1)$$

3.2. Branching backbones

Having defined $\tilde{M}_x$ by a martingale change of measure, we now define a second measure $\tilde{N}_x$. We will show that the laws of the exit measures $X^k$ under $\tilde{M}_x$ agree with those of corresponding exit measures $Y^k$ under $\tilde{N}_x$. In fact, using historical processes as in the last section of SV, one can show that $\tilde{M}_x$ and $\tilde{N}_x$ agree on all of $\mathcal{F}_k$. We will not pursue this here.

$Y^k$ arises from a backbone $\mathcal{T}$ throwing off mass. So to specify the law of $Y^k$ under $\tilde{N}_x$, we need to give two ingredients: the law $Q_x$ of the backbone $\mathcal{T}$, and the measures $\tilde{N}_y$ which describe how the mass thrown off evolves.
First we construct a homogeneous branching process. Our underlying process will be Brownian motion killed at rate $4g$ (that is, with generator $L_{4g}$). Recall that $P_{4g}$ denotes its law. Then $v$ satisfies

$$L_{4g}v = \frac{1}{2}(Ag - Au) - 4gv = 2v(g + u - 2g) = -2v^2.$$  \hspace{1cm} (3.2)

In other words, $v$ is $L_{4g}$-superharmonic, so we can consider the $v$-transform of the $L_{4g}$-process. Recall that its law is denoted by $P_{4g;v}$.

Now form a branching $g$-process, as follows. Start with a single particle, with law $P_{4g;v}x$. When it dies, say at $y$, it is replaced by two independent offspring, each with law $P_{4g;v}y$. In other words, the offspring evolve with the same transition function as their parent, as do all their descendants. Let $n_t$ denote the number of particles alive at time $t$. Label them with $1 \leq i \leq n_t$, and for each one set $x_i(s)$, $0 \leq s \leq t$, to be the history (including the ancestors’ history) of the individual particle up until time $t$.

Define measure-valued branching processes as follows:

$$T_t(dx) = \sum_{i=1}^{n_t} \delta_{x_i(t)}(dx), \quad T^{k}_t(dx) = \sum_{i=1}^{n_t} 1_{\{x_i(t) > t\}} \delta_{x_i(t)}(dx).$$

The process $T^{k}_t$ puts a mass at each particle alive at time $t$ which has not already exited $D_k$. The process $T_t$ is what we call the backbone, and we let $Q_x$ denote its law.

Without comment, we will feel free to refer to $T_t$ and $T^{k}$ in terms of the underlying particles, or as a branching process, although strictly speaking they are measure-valued process.

By Lemma 3.1, $1 - \exp - \langle X^k, y \rangle$ is an $\mathcal{F}_t$-martingale. Thus so is $\exp - \langle X^k, y \rangle$ (even though the latter has infinite first moment under $\mathbb{N}_x$), and we may consistently define a measure $\mathbb{N}_x$ on $\mathcal{F}_t$ by $\mathbb{N}_x(\Phi_t) = \mathbb{N}_x(\Phi_t \exp - \langle X^k, y \rangle)$. Dawson’s Girsanov formula (see also Lemma 4.1 of SV) shows that $\mathbb{N}_x$ is actually the excursion law for the superprocess in $D$ based on the generator $L_{4g}$. In other words, we could realize $\mathbb{N}_x$ by starting with a superprocess with law $\mathbb{N}_x$, and then pruning off particles at rate $4g$.

We can now specify how mass is thrown off (or immigrated) along $Y$. Though one thinks of mass being created continuously along the backbone, only at countably many times will it actually survive, even instantaneously. At each such time, the mass created evolves like a superprocess with “law” a $\mathbb{N}_x$, and, if it survives long enough, produces a contribution to the exit measure. More properly, given the backbone $T^k$, we form a Poisson random measure $N^k(d\mu)$ with intensity $\int_0^\infty dt \int T^k_t(dy)\mathbb{N}_x(\{X^k \in d\mu\})$. We then realize the exit measure under $\mathbb{N}_x$, as $Y^k = \int \mu N^k(d\mu)$. A standard calculation now shows that

$$\mathbb{N}_x(\exp - \langle Y^k, \phi \rangle) = \mathbb{Q}_x \left( \exp - \int_0^\infty dt 4 \langle T^{k}_t, \mathbb{N}_x(1 - \exp - \langle X^k, \phi \rangle) \rangle \right).$$

This formula could therefore equally well be taken to define the law of $Y^k$ under $\mathbb{N}_x$.

Since the branching process making up the backbone is homogeneous, we can partition the particles into classes determined by their having a common ancestor prior to exiting $D_k$. Let $T^k \sim n$ denote the event that there are exactly $n$ distinct ancestors before exiting $D_k$. 


Theorem 3.2. We have, in the notation of this section,
\[ \hat{\mathcal{M}}_x(\exp - \langle X^k, \phi \rangle) = \hat{\mathcal{N}}_x(\exp - \langle Y^k, \phi \rangle). \]

Remark 3.3. Using historical processes, as in the last section of SV, one can show that \( \hat{\mathcal{M}}_x = \hat{\mathcal{N}}_x \) on \( \mathcal{F}_k \).

Proof. We show by induction that
\[ v(x)Q_x \left( \exp - \int_0^\infty dt \, 4(\mathcal{T}_t^k, \hat{\mathcal{N}}_x(1 - e^\phi)), T_k \sim n \right) = \frac{1}{n!} \mathbb{N}_x(e^{\phi}(X^k, v)^n). \quad (3.3) \]

From this it follows, by summing on \( n \), that
\[ \hat{\mathcal{N}}_x(\exp - \langle Y^k, \phi \rangle) = \frac{1}{v(x)} \sum_{n=1}^{\infty} \frac{1}{n!} \mathbb{N}_x(e^{\phi}(X^k, v)^n) \]
\[ = \mathbb{N}_x(e^{\phi}(X^k, v))/(v(x) - 1)) = \hat{\mathcal{M}}_x(e^{\phi}). \]

This will prove the theorem.

We note that by Lemma 2.5, each term above satisfies
\[ \frac{1}{n!} \mathbb{N}_x(e^{\phi}(X^k, v)^n) \]
\[ = \frac{1}{n!} \sum_{j=1}^{n-1} \frac{n}{j} P_x \left( 2 \int_0^\tau e^{\phi} \mathcal{N}_x(e^{\phi}(X^k, v)^j) \mathbb{N}_x(e^{\phi}(X^k, v)^{n-j}) dt \right) \]
\[ = \sum_{j=1}^{n-1} P_x \left( 2 \int_0^\tau e^{\phi} \left( \frac{1}{j!} \mathbb{N}_x(e^{\phi}(X^k, v)^j) \right) \right) \]
\[ \times \left( \frac{1}{(n-j)!} \mathbb{N}_x(e^{\phi}(X^k, v)^{n-j}) \right) dt \]. \quad (3.4)

Now to establish (3.3). First, in the case when \( n = 1 \) we have that \( T_k \) is given by a single \( \varepsilon \)-process which has lifetime greater than \( \tau_0 \). Hence,
\[ v(x)Q_x \left( \exp - \int_0^\infty dt \, 4(\mathcal{T}_t^k, \hat{\mathcal{N}}_x(1 - e^\phi)), T_k \sim 1 \right) \]
\[ = v(x)P^{\phi}(\varepsilon) \left( \exp - \int_0^{\tau_0} dt \, 4\hat{\mathcal{N}}_x(1 - e^\phi), \xi > \tau_0 \right) \]
\[ = P^{\phi}(v(\xi(\tau_0))) \exp - \int_0^{\tau_0} dt \, 4\hat{\mathcal{N}}_x(1 - e^\phi), \xi > \tau_0 \]
\[ = P^{\phi}(v(\xi(\tau_0))) \exp \left( - \int_0^{\tau_0} dt \, 4g(\xi(t)) + \exp \left( - \int_0^{\tau_0} dt \, 4\hat{\mathcal{N}}_x(e^\phi(1 - e^\phi)) \right) \right) \]
\[ P_x \left( v(\bar{\xi}(\tau_k)) \exp - \int_0^{\tau_k} dt \, 4g(\bar{\xi}(t)) - \mathbb{N}_{\bar{\xi}(t)}(1 - e_\phi^k) + \mathbb{N}_{\bar{\xi}(t)}(1 - e_{\phi+g}^k) \right) \]

\[ = P_x(v(\bar{\xi}(\tau_k))) \mathcal{N}_{\bar{\xi}}(e_\phi^k) \]

\[ = \mathbb{N}_{\xi}(e_{\phi+g}^k (X^k, \nu)) \] (3.5)

Where (3.5) follows from (2.3) and (3.6) from (2.5).

When \( n > 1 \) the first particle splits at its lifetime \( \bar{\xi} < \tau_k \). By the Markov property for \( \bar{Y} \) and the conditional independence of the offspring,

\[ v(x)Q_x \left( \exp - \int_0^{\infty} dt \, 4\langle \bar{Y}^k_t, \mathbb{N}_t(1 - e_\phi^k) \rangle, \ T^k \sim n \right) \]

\[ = \sum_{j=1}^{n-1} v(x)P^{x_{\bar{Y},\nu}} \left( \exp - \int_0^{\bar{\xi}} dt \, 4\mathbb{N}_{\bar{\xi}(t)}(1 - e_\phi^k) \right) \]

\[ \times Q_{\bar{\xi}(t)} \left( \exp - \int_0^{\infty} dt \, 4\langle \bar{Y}^k_t, \mathbb{N}_t(1 - e_\phi^k) \rangle, \ T^k \sim n - j \right) \]

\[ \times Q_{\bar{\xi}(s)} \left( \exp - \int_0^{\infty} dt \, 4\langle \bar{Y}^k_t, \mathbb{N}_t(1 - e_\phi^k) \rangle, \ T^k \sim j \right), \ \bar{\xi} < \tau_k \]

\[ = \sum_{j=1}^{n-1} P^{x_{\bar{Y},\nu}} \left( \int_0^{\tau_s} ds \, 2\nu^2(\bar{\xi}(s))1_{\bar{\xi} > s} \right) \]

\[ \times \exp \left( - \int_0^{\tau_s} dt \, 4\mathbb{N}_{\bar{\xi}(t)}(1 - e_{\phi+g}^k) - \mathbb{N}_{\bar{\xi}(t)}(1 - e_\phi^k) \right) \]

\[ \times Q_{\bar{\xi}(t)} \left( \exp - \int_0^{\infty} dt \, 4\langle \bar{Y}^k_t, \mathbb{N}_t(1 - e_\phi^k) \rangle, \ T^k \sim n - j \right) \]

\[ \times Q_{\bar{\xi}(s)} \left( \exp - \int_0^{\infty} dt \, 4\langle \bar{Y}^k_t, \mathbb{N}_t(1 - e_\phi^k) \rangle, \ T^k \sim j \right) \] (3.7)

\[ = \sum_{j=1}^{n-1} P_x \left( 2 \int_0^{\tau_s} ds \exp \left( - \int_0^{s} dt \, 4g(\bar{\xi}(t)) \right) \exp \left( \int_0^{s} dt \, 4g(\bar{\xi}(t)) \right) \mathcal{M}_{\bar{\xi}}(e_{\phi+g}^k) \right) \]

\[ \times v(\bar{\xi}(s))Q_{\bar{\xi}(s)} \left( \exp - \int_0^{\infty} dt \, 4\langle \bar{Y}^k_t, \mathbb{N}_t(1 - e_\phi^k) \rangle, \ T^k \sim n - j \right) \]

\[ \times v(\bar{\xi}(s))Q_{\bar{\xi}(s)} \left( \exp - \int_0^{\infty} dt \, 4\langle \bar{Y}^k_t, \mathbb{N}_t(1 - e_\phi^k) \rangle, \ T^k \sim j \right) \] (3.8)
\[
\frac{1}{n!} N(x(e^k_x, v)^n). \tag{3.10}
\]

Line (3.7) follows from (2.4), (3.8) follows from (2.1) and Lemma 2.3, (3.9) by the inductive hypotheses, and (3.10) by (3.4). \(\Box\)

There is an alternative description of the above backbone, which is in some ways more natural, though it is less closely tied to the approach of SV. In this version, the backbone is a branching diffusion. The diffusion is again a \(v\)-transform, but this time of the process with generator \(L_{2(u+g)}\). We denote this process by \(\bar{\lambda}_t\). Note that now \(L_{2(u+g)}^v=0\), so that \(v\) is \(L_{2(u+g)}\)-harmonic and \(\lambda_t\) survives to reach \(\partial \Omega\). We let \(\lambda\) branch at rate \(2v\). This produces a tree \(T_\lambda\), and we write \(\bar{Q}_x\) for its law. Branches of course evolve independently. On top of this branching process, we immigrate mass exactly as before, to produce a measure \(\mathbb{N}_x\). It turns out to be the same as the measure \(\bar{\mathbb{N}}_x\) given above.

**Theorem 3.4.** For the measure described above one has

\[
\mathbb{N}_x(e^k_x) = \bar{\mathbb{N}}_x(e^k_x) = \bar{\mathbb{N}}_x(e^k_x). \tag{3.11}
\]

**Remark 3.5.** Actually we will show that \(\bar{\mathbb{N}}_x = \mathbb{N}_x\), so that it follows from Remark 3.3 that \(\bar{\mathbb{N}}_x = \mathbb{N}_x\) on \(\mathcal{F}_x\).

**Proof of Theorem 3.4.** There are several ways to prove this. One is to show directly that \(\bar{\mathbb{N}}_x(e^k_x) = \bar{\mathbb{N}}_x(e^k_x)\), in a manner similar to the proof of Theorem 3.2. Another is to use induction and \(h\)-transform arguments to show that

\[
\bar{Q}_x \left( \exp - \int_0^\infty dt \langle T^k_t, \phi_t \rangle; T^k \sim n \right) = Q_x \left( \exp - \int_0^\infty dt \langle T^k_t, \phi_t \rangle; T^k \sim n \right)
\]

for every \(n\) and every measurable \(\phi_t(x) \geq 0\), and to then infer that \(T^k\) has the same law under \(\bar{Q}_x\) as under \(Q_x\), for every \(k\).

But the simplest approach seems to be via generators. In particular, consider the segments of the backbone tree between successive branches. Under \(Q_x\) they are \(v\)-transforms of the process with generator \(L_{4g}\). That is, they have generator \(L_{4g,v}\). Under \(\bar{Q}_x\) they are \(v\)-transforms of the process with generator \(L_{2(u+g)}\), but are then killed at rate \(2v\) (because now we only consider the process between the branches). That is, they have generator \(L_{2(u+g),v} - 2v\). But

\[
[L_{2(u+g),v} - 2v] f = \frac{1}{v} L_{2(u+g)}(vf) - 2vf = \frac{1}{v} \left[ \frac{1}{2} A(vf) - 2(u+g)vf \right] - 2vf
\]

Since the generators agree, and the backbone trees can be built up by binary branching based on this common Markov process, in fact the law of $\mathcal{T}$ under $\tilde{Q}_x$ agrees with that under $Q_x$. The conclusion of the theorem therefore follows from Theorem 3.2.

3.3. Branching numbers

The above tells us how to obtain the law of the $v$-transformed process from that of the backbone. In the reverse direction, we will content ourselves with identifying the law of $\eta^k$ given $X^k$, where $\eta^k$ is the exit measure of $\mathcal{T}$ from $D_k$. In other words, $\eta^k$ puts unit mass at each terminal node of the tree $T^k$. For $\lambda$ a $\sigma$-finite measure, start with a Poisson random measure with intensity $\lambda$, and condition it on being non-zero. Write $\pi(\lambda)$ for the law of the resulting random measure.

**Proposition 3.6.** Under $\tilde{\mathbb{N}}_x$, the law of $\eta^k$ given $X^k$ is $\pi(vX^k)$.

**Proof.** We will compute $\tilde{\mathbb{N}}_x(e^k_{\theta} e^{-\langle \eta^k, \phi \rangle})$. The same inductive argument as in the proof of Theorem 3.2 shows that

$$v(x)\tilde{\mathbb{N}}_x(e^k_{\theta} e^{-\langle \eta^k, \phi \rangle}, T^k \sim n) = v(x)Q_x \left( e^{-\langle \eta^k, \phi \rangle} \exp - \int_0^\infty dt \, 4 \langle T^k_t, \tilde{\mathbb{N}}_x (1 - e^k_{\theta}) \rangle, T^k \sim n \right)$$

$$= \frac{1}{n!} \tilde{\mathbb{N}}_x(e^k_{\theta + g} \langle X^k, ve^{-\psi} \rangle^n).$$

Summing on $n$ then gives that

$$\tilde{\mathbb{N}}_x(e^k_{\theta} e^{-\langle \eta^k, \phi \rangle}) = \frac{1}{v(x)} \sum_{n=1}^\infty \frac{1}{n!} \tilde{\mathbb{N}}_x(e^k_{\theta + g} \langle X^k, ve^{-\psi} \rangle^n)$$

$$= \frac{1}{v(x)} \tilde{\mathbb{N}}_x(e^k_{\theta + g} (e^{\langle X^k, ve^{-\psi} \rangle} - 1))$$

$$= \frac{1}{v(x)} \tilde{\mathbb{N}}_x \left( \mathcal{M}^k e^k_{\theta} e^{\langle X^k, ve^{-\psi} \rangle} - 1 \right)$$

$$= \tilde{\mathbb{N}}_x \left( e^k_{\theta} e^{\langle X^k, ve^{-\psi} \rangle} - 1 \right).$$
But a simple calculation shows that if a random measure $\Pi$ has law $\pi(\lambda)$, then the expectation of $e^{\lambda \psi - \psi}$ is exactly

$$\frac{e^{\lambda \psi - \psi} - 1}{e^{\lambda} - 1},$$

which establishes the proposition. □

The following could be obtained immediately from (3.3), but it is also instructive to derive it from Proposition 3.6.

**Corollary 3.7.**

$$\mathbb{Q}_x(\mathcal{T}^k \sim n) = \frac{1}{v(x)} \mathbb{N}_x \left( e^{k} \frac{\langle X^k, v \rangle^n}{n!} \right).$$

**Proof.** By definition, if $\Pi$ is a random measure with law $\pi(\lambda)$, then the probability that $\Pi$ has total mass $n$ is

$$\frac{e^{\langle \lambda, 1 \rangle n}}{n!} \cdot \frac{1}{1 - e^{\langle \lambda, 1 \rangle}},$$

for every $n \geq 1$. Thus

$$\mathbb{Q}_x(\mathcal{T}^k \sim n) = \mathbb{Q}_x(\langle \eta^k, 1 \rangle = n)$$

$$= \mathbb{N}_x \left( e^{k} \frac{\langle X^k, v \rangle^n}{n!} \cdot \frac{1}{1 - e^{\langle \lambda, 1 \rangle}} \right)$$

$$= \mathbb{N}_x \left( e^{k} \frac{\langle X^k, v \rangle^n}{1 - e^{k} \frac{\langle X^k, v \rangle^n}{n!}} \right)$$

$$= \mathbb{N}_x \left( e^{k} \frac{\langle X^k, v \rangle^n}{M_k} \right)$$

$$= \frac{1}{v(x)} \mathbb{N}_x \left( e^{k} \frac{\langle X^k, v \rangle^n}{n!} \right).$$

We turn to the question of whether the tree $T$ has finitely or infinitely many branches. Write $\beta(\mathcal{T}^k)$ for the number of terminal nodes of $\mathcal{T}^k$. Recalling that the measure $\eta^k$ puts unit mass at each terminal node of $\mathcal{T}^k$, we have that $\beta(\mathcal{T}^k) = \langle \eta^k, 1 \rangle$. Alternatively, $\beta(\mathcal{T}^k) = n$ means exactly that $\mathcal{T}^k \sim n$. Set $\beta(T) = \lim_{k \to \infty} \beta(\mathcal{T}^k)$, so that our question becomes that of the finiteness of $\beta(T)$.

We will need Dynkin’s notion of a stochastic boundary value. Dynkin (1998) showed that if $\frac{1}{2}Ag = 2g^2$ then

$$Z_\gamma = \lim_{k \to \infty} \langle X^k, g \rangle$$

exists $\mathbb{N}_x$-a.s., and $\mathbb{N}_x(Z_\gamma < \infty) > 0$ (actually Dynkin (1998) does not use the excursion laws $\mathbb{N}_x$, but the extension to this case is elementary). Clearly $Z_\gamma$ exists and equals
\[ Z_g - Z_u, \text{ on } \{ Z_u < \infty \}. \] Moreover, by dominated convergence,

\[ \tilde{N}_v(\Phi) = \tilde{N}_v(\Phi) = N_v(\Phi(e^{-Z_v} - e^{-Z_g})), \quad \tag{3.11} \]

so that \( \tilde{N}_v \ll N_v \), and \( Z_u < \infty \), \( \tilde{N}_v \)-almost surely. Thus \( Z_v \) exists \( \tilde{N}_v \)-almost surely.

**Proposition 3.8.** The following conditions are equivalent:

(a) \( \beta(T) = \infty, \tilde{N}_v \)-a.s.

(b) \( v \) is an \( L_{4g} \)-potential.

(c) \( v = U^{4g}(2v^2) \).

(d) \( \int_{0}^{v(\lambda_i)} 2v(\lambda_i) \, dt = \infty, P^{2(u+g),v} \)-a.s.

(e) \( \tilde{N}_v = \infty, \tilde{N}_v \)-a.s.

(f) \( Z_g = \infty, \tilde{N}_v \)-a.s. on \( \{ Z_g > Z_u \} \).

**Proof.** (a) \( \Leftrightarrow \) (b): If \( v \) is an \( L_{4g} \)-potential, then under \( \mathbb{Q}_v \), the branching tree \( T \) will have infinitely many branches, as no single particle survives to reach \( \partial D \). In other words, each particle dies and is replaced by a pair of particles, before reaching \( \partial D \). Conversely, let \( v \) have an \( L_{4g} \)-harmonic component. Then the original particle has positive probability of reaching \( \partial D \) before dying, in which case \( \beta(T) = 1 < \infty \).

(b) \( \Leftrightarrow \) (c): Recall from (3.2) that \( L_{4g} v = -\frac{2}{v(\lambda_i)^2} \). Thus \( v > U^{4g}(2v^2) \), with equality if and only if \( v \) is an \( L_{4g} \)-potential.

(a) \( \Leftrightarrow \) (d): Here we use the branching scheme \( \hat{Q}_v \) for \( T \), and appeal to Theorem 3.4. With this scheme observe that, given a path of the \( L_{2(u+g),v} \) process, the conditional probability of there being no branch along this path is \( \exp -\int_{0}^{v(\lambda_i)} 2v(\lambda_i) \, dt \). If this is non-zero, then there is a positive probability of having a single branch. While if it equals zero then every path branches.

(c) \( \Leftrightarrow \) (d): Of course, given the above arguments, this equivalence is superfluous. Nevertheless, it is instructive to link the conditions directly.

\[
\frac{U^{4g}(2v^2)(x)}{v(x)} = \frac{1}{v(x)} \int_{0}^{v(\lambda_i)^2} e^{-\int_{0}^{4g(\lambda_i)} \, ds} s \, dt
\]

\[
= \frac{1}{v(x)} \int_{0}^{\infty} P_{4g}^{2(u+g)} \left( 2v(\lambda_i)^2 e^{\int_{0}^{2(u+g) - 4g(\lambda_i)} ds}, \lambda_i > \xi \right) dt
\]

\[
= \frac{1}{v(x)} \int_{0}^{\infty} P_{4g}^{2(u+g)} \left( 2v(\lambda_i)^2 e^{\int_{0}^{2(u+g) - 4g(\lambda_i)} ds}, \lambda_i > \xi \right) dt
\]

\[
= \int_{0}^{\infty} P_{4g}^{2(u+g),v} \left( 2v(\lambda_i)^2 e^{-\int_{0}^{2(u+g)} ds}, \lambda_i > \xi \right) dt
\]
\[ = P_x^{2(u+g),v} \left( \int_0^\tau 2v(\lambda_t)e^{-\int_0^t 2\lambda_s} \, ds \, dt \right) \]
\[ = P_x^{2(u+g),v} \left( 1 - e^{\int_0^\tau 2\lambda_s} \right). \]

This equals 1 if and only if \( \int_0^\tau 2v(\lambda_t) \, dt = \infty \).

(a) \( \Leftrightarrow \) (e): Recall from our discussion of stochastic boundary values, that \( Z_v \) is well-defined \( \hat{\mathbb{N}}_x \)-almost surely. For \( r > 0 \), let \( (\rho_r) \) be the law obtained by starting with a Poisson random variable of mean \( r \), and conditioning it to be non-zero. Recall that \( (\rho_k) = (h_k, 1) \). Then by Proposition 3.6, the law of \( \beta(T^k) \) given \( X^k \) is exactly \( (h_{X^k}, v) \). Since \( (\rho_k) \) \( \Leftrightarrow \) \( (\rho_{X^k}) \) and \( (h_{X^k}, v) \), as \( k \to \infty \), we see that in general
\[ f(z) = 1 \quad \text{if and only if} \quad \hat{\mathbb{N}}_x(\rho_z) = 1, \quad \hat{\mathbb{N}}_x \text{-a.s.} \tag{3.12} \]

From this the desired equivalence is immediate.

(e) \( \Leftrightarrow \) (f): If \( \hat{\mathbb{N}}_x(\rho_{Z_u < Z_g < 1}) > 0 \), then by (3.11) we also have that \( \hat{\mathbb{N}}_x(\rho_{Z_g < 1}) > 0 \). Thus condition (e) fails, since \( Z_v \leq Z_g \). Conversely, suppose that \( \hat{\mathbb{N}}_x(\rho_{Z_u < Z_g < 1}) = 0 \). But \( Z_v = \infty \) whenever \( Z_u < \infty = Z_g \). Thus \( \hat{\mathbb{N}}_x(\rho_{Z_u < 1}) = 0 \) by (3.11), so condition (e) holds. \( \square \)

3.4. Examples

We now consider a number of examples of such conditionings.

Example 3.9 (Hitting one point in dimension 2). Let \( D \subset \mathbb{R}^2 \) be a bounded \( C^2 \)-domain. It is shown in Le Gall (1994a) that boundary points get hit with positive probability. Thus, when \( n = 1 \) and \( d = 2 \), the analogue of the transforms of SV would be a conditioning on the event that \( z \in \partial D \). This conditioning is therefore a special case of that in the next example (Example 3.10). As a consequence, it will follow that

\[ \hat{\mathbb{N}}_x(\rho^k_z | z \in \partial D) = \frac{1}{g_z(x)} \hat{\mathbb{N}}_x(\rho^k_z (1 - \exp(-\langle X^k, g_z \rangle))), \]

where \( g_z(x) = \hat{\mathbb{N}}_x(z \in \partial D) \) satisfies \( \Delta g = 4g^3 \) in \( D \) and has boundary value \( 0 \) away from \( z \). Applying Theorem 3.2 or 3.4, with \( u = 0 \) and \( v = \rho_z \), gives a representation of the conditioned process in terms of a tree backbone, throwing off mass which gets killed off at rate \( 4g_z \). For example, in the representation of Theorem 3.4, the tree consists of particles branching at rate \( 2g_z \), and performing a diffusion that is a \( g_z \)-transform of the \( L_2 \) process. In other words, the single-particle motions are Brownian motions with drift \( g_z^{-1}\nabla g_z \). Each branch of the tree converges to \( z \), and we will see that there are actually infinitely many such branches.

Example 3.10 (Hitting a non-polar set). Consider, more generally, a bounded Lipschitz domain \( D \subset \mathbb{R}^d \), and a closed non-polar subset \( \Gamma \) of \( \partial D \). Let \( u = 0 \) and take \( v(x) = g(x) = \hat{\mathbb{N}}_x(\partial D \cap \Gamma \neq \emptyset) \). Thus \( \hat{\mathbb{M}}_x = 1 - \exp(-\langle X^k, g \rangle) \), and our first goal is to show that \( \hat{\mathbb{M}}_x \) arises by conditioning on the event that \( \partial D \cap \Gamma \neq \emptyset \).
Let \( O_n \) be relatively open subsets of \( \partial D \), with \( O_n \downarrow \Gamma \). As in the proof of Proposition 4.4 of Le Gall (1994a),
\[
\mathbb{N}_x(\mathcal{B}^D \cap O_n \neq \emptyset) = \mathbb{N}_x(X^D(O_n) > 0).
\]

Thus by Lemma 2.4,
\[
\mathbb{N}_x(e_\phi^K|\mathcal{B}^D \cap \Gamma \neq \emptyset) = \lim_{n \to \infty} \frac{1}{g(x)} \mathbb{N}_x(e_\phi^K, \mathcal{B}^D \cap O_n \neq \emptyset)
\]
\[
= \lim_{n \to \infty} \lim_{\lambda \to \infty} \frac{1}{g(x)} \mathbb{N}_x(e_\phi^K(1 - \exp - \langle X^D, \lambda 1_{O_n} \rangle))
\]
\[
= \lim_{n \to \infty} \lim_{\lambda \to \infty} \frac{1}{g(x)} \mathbb{N}_x(e_\phi^K(1 - \exp - \langle X^D, \mathbb{N}_x(\partial D \cap \Gamma \neq \emptyset) \rangle))
\]
\[
= \frac{1}{g(x)} \mathbb{N}_x(e_\phi^K(1 - \exp - \langle X^D, \mathbb{N}_x(\mathcal{B}^D \cap \Gamma \neq \emptyset) \rangle))
\]
\[
= \frac{1}{g(x)} \mathbb{N}_x(e_\phi^K(1 - \exp - \langle X^D, g \rangle))
\]
\[
= \mathbb{N}_x(\mathcal{B}^D).
\]

Note that \( g \) satisfies \( \Delta g = 4g^2 \) in \( D \) and has boundary value 0 on \( \partial D \setminus \Gamma \). In fact it is the maximal such solution (see Proposition 4.4 of Le Gall (1994a)).

Applying Theorem 3.2 or Theorem 3.4 lets us represent this conditioned process in terms of a branching backbone \( \mathcal{T} \) throwing off mass. That mass then gets killed off at rate \( 4g \), but otherwise evolves just as it would if there had been no conditioning. In the representation of Theorem 3.4, the backbone branches at rate \( 2g \), and its particles perform diffusions with generator
\[
L_{2g,g}f = \frac{1}{g} L_{2g}(g f) = \frac{1}{2} \Delta f + \frac{1}{g} \nabla g \cdot \nabla f.
\]

Since the mass thrown off by \( \mathcal{T} \) will die before reaching \( \Gamma \), we are entitled to interpret \( \mathcal{T} \) as the historical tree of all those “particles” that survive to hit \( \Gamma \). Note that the above diffusion also arises in Le Gall (1994a), in a somewhat different context. According to p. 305 of that paper, the law of this process is a multiple of the capacitary measure (with respect to the Brownian snake) of the set of paths which hit \( \Gamma \). In other words, the first path of the snake to terminate in \( \Gamma \) will have law \( P^{2g,g}_x \) under \( \mathbb{N}_x \).

In view of Proposition 3.8, it is worth noting the following.

**Corollary 3.11.** For \( D \subset \mathbb{R}^d \) a bounded Lipschitz domain, and \( \Gamma \) a closed non-polar subset of \( \partial D \), the associated tree \( \mathcal{T} \) given above has infinitely many branches, \( \mathbb{N}_x \)-almost surely.
Proof. Clearly, $u=0$ implies that $Z_u=0$. It is shown in Theorem 6.1 of Dynkin (1998), that
\[ Z_g = \infty \cdot 1_{\{g\neq 0 \land g \neq 0\}}. \]
Thus condition (f) of Proposition 3.8 holds, and the desired conclusion now follows. \qed

Example 3.12 (Transforms by moderate functions). We start with a simple special case. Let $D \subset \mathbb{R}^d$ be a bounded domain, and let $f \geq 0$ be a continuous function on $\partial D$.
Consider the solution to $\Delta g = 4g^2$ on $D$, given by
\[ g(x) = N_x(1 - \exp(-g^2, f)). \]
Note that if $\partial D$ is regular, then $g$ has boundary value $f$; see Dynkin (1991c) or Le Gall (1994b). Let $u=0$ and $v=g$, and consider the associated transform $\hat{N}_x$.

Theorem 3.2 or Theorem 3.4 gives a representation of the solution in terms of a branching backbone $\mathcal{T}$. Because $g$ is bounded, the rate at which branches is also bounded, and so it will have finitely many branches almost surely.

More generally, a solution $g \geq 0$ to $\Delta g = 4g^2$ is called moderate if it is dominated by a harmonic (i.e. $L_0$-harmonic) function. See Le Gall (1995) or Dynkin and Kuznetsov (1998a).

Proposition 3.13. Let $g \geq 0$ solve $\Delta g = 4g^2$, and let $\mathcal{T}$ be the tree associated to it, as above. Then $\hat{N}_x(\beta(\mathcal{T})) < \infty$ if and only if $g$ is moderate.

Proof. Recall that $\pi(\lambda)$ is the law obtained by starting with a Poisson random measure with intensity measure $\lambda$, and conditioning it to be non-zero. Thus $\Pi \sim \pi(\lambda)$ implies that $\langle \Pi, f \rangle$ has mean
\[ \frac{\langle \lambda, f \rangle}{1 - e^{-\langle \lambda, f \rangle}}. \]
In Proposition 3.6 the measure $\eta^k$, putting unit mass at the location of every terminal branch of $\mathcal{T}^k$, was shown to have distribution $\pi(g X^k)$ given $X^k$. Thus the mean number of such branches is
\[ \hat{N}_x(\beta(\mathcal{T}^k)) = \hat{N}_x((\eta^k, 1)) = \hat{N}_x \left( \frac{\langle g X^k, 1 \rangle}{1 - e^{-\langle g X^k, 1 \rangle}} \right) = \hat{N}_x \left( \frac{\langle X^k, g \rangle}{M_k} \right) = \frac{1}{g(x)} \hat{N}_x((X^k, g)). \]
Suppose that $g$ is moderate, and let $h$ be harmonic, with $h \geq g$. Then $\hat{N}_x(\beta(\mathcal{T}^k)) \leq \hat{N}_x((X^k, h))/g(x) = h(x)/g(x)$. The last equality follows from the well-known fact that $\langle X^k, h \rangle$ forms a martingale under $\hat{N}_x$, if $h$ is harmonic (or equivalently, by differentiating the Palm formula (2.5)). Letting $k \to \infty$, we see that $\hat{N}_x(\beta(\mathcal{T})) \leq h(x)/g(x) < \infty$, and hence $\mathcal{T}$ has finitely many branches almost surely.

Conversely, suppose that $\hat{N}_x(\beta(\mathcal{T})) < \infty$. Recall that $Z_g = \lim \langle X^k, g \rangle$ denotes the stochastic boundary value of $g$. Then by Fatou’s lemma and monotone convergence,
\[ \hat{N}_x(Z_g) \leq \lim_{k \to \infty} \hat{N}_x((X^k, g)) = \lim_{k \to \infty} g(x) \hat{N}_x(\beta(\mathcal{T}^k)) = g(x) \hat{N}_x(\beta(\mathcal{T})) < \infty. \]
By a straightforward modification of Theorem 3.2 of Dynkin (1998), it follows that \( g \) is moderate. 

Note also that we recover an interpretation of the backbone \( \mathcal{T} \) as a historical tree, as stated in the introduction. The distinction is that the particles whose genealogy it gives are not determined strictly by the exit measure, but are rather chosen at random as in Proposition 3.6.

**Example 3.14** (*Transforms by planar functions with general trace*). Let \( D \subset \mathbb{R}^2 \) be a bounded \( C^2 \)-domain. Le Gall (1997) classifies all solutions to \( \Delta g = 4g^2 \). They are in one-to-one correspondence with pairs \((\Gamma, v)\), where \( \Gamma \) is a closed subset of \( \partial D \), and \( v \) is a Radon measure on \( \partial D \setminus \Gamma \). In dimension 2, the exit measure \( X^D \) will be absolutely continuous with respect to the surface area measure \( \sigma \), and will in fact have a continuous density. Writing \( x^D = dX^D/d\sigma \) for this density, the correspondence is then given by the formula

\[
g(x) = \mathbb{N}_x(\mathcal{A} \cap \Gamma \neq \emptyset) + \mathbb{N}_x(1 - \exp - \langle v, x^D \rangle, \mathcal{A} \cap \Gamma = \emptyset).
\]

In the case that \( v(dz) = f(z)\sigma(dz) \), this can be simplified, to become

\[
g(x) = \mathbb{N}_x(\mathcal{A} \cap \Gamma \neq \emptyset) + \mathbb{N}_x(1 - \langle X^D, f \rangle, \mathcal{A} \cap \Gamma = \emptyset).
\]

Note that moderate solutions correspond to precisely to the case \( \Gamma = \emptyset \), in which case \( v \) is automatically a finite measure.

Let \( u = \mathbb{N}_x(\mathcal{A} \cap \Gamma \neq \emptyset) \), so \( \frac{1}{2} \Delta u = 2u^2 \), and set \( v = g - u \). For \( \Phi \in \mathcal{F}_k \), the \( g \)-transformed measure

\[
\mathbb{N}^g_x(\Phi) = \frac{1}{g(x)} \mathbb{N}_x(\Phi(1 - e^k_g))
\]

becomes a superposition

\[
\mathbb{N}^g_x = \frac{u(x)}{g(x)} \mathbb{N}^u_x + \frac{v(x)}{g(x)} \mathbb{N}^v_x,
\]

where

\[
\mathbb{N}^u_x(\Phi) = \frac{1}{u(x)} \mathbb{N}_x(\Phi(1 - e^k_u)),
\]

\[
\mathbb{N}^v_x(\Phi) = \frac{1}{v(x)} \mathbb{N}_x(\Phi e^k_v(1 - e^k_v)).
\]

The measure \( \mathbb{N}^u_x \) is of the type considered in Example 3.10, and is represented in terms of a tree whose infinitely many branches terminate in \( \Gamma \), and throw off mass which gets killed at rate \( u \). On the other hand, \( \mathbb{N}^v_x \) is also of the form (3.1), with \( g, u, v \) all as described above. Thus Theorem 3.2 or Theorem 3.4 also represent it in terms of a tree throwing off mass.
Now the mass gets killed at rate $4g$. Each branch of the tree follows a $\nu$-transform of the process with generator $L_{2(g+\nu)}$, with branching at rate $2\nu$.

If $\nu$ is bounded, then $\tilde{N}^{\nu}_t(\beta(T))$ will be finite, as the argument of Proposition 3.13 still applies. But in general this may fail, as the measure $\nu$ can be chosen to blow up near $\Gamma$, essentially as badly as we wish. However, it will still be the case that $\beta(T) < \infty$, $\tilde{N}^{\nu}_t$-a.s. The reason is that if $\mathcal{A} \cap \Gamma = \emptyset$, as in the case $\tilde{N}^{\nu}_t$-a.s., then the exit measure $X^D$ will fail to charge some open neighbourhood $O$ of $\Gamma$. Thus all branches of $T$ will terminate in $\partial D \setminus O$. But, for any fixed open neighbourhood $O$ of $\Gamma$, the argument of Proposition 3.13 shows that the mean number of branches terminating in $\partial D \setminus O$ is finite.

The results of Le Gall (1997) have been generalized to higher dimensions in Dynkin and Kuznetsov (1998a,b). But the higher dimensional results remain less complete than their planar analogues (for example, the question of whether every solution is "$\sigma$-moderate" is as yet unresolved), and we have chosen to restrict the above discussion to the planar case.

### 3.5. Relationships with SV

It is natural to ask for relationships between the current results and those of SV. In the following section we will consider how to generalize both simultaneously, but for now we simply wish to connect the two sets of results.

Recall that our basic equation is (3.2), namely that $L_{2g} \nu = -2\nu^2$. The corresponding relation in SV is a relation, not for a single function $\nu$, but for a family $\nu_A$ indexed by non-empty subsets $A \subseteq N = \{1, \ldots, n\}$. Namely that (see Eq. (3.2) of SV):

$$L_{4g} \nu_A = -2 \sum_{B \subseteq A \setminus \emptyset, \emptyset \neq B} \nu_B \nu_{A \setminus B} \quad (3.13)$$

To see one connection between these relations, fix $n$ and set $\nu_A = c_{|A|}\nu$. Then Eq. (3.13) must fail for $|A| = 1$, as $L_{4g}\nu \neq 0$. We can however ensure that (3.13) holds for all $|A| > 1$, provided we choose the $c_j$ to make

$$c_k = \sum_{j=1}^{k-1} \binom{k}{j} c_j c_{k-j} \quad (3.14)$$

for $1 < k \leq n$. As in the proof of Lemma 3.1 of Serlet (1996) (or of Lemma 2.7 of SV), the standard recurrence for the binomial coefficients yields the solution

$$c_{k+1} = a^{k+1} \frac{(2k)!}{k!},$$

where $a$ is arbitrary. In SV, the analogue of the martingale $\tilde{M}_k$ is

$$M^N_k = \sum_{\sigma \in \mathcal{P}(N)} \exp(-\langle X^k, \sigma \rangle) \prod_{A \in \sigma} (X^k, v^A), \quad (3.15)$$

where $\mathcal{P}(N)$ denotes the set of partitions of the set $N$. If (3.13) holds, then $M^N_k$ will be an $\mathcal{F}_k$ martingale. With the above choice of $v^A$ this will not be the case of course,
but it is still interesting to compare $M^N_k$ with $\hat{M}_k$. We obtain that

$$M^N_k = e^g \sum_{\sigma \in \mathcal{P}(N)} \prod_{\alpha \in \sigma} \langle X^k, v^A \rangle = e^g \sum_{m=1}^{n} \langle X^k, v^m \rangle \sum_{\sigma \in \mathcal{P}(N)} \prod_{|\sigma|=m} c_{|\sigma|}$$

$$= e^g \sum_{m=1}^{n} \frac{c_m}{m!} \langle X^k, v^m \rangle,$$

the latter by induction on $m$. Choosing $a$ to make $c_n = 1$ thus makes $M^N_k$ a truncated version of $\hat{M}_k$.

While this does provide a simple link between the two classes of objects, a less tenuous relationship would be preferable. Such a relationship is suggested by the observation that conditioning $\hat{N}_x$ on the event $\beta(T) = n$ should induce a change of measure by a martingale of type (3.15). While this turns out to be the case, at least in the circumstances described below, we will content ourselves with proving less, namely that $\hat{N}_x$ can be obtained as a superposition of such objects. As before, we will restrict attention to the laws of the exit measures $X^k$.

Let $g$ and $v$ be as in Lemma 3.1. Assume that both $g$ and $v$ are bounded and continuous on $\hat{D}$, and define

$$v_j(x) = \mathbb{N}_x(e^g \langle X^D, v \rangle).$$

Then (2.5) shows that

$$v_1(x) = \mathbb{N}_x(e^g \langle X^D, v \rangle) = P_x(v(B_{t_D}) \exp - \int_0^{t_D} ds 4g(B_s))) = P_x^{4g}(v(B_{t_D}))$$

is $L_{4g}$-harmonic, and for $k > 1$, Lemma 2.5 shows that

$$v_k(x) = 2 \sum_{j=1}^{k} \binom{k}{j} U^{4g}(v_jv_{k-j}).$$

In other words, $v^A = v_{|A|}$ satisfies

$$v^A = 2 \sum_{B \subseteq A \neq \emptyset, A \neq B} U^{4g}(v^Bv^A),$$

so that (3.13) holds for every $|A|$. Defining $M^N_k$ by (3.15), Theorem 3.1 of SV establishes that $M^N_k$ is an $\mathcal{F}_k$ martingale, so that we may consistently define

$$\mathbb{M}^N_k(e^g) = \frac{1}{\nu(x)} \mathbb{N}_x(e^g M^N_k).$$

**Proposition 3.15.** Let $g$ and $v$ be as above (so, in particular, $g$ and $v$ are bounded and continuous on $\hat{D}$). Then

$$\mathbb{N}_x(e^g) = \frac{1}{\nu(x)} \sum_{n=1}^{\infty} \nu_n(x) \mathbb{M}^N_n(e^g).$$
Proof. By our continuity hypotheses, \( v^A = 0 \) on \( \partial D \), unless \( |A| = 1 \), in which case \( v_1 = v \) on \( \partial D \). Thus \( M^N_k \rightarrow M^N_\infty = e_g^D(X_\infty, v)^n \) as \( k \rightarrow \infty \). Therefore \( M^N_k = \mathbb{N}_k(M^N_\infty | \mathcal{F}_k) \) by the martingale convergence theorem (using, say, Lemma 2.6 to show uniform integrability).

As a result, \( \mathbb{N}_k(e^k_\phi) = \sum_{n=1}^{\infty} \mathbb{N}_k \left( e^k_\phi \frac{(X_\infty^D, v)_n^n}{n!} \right) \)

\[ = \frac{1}{v(x)} \sum_{n=1}^{\infty} \mathbb{N}_k \left( e^k_\phi \frac{(X_\infty^D, v)_n^n}{n!} \right) \]

\[ = \frac{1}{v(x)} \sum_{n=1}^{\infty} \frac{1}{n!} \mathbb{N}_k(e^k_\phi M^N_k) \]

\[ = \frac{1}{v(x)} \sum_{n=1}^{\infty} \frac{v_n(x)}{n!} M^N_k(e^k_\phi), \]

which finishes the argument. \( \square \)

4. The martingale \( M_k \)

Finally, we will define another transform, combining features of both the transform \( M_x \) of SV, and the \( \mathbb{M}_x \) of (3.1). The example that motivates this (see (4.6)) is that of a bounded smooth domain \( D \subset \mathbb{R}^2 \), with distinct points \( z_1, \ldots, z_n \in \partial D \). We wish to condition on the event that \( z_i \in \partial D \) for every \( i \). In other words, we wish to generalize Example 3.9, and at the same time extend Theorem 5.6 of SV to dimension 2.

We will work below in some generality. The description of the transform will involve several families of functions, indexed by non-empty subsets \( A \subseteq N = \{1, \ldots, n\} \). Since the notation gets somewhat complicated, the reader may wish to keep the motivating example in mind. In that example, the interpretation of these functions will be:

\( u^A(x) = \mathbb{N}_x(z_i \in \partial D \) for some \( i \in A \),

\( v^A(x) = \mathbb{N}_x(z_i \in \partial D \) for every \( i \in A \),

\( v_\Delta(x) = \mathbb{N}_x(z_i \in \partial D \) for every \( i \in A \), and for no \( i \in N \backslash A \).

In general, let \( D \subset \mathbb{R}^d \) be a domain. Let \( n \geq 1 \), and suppose that for every nonempty \( A \subseteq N = \{1, \ldots, n\} \), we are given a solution \( u^A \geq 0 \) to the equation \( \frac{1}{2} Au = 2u^2 \). Define

\[ v^A = \sum_{N \backslash A \subseteq B \subseteq N \atop B \neq \emptyset} (-1)^{|A|+|B|+n+1} u^B. \]
Suppose also that the relations
\[ v_A > 0 \tag{4.1} \]
hold for every \( \emptyset \neq A \subseteq N \). Then

**Lemma 4.1.**

(a) \( u^A = \sum_{\substack{B \subseteq N \\setminus B \neq \emptyset}} v_B \),

(b) \( \frac{1}{2} \Delta v_A = 4u^N v_A - 2 \sum_{\substack{B \cap C = A \\setminus B, C \neq \emptyset}} v_B v_C \).

**Proof.** To show part (a), observe that
\[
\begin{align*}
\sum_{\substack{B \subseteq N \\setminus B \neq \emptyset}} v_B &= \sum_{\substack{B \subseteq N \\setminus B \neq \emptyset}} \sum_{\substack{N \setminus B \subseteq C \subseteq N \\setminus B \neq \emptyset}} (-1)^{|B|+|C|+n+1} u^C \\
&= \sum_{\emptyset \neq C \subseteq N} (-1)^{|C|+n+1} u^C \sum_{\substack{N \setminus C \subseteq B \subseteq N \\setminus C \neq \emptyset}} (-1)^{|B|} \\
&= \sum_{\emptyset \neq C \subseteq N} (-1)^{|C|+n+1} u^C \left( \sum_{\substack{N \setminus C \subseteq B \subseteq N \\setminus B \neq \emptyset}} (-1)^{|B|} - \sum_{\substack{N \setminus C \subseteq B \subseteq N \\setminus A}} (-1)^{|B|} \right) \tag{4.2} \\
&= \sum_{\emptyset \neq C \subseteq N} (-1)^{|C|+n+1} u^C (-1)^{|A|-|A|+|A|} 1_{A \subseteq C} \tag{4.3} \\
&= u^A,
\end{align*}
\]
where (4.2) follows by rewriting the previous summation and (4.3) holds by virtue of Lemma 2.1. Thus,
\[
\begin{align*}
\frac{1}{2} \Delta v_A &= \sum_{\substack{N \setminus A \subseteq B \subseteq N \\setminus B \neq \emptyset}} (-1)^{|A|+|B|+n+1} 2(u^B)^2 \\
&= 2 \sum_{\substack{N \setminus A \subseteq B \subseteq N \\setminus B \neq \emptyset}} (-1)^{|A|+|B|+n+1} \sum_{\substack{C \subseteq C \setminus B \subseteq N \setminus B \neq \emptyset}} v_C v_{C'} \\
&= 2 \sum_{\substack{C \subseteq C \setminus B \subseteq N \\setminus B \neq \emptyset}} v_C v_{C'} (-1)^{|A|+n+1} \sum_{\substack{N \setminus A \subseteq B \subseteq N \setminus B \neq \emptyset}} (-1)^{|B|} \\
&= 2 \sum_{\substack{C \subseteq C \setminus B \subseteq N \\setminus B \neq \emptyset}} v_C v_{C'} (-1)^{|A|+n+1} \sum_{\substack{N \setminus A \subseteq B \subseteq N \setminus B \neq \emptyset}} (-1)^{|B|}.
\end{align*}
\]
\[
= 2 \sum_{C, C' \subseteq N \atop C \neq C'} v_C v_{C'} (-1)^{|A| + |C'| + 1} \left( \sum_{N \setminus A \subseteq B \subseteq N} (-1)^{|B|} - \sum_{N \setminus A \subseteq B \subseteq N \setminus C} (-1)^{|B|} \right) \\
- \sum_{N \setminus A \subseteq B \subseteq N \setminus C'} (-1)^{|B|} + \sum_{N \setminus A \subseteq B \subseteq N \setminus (C \cup C')} (-1)^{|B|} \right) \\
= 2 \sum_{C, C' \subseteq N \atop C \neq C'} v_C v_{C'} (-1)^{|A| + |C'| + 1} (-1)_{C = A} - 1_{C' = A} + 1_{C \cup C' = A} (-1)^{|A| - |A|} \\
= 4v_A \left( \sum_{\emptyset \neq C \subseteq N} v_C \right) - 2 \sum_{C \cup C' = A \atop C, C' \neq \emptyset} v_C v_{C'} \\
= 4u^N v_A - 2 \sum_{C \cup C' = A \atop C, C' \neq \emptyset} v_C v_{C'}. \quad \Box
\]

**Remark 4.2.** Though we will not need it, the analogue of the \( v^A \) of (3.13) are really
\[ v^A = \sum_{\emptyset \neq B \subseteq A} (-1)^{|B| + 1} u^B. \]

The following relations could be proved just as above:
\[
\begin{align*}
  v^A &= \sum_{A \subseteq B \subseteq N} v_B, \\
v_d &= \sum_{A \subseteq B \subseteq N} (-1)^{|A|} v^B, \\
u^A &= \sum_{\emptyset \neq B \subseteq A} (-1)^{|B| + 1} u^B.
\end{align*}
\]

Now set
\[ \tilde{M}_k = 1 + \sum_{\emptyset \neq A \subseteq N} (-1)^{|A|} \exp - \langle X_k, u^A \rangle. \]

It follows immediately that \( \tilde{M}_k \) is a \( \mathbb{N}_x \)-martingale, and so for \( \Phi \in \mathcal{F}_k \) we can define
\[ \mathbb{E}_x(\Phi) = \frac{1}{v_N(x)} \mathbb{E}_x(\Phi \tilde{M}_k). \]

**Lemma 4.3.**
\[ \tilde{M}_k = \exp(-\langle X_k, u^N \rangle) \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{C_1 \cup \cdots \cup C_m = N \atop C_i \neq \emptyset \forall i} \prod_{i=1}^{m} \langle X_k, v_{C_i} \rangle. \]
Proof. Write $u^0 = 0$. Then by Lemma 2.2

$$M_k = \sum_{B \subseteq N} (-1)^{|B|} \exp - \langle X^k, u^B \rangle$$

$$= e^{k} \sum_{B \subseteq N} (-1)^{|B|} \exp \langle X^k, u^N - u^B \rangle$$

$$= e^{k} \sum_{B \subseteq N} \sum_{m=0}^{\infty} \frac{(-1)^{|B|}}{m!} \langle X^k, u^N - u^B \rangle^m$$

$$= e^{k} \left( \sum_{B \subseteq N} (-1)^{|B|} \right) + \sum_{m=1}^{\infty} \frac{(-1)^{|B|}}{m!} \left( \sum_{\emptyset \neq C \subseteq N \setminus B} \langle X^k, v_C \rangle ight)^m$$

$$= e^{k} \sum_{m=1}^{\infty} \frac{(-1)^{|B|}}{m!} \sum_{C_1 \cdots C_m \subseteq N \setminus B} \prod_{i=1}^{m} \langle X^k, v_{C_i} \rangle$$

$$= e^{k} \sum_{m=1}^{\infty} \frac{1}{m!} \prod_{i=1}^{m} \langle X^k, v_{C_i} \rangle \sum_{B \subseteq N \setminus \cup C_i} (-1)^{|B|}$$

$$= e^{k} \sum_{m=1}^{\infty} \frac{1}{m!} \prod_{i=1}^{m} \langle X^k, v_{C_i} \rangle \cdot$$

To describe the probabilistic representation of $\tilde{M}_x$, we construct a measure $\tilde{N}_x$, as before. It has a tree backbone $\mathcal{T}$, and throws off mass which gets killed at rate $u^N$. In other words, we use $\tilde{N}_x(\Phi_k) = \tilde{N}_x(\Phi_k e_x^k)$, for $\Phi_k \in \mathscr{F}_k$. To construct the backbone, we start a single particle off at $x$, following a $v_N$-transform of the process with generator $L_{4u}$. When it dies, say at a point $y$, we choose a pair $(A, A')$ such that $A \cup A' = N$ and $A, A' \neq \emptyset$, according to the law

$$p(A, A'; N)(y) = \frac{v_A(y) v_{A'}(y)}{\sum_{B \cup B' = N, B, B' \neq \emptyset} v_B(y) v_{B'}(y)}.$$

At its death, the $v^N$-particle splits into a $v^A$-particle and a $v^{A'}$ particle. The $v^A$-particle follows a $v_A$-transform of $L_{4u}$, and when it dies, it splits into a $v^B$-particle and a $v^{B'}$-particle, where $(B, B')$ is chosen according to law $p(B, B'; A)$, and so on.

This gives us a tree $\mathcal{T}$ of branching particles, each tagged with a set $A$. We may form $\mathcal{T}^k$ as before, by pruning off all particles (together with their descendants), once they leave $D_k$. We write $\tilde{Q}_x$ for the law of $\mathcal{T}$, and set

$$\tilde{N}(\exp - \langle X^k, \phi \rangle) = \tilde{Q}_x \left( \exp - \int_0^\infty dr \{ \mathcal{T}^k_r, \tilde{N}(1 - e^k_r) \} \right).$$
Theorem 4.4. Assume condition (4.1). Then
\[ \mathbb{M}_x(\exp - \langle X^k, \phi \rangle) = \mathbb{N}_x(\exp - \langle Y^k, \phi \rangle). \]

Remark 4.5. Using historical processes, as in the last section of SV one can show that
\[ \mathbb{M}_x = \mathbb{N}_x \text{ on } \mathcal{F}_k. \]

Proof. In the present context, it is useful to label all the particles of \( T^k \) that exit \( D_k \), by placing an order on them. So let \( F_k \) be the set of such particles, and set \( \gamma_k = |F_k| \). For \( A \subset N \), let
\[ \mathcal{S}_m(A) = \{ (C_1, \ldots, C_m); C_1 \cup \cdots \cup C_m = A, \emptyset \neq C_i \forall i \}. \]
If \( \gamma_k = m \) choose at random an ordering of \( F_k \), and for \( A = (C_1, \ldots, C_m) \in \mathcal{S}_m(N) \), write \( T^k \approx A \) for the event that the \( i \)th particle is tagged with the set \( C_i, i = 1, \ldots, m \). Thus for example,
\[ \hat{Q}_x(\gamma_k = m) = \sum_{A \in \mathcal{S}_m(N)} \hat{Q}_x(T^k \approx A). \quad (4.4) \]

Note that if \( M \subset S \) are sets with \( |S| = m \) and \( |M| = j \), then there are \( \binom{m}{j} \) orderings of \( S \) compatible with any given orders of \( M \) and on \( S \setminus M \). In other words, if \( \sigma \) is any order on \( S \), and if \( \Sigma \) is an order on \( S \) picked at random, then the conditional probability
\[ P(\Sigma = \sigma | \Sigma_M = \sigma_M, \Sigma_{S \setminus M} = \sigma_{S \setminus M}) = 1 / \binom{m}{j} \quad (4.5) \]
(writing \( \sigma_M \) etc. for the restriction of \( \sigma \) to \( M \)). As described initially, the root particle of the tree is always a \( v^N \)-particle. It is convenient, for purposes of induction, to allow the same notation to cover the situation that we start with our root being a \( v^A \)-particle for some \( A \subset N \). In this case, \( (4.4) \) still holds, but with \( A \in \mathcal{S}_m(N) \) replaced by \( A \in \mathcal{S}_m(A) \). With this in mind, we may define another restriction operation as follows. For \( 1 \leq i_1 < \cdots < i_k \leq m \), set
\[ (C_1, \ldots, C_m)_{\{i_1, \ldots, i_k\}} = (C_{i_1}, \ldots, C_{i_k}). \]
Thus, if \( A = (C_1, \ldots, C_m) \in \mathcal{S}_m(A) \) and \( M \subset \{1, \ldots, m\} \), we will have that \( A|_M \in \mathcal{S}_m(B) \) for \( B = \bigcup_{i \in M} C_i \). As a shorthand for the latter, we write \( A(M) = \bigcup_{i \in M} C_i \).

We will show, by induction on \( m \geq 0 \), that for \( \emptyset \neq A \subset N \), and \( (C_1, \ldots, C_m) \in \mathcal{S}_m(A) \),
\[ \hat{Q}_x \left( \exp - \int_0^\infty dt \, 4(\tau_t^k, \mathbb{N}_x; (1 - \phi^k) \tau^k \approx (C_1, \ldots, C_m)) \right) \]
\[ = \frac{1}{m! v_A(x) \mathbb{N}_x} \left( \phi^k_{\phi + \psi^k} \prod_{i=1}^m (X^k, v_{C_i}) \right). \quad (4.6) \]
Taking \( A = N \) and summing over \( \mathcal{S}_m(N) \) will then establish the theorem.

The initial stage of the induction, with \( m = 1 \) follows exactly as in the proof of Theorem 3.2. So let \( m > 1 \) and assume the inductive hypothesis for all \( A \subset N \), and for all values smaller than \( m \). For simplicity, we will verify \( (4.6) \) in the case \( A = N \).
For $\zeta$ the lifetime of the initial particle, and $A = (C_1, \ldots, C_m)$, we have that

\[
\mathcal{Q}_\lambda \left( \exp - \int_0^\infty dt \dot{4}(\mathcal{T}_t, \mathcal{N}_t, 1 - e_0^k), T^k \approx A \right)
\]

\[
= \sum_{j=1}^{m-1} \sum_{M \subset \{1, \ldots, m\} \atop |M| = j} P_{\lambda, \nu}^{\lambda u^{+}, \nu} \left( 1 \leq \tau, \exp - \int_0^\tau dt \dot{4}(\mathcal{N}_t, 1 - e_0^k) \right)
\]

\[
\times p(A(M), A(M^c); N)(\zeta) \left( \frac{m}{j} \right)^{-1}
\]

\[
\times \mathcal{Q}_{\zeta_1} \left( \exp - \int_0^\zeta dt \dot{4}(\mathcal{T}_t, \mathcal{N}_t, 1 - e_0^k), T^k \approx A|_M \right)
\]

\[
\times \mathcal{Q}_{\zeta_1} \left( \exp - \int_0^\zeta dt \dot{4}(\mathcal{T}_t, \mathcal{N}_t, 1 - e_0^k), T^k \approx A|_{\{1, \ldots, m\} \setminus M} \right)
\]

(4.7)

\[
= \sum_{j=1}^{m-1} \sum_{M \subset \{1, \ldots, m\} \atop |M| = j} \frac{1}{v_N(x)} P_{\lambda, \nu}^{\lambda u^{+}, \nu} \left( \int_0^{\tau_s} ds \sum_{(A, A') \in \mathscr{F}(N)} e_{A, A'}(\zeta_s) e_{A'}(\zeta_s) \left( 1_{\zeta > s} \right) \right)
\]

\[
\times \frac{j!(m-j)!}{m!} p(A(M), A(M^c); N)(\zeta)
\]

\[
\times \exp \left( - \int_0^s dt \dot{4}(\mathcal{N}_t, 1 - e_0^k) - \mathcal{N}_t(1 - e_0^k) \right)
\]

\[
\times \mathcal{Q}_{\zeta_1} \left( \exp - \int_0^\zeta dt \dot{4}(\mathcal{T}_t, \mathcal{N}_t, 1 - e_0^k), T^k \approx A|_M \right)
\]

\[
\times \mathcal{Q}_{\zeta_1} \left( \exp - \int_0^\zeta dt \dot{4}(\mathcal{T}_t, \mathcal{N}_t, 1 - e_0^k), T^k \approx A|_{\{1, \ldots, m\} \setminus M} \right)
\]

(4.8)

\[
= \sum_{j=1}^{m-1} \sum_{M \subset \{1, \ldots, m\} \atop |M| = j} \frac{2}{m! v_N(x)} P_{\lambda} \left( \int_0^{\tau_s} ds \right)
\]

\[
\times \exp \left( - \int_0^s dt \dot{4} u^{+}_N(\zeta_s) \right) \exp \left( \int_0^s dt \dot{4} u^{+}_N(\zeta_s) \right) \mathcal{A}(e_0^k + u^{+}_N)
\]

\[
\times j! v_{A(M)}(\zeta_s) \mathcal{Q}_{\zeta_1} \left( \exp - \int_0^\zeta dt \dot{4}(\mathcal{T}_t, \mathcal{N}_t, 1 - e_0^k), T^k \approx \Gamma|_M \right)
\]

\[
\times (m-j)! v_{A(M^c)}(\zeta_s) \mathcal{Q}_{\zeta_1} \left( \exp - \int_0^\zeta dt \dot{4}(\mathcal{T}_t, \mathcal{N}_t, 1 - e_0^k), T^k \approx \Gamma|_{M^c} \right)
\]

(4.9)
\[ = \sum_{M \subset \{1, \ldots, m\}} \frac{2}{m! v_N(x)} P_x \left( \int_0^{\tau_k} ds \, \mathcal{N}_s (e_{\phi_u \psi}^k) \mathcal{N}_s \left( \prod_{i \in M} \langle X_t^k, v_C \rangle \right) \right) \]

\[ \times \mathcal{N}_x \left( e_{\phi_u \psi}^k \prod_{i \in \{1, \ldots, m\} \setminus M} \langle X_t^k, v_C \rangle \right) \]

\[ = \frac{1}{m! v_N(x)} \mathcal{N}_x \left( e_{\phi_u \psi}^k \prod_{i \in \{1, \ldots, m\} \setminus M} \langle X_t^k, v_C \rangle \right). \]

In the above line (4.7) follows from (4.5) and the definition of \( \hat{Q} \), (4.8) follows from (2.4), (4.9) from (2.1) and Lemma 2.3, (4.10) by the inductive hypothesis, and (4.11) from (3.4). \( \square \)

**Example 4.6 (Hitting \( n \) non-polar sets).** Let \( \Gamma_1, \ldots, \Gamma_n \) be disjoint closed non-polar subsets of \( \partial D \), where \( D \subset \mathbb{R}^d \) is bounded and Lipschitz. Set

\[ u^A(x) = \mathcal{N}_x \left( \mathcal{R}^D \cap \bigcup_{i \in A} \Gamma_i \neq \emptyset \right). \]

Then, by inclusion–exclusion,

\[ v_A(x) = \mathcal{N}_x \left( \mathcal{R}^D \cap \Gamma_i \neq \emptyset \ \forall i \in A, \mathcal{R}^D \cap \Gamma_i = \emptyset \ \forall i \in N \setminus A \right). \]

Thus (4.1) hold (and, referring to Remark 4.2, \( v_A(x) = \mathcal{N}_x (\mathcal{R}^D \cap \bigcup_{i \in A} \Gamma_i \neq \emptyset) \)). By (5.22) and (5.6) of SV we have that in fact,

\[ \mathcal{N}_x (\phi_A) = \mathcal{N}_x (\mathcal{R}^D \cap \Gamma_i \neq \emptyset \ \forall i \in N), \]

so that Theorem 4.4 provides a particle representation of the process conditioned to hit each \( \Gamma_i \). Note that if \( d = 2 \) and \( \Gamma_1 = \{z_1\}, \ldots, \Gamma_m = \{z_n\} \), then this becomes an extension to dimension 2 of the representation given in SV, as well as a generalization of Example 3.9.

**Acknowledgements**

We would like to thank the referee, whose comments improved the exposition of the paper.

**References**

