Population-size-dependent and age-dependent branching processes

Peter Jagers\textsuperscript{a, \textasteriskcentered}, Fima C. Klebaner\textsuperscript{b}
\textsuperscript{a}School of Mathematical and Computing Sciences, Chalmers University of Technology, Gothenburg, Sweden
\textsuperscript{b}Department of Mathematics and Statistics, University of Melbourne, Parkville, Victoria, Australia

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Abstract

Supercritical branching processes are considered which are Markovian in the age structure but where reproduction parameters may depend upon population size and even the age structure of the population. Such processes generalize Bellman–Harris processes as well as customary demographic processes where individuals give birth during their lives but in a purely age-determined manner. Although the total population size of such a process is not Markovian the age chart of all individuals certainly is. We give the generator of this process, and a stochastic equation from which the asymptotic behaviour of the process is obtained, provided individuals are measured in a suitable way (with weights according to Fisher’s reproductive value). The approach so far is that of stochastic calculus. General supercritical asymptotics then follows from a combination of $L^2$ arguments and Tauberian theorems. It is shown that when the reproduction and life span parameters stabilise suitably during growth, then the process exhibits exponential growth as in the classical case. Application of the approach to, say, the classical Bellman–Harris process gives an alternative way of establishing its asymptotic theory and produces a number of martingales.

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1. Introduction

In age-dependent, Bellman–Harris branching processes a particle lives for a random length of time and on its death produces a random number of offspring, all of which live and reproduce independently, with the same laws as the original particle. In classical demographic theory (female) individuals give birth according to age-specific birth rates. What unites these two types of processes is that they are Markovian in the age structure, cf. Jagers (1975, p. 208).

Now, imagine a collection of individuals with ages $(a_1, \ldots, a^n) = A$. In a population-size-dependent process such an individual of age $a$ has a random life span with hazard rate $h_A(a)$. During her life she gives birth with intensity $b_A(a)$, both rates dependent
on the individual’s age as well as the whole setup of ages. Finally, when she dies, she splits into a random number $Y(a)$ of off-springs with distribution that may depend upon the whole collection $A$ of ages as well as the age $a$ of the individual splitting. Childbearing and life length may thus be affected by population size and age structure, but apart from this individuals live and reproduce independent of each other. We write $m_A(a) = \mathbb{E}[Y(a)]$.

The type of dependence we have in mind is the dependence on the total population size $z = |A| = (1, A)$, so that $h_a = h_z$, $b_a = b_z$, and similarly for the off-spring at splitting. But a more general influence pattern is also allowed. The aim is to obtain results on the asymptotic behaviour of such population-size-dependent processes in the case of stabilizing reproduction and lifelengths of individuals, namely when $h_z \to h$, $b_z \to b$, and $m_z = m_A \to m$, as $z \to \infty$.

We consider only the supercritical case, so that $m > 1$ in the case of pure splitting. Such results were obtained by Klebaner (1984, 1989) for Galton–Watson processes and Klebaner (1994) for Markov population-size-dependent branching process, where life spans are exponential with parameters depending on $z$. In particular, it was shown that the condition $\sum |m_z - m|/z < \infty$ is essentially necessary and sufficient for the process to grow at an exponential rate. Jagers (1997), using a coupling argument, obtained results for population-size-dependent branching processes which were more general than presently considered, however a sufficient condition for the exponential growth obtained by this method is $\sum |m_z - m| < \infty$. In Jagers (2000) the sharp necessary and sufficient condition was recovered, but only for processes that possess a symmetric growth property. Here we pursue the line of analysis as for the Markov case by considering the age process.

Our approach combines a stochastic calculus and Markov processes analysis with the method of random characteristics for general branching processes. We identify the compensator of the process and the martingale in Section 2. In Section 3 we show that this approach leads to new results, as well as recovers known results, for classical branching processes. In Section 4 we show the convergence of the Malthusian normed population-size-dependent and age-dependent branching processes with stabilizing reproduction, provided individuals are counted in a special way, through Fisher’s reproductive value. In Section 5 we use the population tree formulation and random characteristics to obtain exponential growth of the population size as measured in the usual straightforward way, by means of a quadratic mean argument combined with classical Tauberian analysis.

2. Population-size-dependent processes as Markov processes of ages

It has been known for a long time that the process of ages in a Bellman–Harris process constitutes a Markov process. It is not difficult to see that the most general classical branching processes that are Markovian in the age structure are those outlined in the introduction, combining a Sevastyanov-type splitting (life length and off-spring number at splitting are not necessarily independent) with an age-dependent propensity to child-birth during life (or a fertile subinterval thereof).
To be precise one needs to introduce the appropriate state space and topology, for the standard theory of Markov processes cf. Ethier and Kurtz (1986). This has been done in a number of ways in the past. We take the state space to be the finite positive Borel measures on $\mathbb{R}$ with the topology of weak convergence, i.e. $\lim_{n \to \infty} \mu_n = \mu$ if and only if $\lim_{n \to \infty} (f, \mu_n) = (f, \mu)$ for any bounded and continuous function $f$ on $\mathbb{R}$, see Ethier and Kurtz (1986, Section 9.4) and Dawson (1993).

Motivated by sequences of scaled measure-valued branching processes and their limits (superprocesses) Métivier (1987) and Borde-Boussion (1990) imbedded the space of measures into a weighted Sobolev space. Oelschl/DELager (1990) took for statespace the signed measures with yet another topology. All of the above works defined the process as the solution to the appropriate martingale problem. In our case we study a single process by means of martingale techniques, so all we need is the basic representation given by Dynkin’s formula. Most of the results below are standard, and we shall not go into the details.

Let $A = (a^1, \ldots, a^n)$, where the points $a^i \geq 0$, and $n$ is an integer. The corresponding counting measure $A$ is defined by $A(B) = \sum_{i=1}^{n} 1_B(a^i)$, for any Borel set $B$ in $\mathbb{R}^+$. For a function $f$ on $\mathbb{R}$ the following notations are used interchangeably throughout the paper:

$$ (f, A) = \int f(x)A(dx) = \sum_{i=1}^{n} f(a^i). $$

Let $z_t$ be the size of the population at $t$, i.e. the number of individuals alive. If $A_t = (a^1_t, \ldots, a^n_t)$ denotes the age chart of the particles, we shall study processes $(f, A_t)$. The population size process is simply $z_t = (1, A_t)$. 

Test functions used on the space of measures are of the form $F((f, \mu))$, where $F$ and $f$ are functions on $\mathbb{R}$. In order not to overburden the presentation we assume, throughout this paper, that births during a mother’s life are never multiple and the populations are non-explosive in the sense that only a finite number of births can occur in finite time. A sufficient condition for the latter is that the functions $b, h, m$ are uniformly bounded.

Indeed, assume that the process starts from $z$ individuals aged $(a^1, \ldots, a^n) = A$. We use $P_0$ and $E_0$ for the probability measure and expectation with this initial condition. Sometimes the suffix will be omitted. Let $X_i(u)$ denote the number of children the $i$th individual of these obtains during the first $u$ time units.

The boundedness of $m_i$ and the two rates implies that there is a $c$ such that

$$ E_0[X_i(u)] \leq cu. $$

Take a $u < 1/c$. If $y(u)$ is the total number of individuals born by $u$, then

$$ E_0[y(u)] \leq z/(1 - cu) < \infty. $$

Generally, for $t \geq u$

$$ E_0[y(t)] \leq (z/(1 - cu))^{t/u} < \infty $$

by repeated conditioning.
Theorem 2.1. For a bounded differentiable function $F$ on $\mathbb{R}^+$ and a continuously differentiable function $f$ on $\mathbb{R}^+$, the following limit exists
\[ \lim_{t \to 0} \frac{1}{t} \mathbb{E}_A \{ F((f, A_t)) - F((f, A)) \} = \mathcal{G}F((f, A)), \] (2.1)
where
\[ \mathcal{G}F((f, A)) = F'((f, A))(f', A) + \sum_{j=1}^{z} b_{A}(a^j)(F(f(0) + (f, A)) - F((f, A))) \]
\[ + \sum_{i=1}^{z} h_{A}(a^i)\{ \mathbb{E}_A[F(Y(a^i)f(0) + (f, A) - f(a^i))] - F((f, A)) \}, \] (2.2)
and $Y(a)$ denotes the number of children at death of a mother, dying at age $a$.

Proof. Direct calculations. □

Remark. If we were to allow for the possibility of $X(a)$ children if there is a bearing during life and at age $a$, we would have to replace $F(f(0) + (f, A))$ by $\mathbb{E}_A[F(Y(a^i)f(0) + (f, A))]$. $\mathcal{G}$ in (2.2) defines a generator of a measure-valued branching process, in which the movement of the particles is deterministic, namely shift.

The following result is often referred to as Dynkin’s formula.

Lemma 2.1. For a bounded $C^1$ function $F$ on $\mathbb{R}$ and a $C^1$ function $f$ on $\mathbb{R}^+$
\[ F((f, A_t)) = F((f, A_0)) + \int_0^t \mathcal{G}F((f, A_s)) \, ds + M_t^{F,f}, \] (2.3)
where $M_t^{F,f}$ is a local martingale with the sharp bracket given by
\[ \langle M^{F,f}, M^{F,f} \rangle_t = \int_0^t \mathcal{G}^2 F((f, A_s)) \, ds - 2 \int_0^t F((f, A_s)) \mathcal{G} F((f, A_s)) \, ds. \] (2.4)
Consequently,
\[ \mathbb{E}_A(M_t^{F,f})^2 = \mathbb{E}_A \left( \int_0^t \mathcal{G}^2 F((f, A_s)) \, ds - 2 \int_0^t F((f, A_s)) \mathcal{G} F((f, A_s)) \, ds \right), \]
provided $\mathbb{E}_A(M_t^{F,f})^2$ exists.

Proof. The first statement is obtained by Dynkin’s formula. Expression (2.4) is obtained by letting $U_t = F((f, A_t))$, and an application of the following lemma. □

Lemma 2.2. Let $U_t$ be a locally square-integrable semi-martingale, such that $U_t = U_0 + A_t + M_t$, where $A_t$ is a predictable process of locally finite variation and $M_t$ is a locally square-integrable local martingale, $A_0 = M_0 = 0$. Let $U_t^2 = U_0^2 + B_t + N_t$, where $B_t$ is a predictable process and $N_t$ is a local martingale. Then
\[ \langle M, M \rangle_t = B_t - 2 \int_0^t U_{s-} \, dA_s - \sum_{s \leq t} (A_s - A_{s-})^2. \] (2.5)
Of course, if \( A \) is continuous (as in our applications) the last term in the above formula (2.5) vanishes.

**Proof.** By the definition of the quadratic variation process (or integration by parts)
\[
U_t^2 = U_0^2 + 2 \int_0^t U_s^- dU_s + [U, U]_t = U_0^2 + 2 \int_0^t U_s^- dA_s + 2 \int_0^t U_s^- dM_s + [U, U]_t.
\]

Using the representation for \( U_t^2 \) given in the conditions of the lemma, we obtain that \([U, U]_t - B_t + 2 \int_0^t U_s^- dA_s \) is a local martingale. Since \([A, A]_t = \sum_{s \leq t} (A_s - A_{s-})^2\), the result follows.

Let \( v^2_A(a) = \mathbb{E}_a[Y^2(a)] \) be the second moment of the offspring-at-splitting distribution in a population with composition \( A \). Applying Dynkin’s formula to the function \( F(u) = u \) (and writing \( M^f \) for \( M^{u^1/f} \)), we obtain the following:

**Theorem 2.2.** For a \( C^1 \) function \( f \) on \( \mathbb{R}^+ \)
\[
(f, A_t) = (f, A_0) + \int_0^t (L_A f, A_s) ds + M^f_t,
\]
where the linear operators \( L_A \) are defined by
\[
L_A f = f' - h_A f + f(0)(b_A + h_A m_A)
\]
and \( M^f_t \) is a local square-integrable martingale with the sharp bracket given by
\[
\langle M^f, M^f \rangle_t = \int_0^t (f^2(0)b_{t, b} + f^2(0)v^2_A h_{t, h} + h_A f^2 - 2f(0)m_A h_{t, f}, A_s) ds.
\]

**Proof.** The first statement is Dynkin’s formula for \( F(u) = u \). This function is unbounded and the standard formula cannot be applied directly. However, the statement follows from the formula for bounded functions by taking smooth bounded functions that agree with \( u \) on bounded intervals, \( F_n(u) = u \) for \( u \leq n \), moreover the sequence of stopping times \( T_n = \inf \{ (f, A_t) > n \} \) serves as a localizing sequence, as was done in, for example, Oelschl/DELLager (1984). The form of the operator \( L_A \) follows from (2.2). Note that with \( F(u) = u^2 \),
\[
\mathbb{E}F((f, A)) = 2(f, A)(f', A) + (h_A, A)(f^2(0) + 2f(0)(f, A))
+ f^2(0)(v^2_A h, A) + (h_A f^2, A) + 2f(0)(m_A h_A, A)(f, A)
- 2f(0)(m_A h_A f, A) - 2(h_A f, A)(f, A),
\]
so that (2.8) follows from (2.4). An argument similar to that given above but for \( F(u) = u^2 \) shows that \( M^f_t \) is locally square integrable.

By taking \( f \) to be a constant, \( f(u) = 1 \), Theorem 2.2 yields the following corollary. Recall that \( z_t = (1, A_t) \) is the population size at time \( t \).
Corollary 2.1. The compensator of $z_t$ is given by $\int_0^t (b_{A_s} + h_{A_s}(m_{A_s} - 1)) \, ds$.

An equation similar to (2.6) is given in Métivier (1987) and Borde-Boussion (1990) for $(f, A_t)$ with $f \in C^{\infty}_K$, infinitely differentiable functions with a compact support. Since a smooth function can be approximated on a finite interval by $C^{\infty}_K$ functions, Eq. (2.6) can also be deduced from that equation.

It is important to give conditions that assure the integrability of the processes appearing in Dynkin’s formula (2.6). Typically, to achieve integrability it is assumed that functions are bounded, and in this case the local martingales appearing in Dynkin’s formula are true martingales. However, integrability may hold also for some unbounded functions. In the case of pure jump processes not too stringent conditions for it to hold were given in Hamza and Klebaner (1995). However, the age process considered here includes a deterministic motion, and it is not a pure jump process, therefore the condition in the above paper cannot be used directly. But a similar condition (H1) is given below. We restrict ourselves to positive functions, since these are the ones we use.

Theorem 2.3. Let $f \geq 0$ be a $C^1$ function on $\mathbb{R}^+$ that satisfies

$$|(L_A f, A)| \leq C(1 + (f, A))$$

(H1)

for some constant $C$ and any $A$, and assume that $(f, A_0)$ is integrable. Then $(f, A_t)$ and $M^f_t$ in (2.6) are also integrable with $\mathbb{E}M^f_t = 0$.

Proof. Let $T_n$ be a localizing sequence, then from (2.6)

$$(f, A_{t \wedge T_n}) = (f, A_0) + \int_0^{t \wedge T_n} (L_A f, A_s) \, ds + M^f_{t \wedge T_n},$$

(2.9)

where $M^f_{t \wedge T_n}$ is a martingale. Taking expectations we have

$$\mathbb{E}(f, A_{t \wedge T_n}) = \mathbb{E}(f, A_0) + \mathbb{E} \int_0^{t \wedge T_n} (L_A f, A_s) \, ds.$$  

(2.10)

Using condition (H1)

$$\left| \mathbb{E} \int_0^{t \wedge T_n} (L_A f, A_s) \, ds \right| \leq \mathbb{E} \int_0^{t \wedge T_n} |(L_A f, A_s)| \, ds$$

$$\leq C t + C \mathbb{E} \int_0^{t \wedge T_n} (f, A_s) \, ds.$$  

Thus, we have from (2.10)

$$\mathbb{E}(f, A_{t \wedge T_n}) \leq \mathbb{E}(f, A_0) + C t + C \mathbb{E} \int_0^{t \wedge T_n} (f, A_s) \, ds$$

$$\leq \mathbb{E}(f, A_0) + C t + C \mathbb{E} \int_0^t I(s \leq T_n)(f, A_s) \, ds$$

$$\leq \mathbb{E}(f, A_0) + C t + C \int_0^t \mathbb{E}(f, A_{s \wedge T_n}) \, ds.$$
It now follows by Gronwall’s inequality (cf. e.g. Dieudonné, 1960) that

\[ E(f, A_t) \leq E(f, A_0) + Ct + re^{Ct} < \infty. \tag{2.11} \]

Taking \( n \to \infty \) we obtain by Fatou’s lemma that \( E(f, A_t) < \infty \), thus \( (f, A_t) \) is integrable, as it is nonnegative. Now by condition (H1)

\[ E \left| \int_0^t (L_A f, A_s) \, ds \right| \leq \int_0^t E|L_A f, A_s| \, ds \leq \int_0^t C(1 + E(f, A_s)) \, ds < \infty. \tag{2.12} \]

It follows from (2.12) that \( \int_0^t (L_A f, A_s) \, ds \) and its variation process \( \int_0^t |L_A f, A_s| \, ds \) are both integrable. It now follows from (2.6) that

\[ M^f_t = (f, A_t) - (f, A_0) - \int_0^t (L_A f, A_s) \, ds \tag{2.13} \]

is integrable with zero mean. \( \square \)

Before considering populations with stabilizing reproductions, we shall have a look at classical, population-size-independent branching processes, which have a Markovian age structure.

3. Classical branching processes which are Markovian in the age structure

In traditional branching processes, as well as demography or population dynamics, the reproduction rate, hazard function and offspring-at-splitting distribution are all population independent. We write \( b \) and \( h \) for the former two, and \( G \) for the lifespan distribution. Its density is denoted by \( g \) and the first two moments of offspring-at-splitting distribution by \( m \) and \( v^2 \). Note that the latter may still be functions of age at split, unless we are in the Bellman–Harris case of independence between life span and reproduction.

This section is confined to the case of i.i.d. individuals. For simplicity, we assume that \( G(u) < 1 \) for all \( u \in \mathbb{R}^+ \), and write \( S = 1 - G \) for the survival function, recall that \( S' = -hS \). The results of the previous section are summarized in the following theorem.

**Theorem 3.1.** (1) For a \( C^1 \) function \( f \) on \( \mathbb{R}^+ \)

\[ (f, A_t) = (f, A_0) + \int_0^t (Lf, A_s) \, ds + M^f_t, \tag{3.1} \]

where the linear operator \( L \) is defined by

\[ Lf = f' - hf + f(0)b + f(0)mh \tag{3.2} \]

and \( M^f_t \) is a local square-integrable martingale with the sharp bracket given by

\[ \langle M^f_t, M^f_s \rangle = \int_0^t f^2(0)b + h(f^2 + f^2(0)v^2 - 2f(0)m\lambda, A_s) \, ds. \tag{3.3} \]

(2) If \( f \geq 0 \) satisfies

\[ |(Lf, A)| \leq C(1 + (f, A)) \] (H2)
for some constant \( C \), and \((f, A_0)\) is integrable, then \((f, A_t)\) and \( M^f_t \) in (3.1) are also integrable with \( \mathbb{E} M^f_t = 0 \).

Eq. (3.1) can be analyzed through the eigenvalue problem for the operator \( L \) given in (3.2).

**Theorem 3.2.** Let \( L \) be the operator in (3.2). Then the equation

\[
Lq = rq
\]

has a solution \( q_r \) for any \( r \). The corresponding eigenfunction (normed so that \( q(0) = 1 \)) is given by

\[
q_r(u) = e^{ru} S(u) \left( 1 - \int_0^u e^{-rs} \{ m(s) g(s) + b(s) S(s) \} \, ds \right).
\]

**Proof.** Since eigenfunctions are determined up to a multiplicative constant, we can take \( q(0) = 1 \). Eq. (3.4) is a first-order linear differential equation, and solving it we obtain solution (3.5).

**Theorem 3.3.** Let \( q_r \) be a positive eigenfunction of \( L \) corresponding to the eigenvalue \( r \). Then \( Q_r(t) = e^{-rt}(q_r, A_t) \) is a positive martingale.

**Proof.** Using (3.1) and the fact that \( q_r \) is an eigenfunction for \( L \), we have

\[
(q_r, A_t) = (q_r, A_0) + r \int_0^t (q_r, A_s) \, ds + M^{q_r}_t,
\]

where \( M^{q_r}_t \) is a local martingale. The functions \( q_r \) clearly satisfy condition (H2). Therefore \((q_r, A_t)\) is integrable, and it follows from (3.6) by taking expectations that

\[
\mathbb{E}(q_r, A_t) = e^{rt} \mathbb{E}(q_r, A_0).
\]

Using integration by parts for \( e^{-rt}(q_r, A_t) \), we obtain from (3.6) that

\[
dQ_r(t) = d(e^{-rt}(q_r, A_t)) = e^{-rt} \, dM^{q_r}_t
\]

and

\[
Q_r(t) = (q_r, A_0) + \int_0^t e^{-rts} \, dM^{q_r}_t
\]

is a local martingale as an integral with respect to the local martingale \( M^{q_r}_t \). Since a positive local martingale is a super-martingale, and \( Q_r(t) \geq 0 \), \( Q_r(t) \) is a super-martingale. But from (3.7) it follows that \( Q_r(t) \) has a constant mean. Thus the super-martingale \( Q_r(t) \) is a martingale.

The Malthusian parameter \( \alpha \) is defined as the value of \( r \) which satisfies

\[
\int_0^\infty e^{-ru} \{ m(u) g(u) + b(u) S(u) \} \, du = 1.
\]

In what follows, we make the assumption that the Malthusian parameter \( \alpha \) exists and is positive, in other words that the limiting process is supercritical. The natural
condition for this being the case is that
\[ \int_0^\infty \{ m(u)g(u) + b(u)S(u) \} \, du > 1. \]

**Theorem 3.4.** Provided \( b \) and \( m \) are bounded, there is only one bounded positive eigenfunction \( V \), the reproductive value function, corresponding to the eigenvalue \( \lambda \) which is the Malthusian parameter,
\[ V(u) = \frac{e^{au}}{S(u)} \int_u^\infty e^{-\lambda s} \{ m(s)g(s) + b(s)S(s) \} \, ds. \]  

**(3.10)**

**Proof.** It follows that for \( r > \lambda \), \( \int_0^\infty e^{-ru} \{ m(u)g(u) + b(u)S(u) \} \, du < 1 \) and the eigenfunction \( q_r \) in (3.5) is positive and grows exponentially fast or faster. For \( r < \lambda \), \( \int_0^\infty e^{-ru} \{ m(u)g(u) + b(u)S(u) \} \, du > 1 \) and the eigenfunction \( q_r \) in (3.5) takes negative values. When \( r = \lambda \), \( q^* = V \) in (3.10) is the eigenfunction. To see that it is bounded, write
\[ V(u) = \frac{e^{au}}{S(u)} \int_u^\infty e^{-\lambda s} m(s)g(s) \, ds + \frac{e^{au}}{S(u)} \int_u^\infty e^{-\lambda s} b(s)S(s) \, ds \leq m^* + b^*, \]  

**(3.11)**

where \( m^* = \sup m(s) \) and \( b^* = \sup b(s) \). Replace \( e^{-\lambda s} \) by its largest value \( e^{-au} \) in the first integral to see that it does not exceed \( m^* \) and replace \( b(s) \) by \( b^* \) and \( S(s) \) by \( S(u) \) in the second integral to see that it does not exceed \( b^* \). \( \square \)

Taking \( r = \lambda \), we obtain an important corollary:

**Theorem 3.5.** Let \( V \) be the reproductive value function. Then \( W_t = e^{-at}(V, A_t) \) is a positive martingale, which converges almost surely to a limit \( W \geq 0 \).

By the martingale convergence theorem a positive martingale converges almost surely to a non-negative limit, but the limit \( W \) may be degenerate, \( P(W > 0) = 0 \). However, under additional assumptions \( W \) is non-degenerate.

**Theorem 3.6.** Assume that \( \sigma^2(a) = \Var(Y(a)) < \infty \) and for some \( r, \lambda \leq r \leq 2\lambda \), and some constant \( C \)
\[ b(a) + (\sigma^2(a) + (V(a) - m(a))^2)h(a) \leq Cq_r(a). \]  

**(3.12)**

(which holds in particular when \( b, h \) and \( m, \sigma^2 \) are bounded). Then \( W_t \) is a square-integrable martingale, and therefore converges almost surely and in \( L^2 \) to the non-degenerate limit.

**Proof.** It follows from (3.3) that
\[ \langle M^V, M^V \rangle_t = \int_0^t (b + (\sigma^2 + (m - V)^2)h, A_t) \, ds. \]  

**(3.13)**

Since (see (3.8))
\[ W_t = (V, A_0) + \int_0^t e^{-\lambda s} \, dM^V_s, \]  

**(3.14)**
we obtain that
\[
\langle W_t, W_t \rangle_t = \int_0^t e^{-2s} d\langle M^V, M^V \rangle_s = \int_0^t e^{-2s}(b + (\sigma^2 + (m - V)^2)h_s)ds.
\]
(3.15)

It follows by assumption (3.12) that
\[
(b + (\sigma^2 + (m - V)^2)h_s) \leq C(q_r, A_s)
\]
(3.16)
and by Theorem 3.3 that
\[
\mathbb{E} \int_0^\infty e^{-2s}(b + (\sigma^2 + (m - V)^2)h_s)ds \leq C \int_0^\infty e^{-2s}\mathbb{E}(q_r, A_s)ds
\]
\[
< C \int_0^\infty e^{(r - 2z)s} < \infty,
\]
(3.17)
where for the last inequality the assumption \(r < 2\sigma\) was used. This implies from (3.15) that \(\mathbb{E}\langle W_t, W_t \rangle_\infty < \infty\). Therefore, \(W_t\) is a square-integrable martingale, e.g. Protter (1992), Klebaner (1998), and the assertion follows. 

**Remark.** For Bellman–Harris processes the martingale \(\{W_t\}\) was given in Harris (1963) and Athreya and Ney (1972). For the processes we consider, with a Markovian age structure, it appeared in Jagers (1975, p. 213). It is the conditional expectation, given the age chart, of Nerman’s (1981) martingale intrinsic in general (Crump–Mode–Jagers) branching processes.

### 4. Processes with stabilizing reproduction

Consider now population-size-dependent and age-dependent branching processes.

Assume that reproduction stabilizes as the population size becomes large, that is, \(m_A \to m, h_A \to h, b_A \to b\), as \(z = (1, A) \to \infty\).

Then the operator \(LA\) in (2.7) can be represented as
\[
LAf = Lf + DAf,
\]
(4.1)
where
\[
DAf = LAf - Lf = m(h_A - h)f(0) + (m_A - m)h_Af(0)
\]
\[
+ (h - h_A)f + (b_A - b)f(0),
\]
(4.2)
so that all the terms are small when \(z \to \infty\).

Suppose now that the limiting operator \(L\) has the eigenfunction \(V\) given in (3.10) with the corresponding eigenvalue \(x\). Then it follows from Theorem 2.2 that
\[
(V, A_t) = (V, A_0) + x \int_0^t (V, A_s)ds + \int_0^t (D_A, V, A_s)ds + M^V_t,
\]
(4.3)
where \(M^V_t\) is a local martingale with the sharp bracket given by
\[
\langle M^V, M^V \rangle_t = \int_0^t (b_A + h_A(V^2 + v_A^2 - 2m_A V), A_s)ds.
\]
(4.4)
In fact, it is easy to see that, under condition (A1) given below (coefficients of $L_A$ and $L$ are bounded), condition (H1) of Theorem 2.3 is satisfied, and $M_t^V$ is integrable with zero mean. Let

$$W_t = e^{-2t}(V, A_t).$$

Using integration by parts for $e^{-2t}(V, A_t)$, we obtain

$$W_t = W_0 + \int_0^t e^{-2s}(D_A V, A_s) \, ds + N_t,$$

where the local martingale $N_t$ is given by

$$N_t = \int_0^t e^{-2s} \, dM_s^V.$$  \hfill (4.5)

We show below, in a way similar to the case of a Markov branching process as in Klebaner (1994), that if the convergence to the limit of $L_A$ to $L$ (or $D_A$ to zero) is fast enough, see Assumption (A3), then $W_t$ converges almost surely (and under an additional assumption on second moments also in $L^2$) as $t \to \infty$ to a limit.

**Assumptions.**

(A1) The functions $b, b_A, m, m_A, h, h_A$ are bounded from above.

(A2) $V$ is bounded from below, $V \geq c > 0$.

(A3) There is a function $\alpha$ such that $\sum_1^\infty \alpha(z)/z < \infty$ and

$$\sup_u |m_A(u) - m(u)| + \sup_u |h_A(u) - h(u)| + \sup_u |b_A(u) - b(u)| \leq \alpha(z), \quad z = (1, 4).$$

Of course, (A1) follows from the blanket assumption that the functions $b_A, m_A$ and $h_A$ are bounded, and (A3) implies (A1).

**Theorem 4.1.** Assume that (A1)–(A3) hold. Then

1. $w(t) = EW_t$ has a limit as $t \to \infty$ and this limit is positive, provided $z_0$ is large enough.
2. $W_t$ converges almost surely to a limit $W \geq 0$.
3. If in addition $\nu_2$ is bounded, then $W_t$ converges also in $L^2$, and the limit $W$ is nondegenerate. Moreover, $P(z_t \to \infty) > 0$, and $\log z_t/t$ converges almost surely to $z$ on the set $\{z_t \to \infty\}$.

The proof uses the following lemmas.

**Lemma 4.1.** Let $f(x), g(x)$ and $a(x)$ be non-negative functions defined for $x \geq 0$, such that $f(x)$ is non-increasing and continuous, $xf(x)$ non-decreasing, $\int_1^\infty (f(x)/x + g(x)) \, dx < \infty$. Suppose that $a(x)$ is differentiable, and satisfies for some $r > 0$ and all $x \geq 0$

$$|a'(x)| \leq a(x)f(a(x)e^x) + g(x).$$  \hfill (4.7)

Then $\lim_{x \to \infty} a(x) = a$ exists and $a > 0$ if $a(0)$ is large enough.
A particular case (when \( g(x) = 0 \)) is given in Klebaner (1994).

**Proof.** Let \( B = \int_0^\infty (f(e^x) + g(x)) \, dx \). It is clear that \( 0 < B < \infty \). Suppose first that there is a value \( T \geq 0 \), such that \( a(T) > e^a \). Then it is easy to see that for all \( x \geq T \), \( a(x) > 1 \). Indeed, if \( \tau = \inf \{ x > T : a(x) = 1 \} \), then for \( T \leq x < \tau \), \( a(x) > 1 \). It then follows from (4.7) that

\[
|a'(x)| \leq a(x) f(e^x) + g(x) \tag{4.8}
\]

and that (using \( g(x)/a(x) \leq g(x) \), since \( a(x) \geq 1 \))

\[
|\log a(x)'| \leq f(e^x) + g(x) \tag{4.9}
\]

Thus, we obtain that for all \( x \in [T, \tau] \), \( \log a(T) - \int_T^x (f(e^y) + g(y)) \, dy \leq \log a(x) \). By taking \( x = \tau \) a contradiction is obtained unless \( \tau = \infty \). Thus, the inequality in (4.9) holds for all \( x \geq T \). This together with the convergence of the integral implies that \( |\log a(x+y) - \log a(x)| \leq \int_x^\infty (f(e^y) + g(y)) \, dy \to 0 \) as \( x \to \infty \) uniformly in \( y > 0 \), which implies the existence of the \( \lim_{x \to \infty} \log a(x) = a \). Since \( a(x) \geq 1 \) for all \( x > T \), \( a(x) \geq 1 \).

The case where for all \( x \geq 0 \), \( a(x) \leq B' = e^a \) remains to be considered. Since \( x f(x) \) is nondecreasing, it follows from (4.7) that \( |a'(x)| \leq B' f(e^x) + g(x) \). This bound implies that \( \lim_{x \to \infty} \log a(x) = a \) exists and that \( a > 0 \) if \( a(0) \) is large enough. \( \square \)

The following Lemma 4.2 can be found in Klebaner (1994,1989), where its proof is given.

**Lemma 4.2.** Let \( \delta(x) > 0 \), defined for \( x \geq 0 \), be such that \( \delta(x)/x \) is non-increasing, \( \lim_{x \to \infty} \delta(x)/x = 0 \) and \( \int_1^\infty \delta(x)/x^2 \, dx < \infty \). Then there exists a \( \tilde{\delta}(x) \geq \delta(x) \), such that \( \tilde{\delta}(x)/x \) is non-increasing, \( \lim_{x \to \infty} \tilde{\delta}(x)/x = 0 \), \( \tilde{\delta}(x) \) is non-decreasing, concave and \( \int_1^\infty \tilde{\delta}(x)/x^2 \, dx < \infty \).

Now we are in a position to prove Theorem 4.1.

**Proof.** In what follows, \( C \) stands for a positive constant that may be different in different formulae.

1. We have from (4.5) that

\[
w'(t) = e^{-At} E(D_A V, A_t). \tag{4.10}
\]

It is easy to see that under the stated assumptions the function \( D_A V \) satisfies

\[
|D_A V| \leq C(|m_A - m| + |h_A - h| + |b_A - b|) \leq Cm(z). \tag{4.11}
\]

Therefore,

\[
|(D_A V, A_t)| \leq Cz_t m(z_t). \tag{4.12}
\]

By Lemma 4.2 we can replace the function \( \delta(x) = xz(x) \) by a dominating concave non-decreasing function \( \tilde{\delta}(x) \) that satisfies \( \int_1^\infty \tilde{\delta}(x)/x^2 \, dx < \infty \). By Jensen’s inequality

\[
\mathbb{E}|(D_A V, A_t)| \leq C \tilde{\delta}(z_t) \leq C \tilde{\delta}(z) \leq C \tilde{\delta}(\mathbb{E} z) \leq C \tilde{\delta}(e^{At} w(t)/c), \tag{4.13}
\]

where \( C \) is a constant. This completes the proof of (4.1).
where the lower bound \( V \geq c > 0 \) was used to obtain \( (V, A_t) \geq cz_t \). Thus we have from (4.10) that
\[
|w'(t)| \leq e^{-zt} E[(D_{v}, V, A_t)] \leq e^{-zt} C \hat{\delta}(e^{zt} w(t)/c) = Cw(t) \hat{\delta}(e^{zt} w(t)/c),
\]
where \( \hat{\delta}(x) = \hat{\delta}(x)/x \). Since \( \int_0^\infty \hat{\delta}(x)/x^2 \, dx < \infty \), \( \hat{\delta}(x) \) satisfies (A3), \( \int_1^\infty \hat{\delta}(x)/x \, dx < \infty \).

By Lemma 4.1 we have convergence of \( w(t) \), and \( \lim_{t \to \infty} w(t) = 0 \) provided \( w(0) \) is large enough. But \( w(0) = (V, A_0) = \sum_{i=1}^\infty V(a_i') \geq cz_0 \) and the statement is proved.

2. By the estimate in (4.13) it follows that
\[
E \int_0^\infty e^{-zt} |(D_{v}, V, A_t)| \, dt \leq C \int_0^\infty e^{-zt} \hat{\delta}(e^{zt} w(t)/c) \, dt
\]
\[
< C \int_0^\infty e^{-zt} \hat{\delta}(C e^{zt}) \, dt < \infty,
\]
where the last inequality holds since \( \lim w(t) > 0 \), \( \hat{\delta} \) is increasing and \( \int_1^\infty \hat{\delta}(x)/x^2 \, dx < \infty \). This implies (by using completeness of \( L_1 \)) that \( \int_0^t e^{-zs}(D_{v}, V, A_s) \, ds \) converges in \( L_1 \) to \( \int_0^\infty e^{-zs}(D_{v}, V, A_s) \, ds \), and also that \( Y := \int_0^\infty e^{-zs} |(D_{v}, V, A_s)| \, dt < \infty \) almost surely. Thus, the integral \( \int_0^t e^{-zs}(D_{v}, V, A_s) \, ds \) converges absolutely almost surely as \( t \to \infty \), which implies its almost sure convergence.

It follows now from (4.5), since \( W_t \geq 0 \), that the local martingale \( N_t \) in (4.6) is bounded below by the integrable random variable \( -W_0 - Y \). It is well known that this implies that \( N_t \) is a supermartingale. But supermartingales bounded from below by an integrable random variable converge almost surely. The almost sure convergence of \( W_t \) now follows from (4.5).

3. We show first that if \( v^2_t \) is bounded, then \( N_t \) is a square-integrable martingale. Using (4.4) we have
\[
\langle N, N \rangle_t = \int_0^t e^{-2s} \, d\langle M^V, M^V \rangle_s = \int_0^t e^{-2s} (b_{t, d} + h_{t, d}(V^2 + v^2_{t, d} - 2m_{t, d} V), A_s) \, ds.
\]
Under the imposed assumptions \( b_{t, d} + h_{t, d}(V^2 + v^2_{t, d} - 2m_{t, d} V) \leq C \), hence
\[
\langle N, N \rangle_t \leq C \int_0^t e^{-2s} (1, A_s) \, ds \leq (C/c) \int_0^t e^{-2s} (V, A_s) \, ds = (C/c) \int_0^t e^{-2s} W_s \, ds,
\]
where the last inequality is by (A2), \( V \geq c \). Thus,
\[
E\langle N, N \rangle_t \leq C \int_0^t e^{-2s} w(s) \, ds,
\]
where \( w(s) = E W_s \). Since \( w(s) \) converges (statement 1 of the Theorem), it follows from (4.18) that
\[
E\langle N, N \rangle_\infty \leq C \int_0^\infty e^{-2s} w(s) \, ds < \infty,
\]
and \( N_t \) is a square-integrable martingale. (4.19) implies that (e.g. Klebaner, 1998)
\[
E[N, N]_t = E\langle N, N \rangle_t < E\langle N, N \rangle_\infty = E[N, N]_\infty < \infty.
\]
Note also that from (4.5) it follows that \([W, W]_t = [N, N]_t\), and therefore \(\mathbb{E}[W, W]_t < \mathbb{E}[W, W]_\infty < \infty\).

We show next that \(W_t\) has finite second moments. Consider the equation for \(W_t^2\), which we obtain from (4.5) by Itô’s formula
\[
W_t^2 = W_0^2 + 2 \int_0^t e^{-x_s} W_s(D_A V, A_s) ds + 2 \int_0^t W_{s-} dN_s + [W, W]_t.
\]
(4.20)

Let \(T_n = \inf \{t: W_t > n\}\). Then clearly \(\mathbb{E}W_{t\wedge T_n}^2 < n^2 + C\), due to the assumption that the jumps have bounded second moments, \(\mathbb{E}Y_A^2 < C\). Using \(T_n\) as a localising sequence in (4.20) we can see that \(\mathbb{E}W_t^2 < 1\).

The second-last term above is a martingale, so by taking expectations and using (4.12) and (4.18), we obtain
\[
\mathbb{E}W_{t\wedge T_n}^2 \leq \mathbb{E}W_0^2 + C \int_0^t \mathbb{E}W_{s\wedge T_n}^2 ds + C \int_0^t e^{-x_s} w(s) ds.
\]
(4.22)

Now by Gronwall’s inequality we have
\[
\mathbb{E}W_{t\wedge T_n}^2 \leq \mathbb{E}W_0^2 + C \int_0^t e^{C(t-s)} \int_0^s e^{-uw} u du ds.
\]
(4.23)

Taking the limit as \(n \to \infty\) establishes that \(\mathbb{E}W_t^2 < \infty\).

Finally, we show that \(\lim_{t \to \infty} \mathbb{E}W_t^2\) exists. We can now deduce from Eq. (4.20) that \(\mathbb{E}W_t^2\) is differentiable, and that its derivative satisfies conditions of Lemma 4.1 (again by using a dominating concave function). Therefore \(\lim_{t \to \infty} \mathbb{E}W_t^2\) exists. The \(L^2\) convergence of \(W_t\) now follows. \(L^2\) convergence implies that \(\mathbb{E}W = \lim_{t \to \infty} \mathbb{E}W_t > 0\), hence \(P(W > 0) > 0\). Since by assumption \(V \geq c, Cz_t > (V, A_t) > cz_t\), the rest of the statement follows from this and convergence of \(W_t\) to the non-degenerate limit.

**Remark.** Convergence of a suitably normed process follows directly from the martingale property of \(W_t\) for simple branching models, such as Markov branching processes (in this case the function \(V\) is a constant and the \((V, A_t)\) is proportional to the population size). However, in Bellman–Harris and general branching processes it is not straightforward to obtain such convergence by martingale methods. For such methods see Athreya and Ney (1972), and Schuh (1982) for Bellman–Harris processes, and Nerman (1981) for general processes. Note here that direct analysis of the stochastic equation (2.6) for \(z_t = (1, A_t)\) does not seem to yield the convergence of \(e^{-zt}z_t\), and this is achieved by a different method in the next section.

5. Asymptotics of the population size

The last section established that \((V, A_t) \sim e^{-zt}W\). From the point of view of general branching processes (Jagers, 1975) this process is obtained from measuring the
population in one of many possible ways: at any time \( t \) an individual, born into the population, is counted if she is still alive, and her weight is \( V \), evaluated at her age. This is a particular random characteristic (Jagers, p. 167 ff., for general multi-type processes cf. Jagers, 1989).

In order to proceed to other characteristics and more natural population sizes, like the number of individuals alive, \( z_t = (1, A_t) \), we shall have to rely upon the traditional population tree definition of branching populations. Thus we quickly review it, in the single-type case.

With any individual \( x \in I \) in a Ulam–Harris family tree space

\[
I := \bigcup_{n=0}^{\infty} N^n, \quad N^0 := \{0\}, \quad N = \{1, 2, 3, \ldots\}
\]

there is associated a reproduction point process indicating the ages at which \( x \) begets children. Those are numbered \( x_1, x_2, \) etc. according to the Ulam–Harris convention. The population starts from Eve = 0 at time zero (or from another conventional set of ancestors though the family tree space has then to be trivially modified), and the birth-times \( \tau_x \) are then recursively defined (as \( x \)'s mother’s birth time plus her age at \( x \)'s birth, \( \tau_x = \infty \) meaning that \( x \) is never born).

The basic process is the total number of births by \( t \geq 0 \),

\[
y(t) := \#\{x \in I; \tau_x \leq t\}.
\]

A host of other “population sizes” can now be defined through the mentioned additive functionals called random characteristics: a random characteristic \( \chi := \{\chi_x; x \in I\} \) is a set of \( D \)-valued, stochastic processes \( \chi_x(u) \) vanishing for negative arguments, and measurable with respect to the \( \sigma \)-algebra generated by the complete life of \( x \) and all her progeny, i.e., \( x \)'s daughter process. We shall assume throughout that the populations are non-explosive in the sense that only a finite number of births can occur in finite time, and also that characteristics are bounded.

The \( \chi \)-counted or -weighted population size at \( t \) is defined as the sum of all \( \chi \) values of those born, evaluated at their actual ages \( t - \tau_x \) now at time \( t \),

\[
z^{\chi}_t := \sum_{x \in I} \chi_x(t - \tau_x) = \sum_{\tau_x \leq t} \chi_x(t - \tau_x).
\]

Clearly,

\[
y(t) = z^{1[x]}_t
\]

and if \( x \in I \) has a life span \( \lambda_x \), and we allow ourselves to write \( A := \{1_{[0, \lambda_x]}, x \in I\} \), then

\[
z_t = z^A_t = \sum_{x \in I} 1_{[0, \lambda_x]}(t - \tau_x)
\]

and \( (f, A_t) = z^{fA}_t \) in the obvious notation

\[
z^{fA}_t = \sum_{x \in I} f(t - \tau_x) 1_{[0, \lambda_x]}(t - \tau_x).
\]

Thus in the symbols of the tree formulation, Theorem 4.1 says that \( e^{-ut}z^{fA}_t \to W \) a.s. and in \( L^2 \). The question is what other characteristics we can have instead of \( VA \) (and what changes in \( W \) we are then lead to).
For any \( x \in I \) let \( \mathcal{B}_x \) denote the \( \sigma \)-algebra generated by the complete lives of all individuals not stemming from \( x \) (with the convention that \( x \) stems from herself).

**Lemma 5.1.** Let \( \chi = \{ \chi_x \} \) be a characteristic such that \( \mathbb{E}[\chi_x(a)|\mathcal{B}_x] = 0 \) for any \( x \in I \) and \( a \geq 0 \). Then
\[
\mathbb{E}(\chi_x^2) = \mathbb{E}[\chi_x^2].
\]

**Proof.** If \( x \neq x' \), then one of the two is not in the daughter process of the other. Say that \( x' \) does not stem from \( x \). Then
\[
\mathbb{E}[\chi_x(t - \tau_x)\chi_x(t - \tau_x)] = \mathbb{E}[\chi_x(t - \tau_x)\mathbb{E}[\chi_x(t - \tau_x)|\mathcal{B}_x]] = 0,
\]
since \( \tau_x \) is measurable with respect to \( \mathcal{B}_x \). Hence,
\[
\mathbb{E}(\chi_x^2) = \mathbb{E}\left[\sum_x \chi_x(t - \tau_x)^2\right] = \mathbb{E}\left[\sum_x \chi_x^2(t - \tau_x)\right] = \mathbb{E}[\chi_x^2].
\]

**Corollary 5.1.** For any bounded characteristic \( \chi \) write \( \mathbb{E}[\chi|\mathcal{B}] \) for the characteristic \( \{\mathbb{E}[\chi_x|\mathcal{B}_x]\} \). Then,
\[
\mathbb{E}(\chi_x^2 - \chi_x^2) = \mathbb{E}[\chi_x^{(\chi_x|\mathcal{B})}]^2.
\]

In particular, if \( S \) denotes the characteristic \( S_x(u) = \mathbb{P}(\lambda_u > u|\mathcal{B}_x) \), then
\[
\mathbb{E}(\gamma_x^2 - \gamma_x^2) = \mathbb{E}[\gamma_x^{(\gamma_x|\mathcal{B})}]^2.
\]

**Theorem 5.1.** Assume Conditions (A1)–(A3), and also that \( v^2 \) is bounded, so that
\[
w_x^{VA} := W_x = e^{-\gamma_x^{VA}} \rightarrow W,
\]
a.s. and in \( L^2 \), as \( t \to \infty \). Then, for some constant \( C \),
\[
\mathbb{E}(w_x^{VA} - w_x^{VS})^2 \leq Ce^{-\gamma t}
\]
and thus
\[
w_x^{VS} := e^{-\gamma_x^{VS}} \rightarrow W
\]
in mean square.

**Proof.** Since \( \sup V < \infty \) by (A1) and Theorem 3.4, Corollary 5.1 yields
\[
\mathbb{E}(w_x^{VA} - w_x^{VS})^2 = e^{-2\gamma_x^{VA}}\mathbb{E}[\gamma_x^{(\gamma_x|\mathcal{B})}] \leq \sup V e^{-2\gamma}\mathbb{E}[\gamma_x^{(\gamma_x|\mathcal{B})}].
\]
But
\[
\mathbb{E}[\gamma_x^{(\gamma_x|\mathcal{B})}] = \mathbb{E}\left[\sum_x V(t - \tau_x)\mathbb{E}[(1_{[0,\lambda_x]}(t - \tau_x) - S_x(t - \tau_x))^2|\mathcal{B}_x]\right]
\]
\[
= \mathbb{E}\left[\sum_x V(t - \tau_x)(1 - S_x(t - \tau_x))S_x(t - \tau_x)\right].
\]
for some $C < \infty$, by Theorem 4.1(1). Hence,

$$
E[(w_i^{FA} - w_i^{FS})^2] \leq Ce^{-2t}.
$$

Since by Theorem 4.1 $W_t = w_i^{FA}$ → $W$ a.s. and in $L^2$, we can conclude the same in $L^2$ for $w_i^{FS}$, as $t \to \infty$. □

In order to proceed to a.s. convergence, note that

$$
\int_0^\infty E[(w_i^{FA} - w_i^{FS})^2] dt \leq K \int_0^\infty e^{-2t} dt < \infty.
$$

Then, use Fubini’s theorem to see that

$$
\int_0^\infty (w_i^{FA} - w_i^{FS})^2 dt < \infty
$$
a.s.

Since $w_i^{FA} \to W$ a.s., it seems plausible that if $w_i^{FS}$ does not oscillate too wildly, then the convergence of the integral implies the a.s. convergence of $w_i^{FS}$. Indeed, make the assumption that asymptotically reproduction decreases slowly:

(A4) There are constants $a, u_0 > 0$ such that the reproduction intensity $\phi(t + u) \geq (1 - au) \phi(t)$ for all $t \geq 0$ and all $0 \leq u \leq u_0$.

**Theorem 5.2.** If life spans are not affected by the total population, so that $S_x$ is the same function $S$ for all $x$ and all outcomes, and (A4) holds besides the conditions of the preceding theorem, then $w_i^{FS} \to W$ also a.s.

**Proof.** Write

$$
w_i^{FS} = \int_0^t e^{\phi(t-u)} V(t-u) S(t-u) e^{-2u} y(du)
$$

and insert

$$
e^{-2t} V(t) S(t) = \int_t^\infty e^{-2u} \phi(u) du
$$

into this. Assumption (A4) leads to

$$
w_i^{FS} \geq (1 - cu) w_i^{FS}
$$

for a $c > 0$ and $0 \leq u \leq$ some $\gamma > 0$. Continue as in Harris (1963, p. 148), and assume that for some $\delta > 0 w_i^{FS} \geq (1 + \delta) W, t_{i+1} - t_i \geq \gamma$. Let $\varepsilon > 0$ and assume that
\(w^{VA}_t \leq (1 + \varepsilon)W\) already for \(t \geq t_1\). Then, provided only
\[
1 + \varepsilon < (1 - c\gamma)(1 + \delta),
\]
\[
\int_0^\tau (w^{VS}_{t+u} - w^{VA}_{t+u})^2 \, du \geq \int_0^\gamma ((1 - cu)(1 + \delta)W - (1 + \varepsilon)W)^2 \, du \geq \varepsilon'W^2 > 0
\]
for \(\varepsilon' = \gamma((1 - c\gamma)(1 + \delta) - 1 - \varepsilon)^2 > 0\), which can always be achieved by choice of \(\varepsilon, \gamma\) close to zero. It follows that
\[
\int_0^\infty (w^{VS}_t - w^{VA}_t)^2 \, dt \geq \sum_i \int_0^\gamma (w^{VS}_{t+u} - w^{VA}_{t+u})^2 \, du = \infty,
\]
provided \(W > 0\). Hence, under (A4) \(\limsup w^{VS}_t > W > 0\) implies divergence of the integral.

In the same vein we can mimic Harris's argument that \(\liminf w^{VS}_t\) cannot be strictly smaller than \(W\), and also show that \(w^{VS}_t \to 0\) on \(\{W = 0\}\). The theorem follows. \(\square\)

In convolution notation we have thus established that
\[
\int_0^\tau e^{-\tau(t-u)} V(t-u)S(t-u)e^{-2u} y(du) \to W
\]
a.s. and in mean square. We wish to proceed to conclusions about the growth of \(y(t)\), or about integrals where \(V^S\) has been replaced by other functions \(f\), and in a later step characteristics with conditional expectation equal or tending to some function \(f\). If we disregard the randomness of \(y\), this is the topic of Wiener's general Tauberian theorem (Wiener, 1933, or Widder, 1946, p. 214), specialized to \(\mathbb{R}_+\).

**Theorem 5.3.** Let \(g_1\) and \(g_2\) be continuous, directly Riemann integrable functions, and assume that \(g_1\)'s Fourier transform does not vanish. Let \(\mu\) be a measure on the positive half-line with \(\sup_{u} \mu((u,u + 1]) < \infty\). If further
\[
\lim_{t \to \infty} \int_0^\tau g_1(t-u) \mu(du) = A \int_0^\infty g_1(t) \, dt,
\]
then also
\[
\lim_{t \to \infty} \int_0^\tau g_2(t-u) \mu(du) = A \int_0^\infty g_2(t) \, dt.
\]

We write \(\hat{f}(z) = \int_0^\infty e^{-zt} f(t) \, dt\) for the Laplace transform of \(f\) in \(z\) and norm \(W\) to
\[
w = W\int_0^\infty e^{-zt} V(t)S(t) \, dt = W/\hat{V}S(z).
\]

**Corollary 5.2.** Under the conditions of Theorem 5.2,
\[
e^{-zt}\hat{f}(z)w
\]
a.s., provided only \(e^{-zt} f(t)\) is continuous and directly Riemann integrable. In particular, \(e^{-zt} y(t) \to w/\hat{S}(z)w\) and \(w_i^S = e^{-zt} z^S(t) \to \hat{S}(z)w\) a.s.
Proof. The integrand corresponding to \( g_1 \),

\[
e^{-zt}V(t)S(t) = \int_t^\infty e^{-zu} \phi(u) \, du
\]

is clearly continuous and directly Riemann integrable. Further,

\[
\int_0^\infty e^{zu} e^{-zu} V(u) S(u) \, du = \int_0^\infty e^{-zs} \phi(s) (e^{zs} - 1) \, ds
\]

for \( t \neq 0 \). Thus, the condition of a non-vanishing Fourier transform reduces to

\[
\int_0^\infty e^{ids} \phi(s) \, ds \neq 1, \quad t \neq 0,
\]

always satisfied, since \( e^{-zs} \phi(s) \) is a probability density, cf. Durrett (1996, p. 131).

Finally,

\[
\int_t^{t+1} e^{-zu} y(du) \leq \int_t^{t+1} \frac{e^{-s(t+1-u)} V(t+1-u) S(t+1-u) e^{-zu} y(du)}{e^{-s} V(1) S(1)}
\]

which is bounded by some (stochastic) constant, since

\[
\int_t^{t+1} e^{-s(t+1-u)} V(t+1-u) S(t+1-u) e^{-zu} y(du)
\]

converges a.s. as \( t \to \infty \). By Wiener’s theorem we can conclude that

\[
e^{-zt} z_t = \int_0^t e^{-s(t-u)} f(t-u) e^{-zu} y(du) \to \hat{f}(x)w\]

for functions \( f \) satisfying the relevant conditions, like the two special functions mentioned, \( f = 1 \) and \( f = S \), which clearly are continuous (the latter by the existence of an intensity \( h \)) and become directly Riemann integrable after multiplication by \( e^{-zt} \). \( \Box \)

In order to conclude convergence of the process of primary interest, \( w_t = e^{-zt} z_t = e^{-zt}(1, A_t) \), the normed natural population size, we first observe that \( w_t^S \leq w_t^S / c \), \( c \) being the minimal value of \( V \). From Theorem 5.1 and Corollary 5.2 we conclude that \( w_t^S \) must converge in mean square as well as almost surely (under the assumptions given).

Since

\[
E[(w_t^S - w_t)^2] = e^{-zt} E[w_t^S(S-A_t)^2] \leq Ce^{-zt},
\]

\( L^2 \)-convergence of \( w_t \) follows. Finally, Harris’s (1963, p. 148) already quoted argument yields a.s. convergence. Thus, we have:

**Corollary 5.3.** Under the conditions of Theorem 5.2,

\[
e^{-zt} z_t \to \hat{S}(x)w
\]

a.s. and in mean square.
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