On stationary solutions of delay differential equations driven by a Lévy process

Alexander A. Gushchin\textsuperscript{a}, Uwe Küchler\textsuperscript{b, *}

\textsuperscript{a}Steklov Mathematical Institute, Gubkina 8, 117966 Moscow GSP-1, Russia
\textsuperscript{b}Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany

Received 24 March 1999; received in revised form 4 October 1999; accepted 22 December 1999

Abstract

The stochastic delay differential equation
\[ dX(t) = \int_{[-r,0]} X(t + u) \, d\mu(u) \, dt + dZ(t), \quad t \geq 0 \]
is considered, where \( Z(t) \) is a process with independent stationary increments and \( \mu \) is a finite signed measure. We obtain necessary and sufficient conditions for the existence of a stationary solution to this equation in terms of \( \mu \) and the Lévy measure of \( Z \). © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Lévy processes; Processes of Ornstein–Uhlenbeck type; Stationary solution; Stochastic delay differential equations

1. Introduction

Let \( \mu \) be a finite signed measure on a finite interval \( J = [-r,0], \, r \geq 0 \). Consider the equation

\[ X(t) = \begin{cases} 
X(0) + \int_0^t X(s + u) \, d\mu(u) \, ds + Z(t), & t \geq 0, \\
X_0(t), & t \in J.
\end{cases} \]  
\hspace{1cm} (1.1)

Here \( Z = (Z(t), \, t \geq 0) \) is a real-valued process with independent stationary increments starting from 0 and having càdlàg trajectories, i.e. \( Z \) is a Lévy process, and \( X_0 = (X_0(t), \, t \in J) \) is an initial process with càdlàg trajectories, independent of \( Z \). The question treated in this note concerns the existence of stationary solutions to (1.1).

If \( r = 0 \), the answer to this question is known. The equation

\[ X(t) = X(0) + \rho \int_0^t X(s) \, ds + Z(t), \quad t \geq 0 \] 
\hspace{1cm} (1.2)
(X(0) and Z are independent) admits a stationary solution if and only if
\[
\rho < 0
\]  
and
\[
\int_{|y|>1} \log |y| F(dy) < \infty,
\]
where F denotes the Lévy measure of Z. This stationary solution X is called a stationary process of Ornstein–Uhlenbeck type. Its distribution is uniquely determined by \( \rho \) and the Lévy–Khintchine characteristics of Z, in particular, the law of \( X(t) \) is the distribution of
\[
U = \int_0^\infty e^{\rho t} dZ(t).
\]

In this paper we show that a stationary solution of (1.1) exists if and only if the equation
\[
h(\lambda) := \lambda - \int e^{iu} a(du) = 0
\]  
has no complex solutions \( \lambda \) with Re \( \lambda \geq 0 \), and condition (1.4) holds. Thus, in comparison with the Ornstein–Uhlenbeck case, condition (1.3) is replaced by
\[
\{ \lambda \in \mathbb{C} | h(\lambda) = 0, \ \text{Re} \ \lambda \geq 0 \} = \emptyset.
\]
The distribution of a stationary solution X is unique for given a and the characteristics of Z, and the law of \( X(t) \) is the distribution of
\[
U = \int_0^\infty x_0(t) dZ(t),
\]
where \( x_0(t) \) is the so-called fundamental solution of the corresponding (1.1) deterministic homogeneous equation (see the definition in Section 2). If Z is a Wiener process and a is concentrated in points 0 and r, these results were proved by Küchler and Mensch (1992).

As in the case of Eq. (1.2), a stationary solution of (1.1) exists if and only if the integral in (1.7) converges in an appropriate sense. But, unlike the Ornstein–Uhlenbeck case (where \( x_0(t) = e^{\rho t} \)), the fundamental solution \( x_0(t) \) is not necessarily a positive monotone function, for example, it may oscillate around 0 under (1.6), see Fig. 1. Thus, the proof of the necessity of (1.6) and (1.4) for the convergence of the integral in (1.7) is not so straightforward as in the case \( r = 0 \).

Stochastic differential equations of type (1.1) can be considered as linear stochastic differential equations in some Hilbert space \( \mathcal{H} \):
\[
dX_t = AX_t \, dt + dZ_t, \quad t \geq 0,
\]
where A is the infinitesimal generator of a strongly continuous semigroup \((T_t)_{t \geq 0}\) of bounded linear operators on \( \mathcal{H} \) and \((Z_t)_{t \geq 0}\) is an \( \mathcal{H} \)-valued Lévy process, see e.g.
Da Prato and Zabczyk (1992) for details. Chojnowska-Michalik (1987) studied the problem of the existence of stationary distributions for the solutions of (1.8) and obtained the sufficiency of conditions similar to (1.6) and (1.4). Under an additional assumption on the semigroup \((T_t)_{t \geq 0}\) ((\(T_t\) can be extended to a group on \(\mathbb{R}\)), which is not satisfied in our case, she proved also the necessity of these conditions.

The assumption that the initial process \(X_0\) and \(Z\) are independent is important for the above result. Otherwise, (1.6) is not necessary for the existence of a stationary solution, cf. Theorem 3.1 in Jacod (1985) and Theorem 20 in Mohammed and Scheutzow (1990).

2. Preliminaries

The aim of this section is twofold: to establish our notation and to recall some basic facts concerning Lévy processes and deterministic delay differential equations of the considered type.
2.1. Deterministic delay differential equations

Since Eq. (1.1) involves no stochastic integrals and is treated pathwise, we will formulate a number of results for solutions of Eq. (1.1) with deterministic \( Z \) and \( X_0 \), for which we refer to Hale and Verduyn Lunel (1993), Diekmann et al. (1995), Myschkis (1972), and also to Mohammed and Scheutzow (1990).

A real-valued function \( X(t) \), \( t \geq -r \), is called a solution of the equation (1.1), if it is locally integrable and satisfies (1.1) for all \( t \geq -r \) or only for \( t \geq 0 \) if the initial condition is not specified (here and below “integrable” means “integrable with respect to the Lebesgue measure”; the double integral in (1.1) exists for such functions by the Fubini theorem).

Assume that a finite signed measure \( a \) on \( J \), a real-valued locally integrable function \( Z \) on \( \mathbb{R}_+ \) satisfying \( Z(0) = 0 \), and a real-valued integrable function \( X_0 \) on \( J \) are given (only such \( a, Z, \) and \( X_0 \) will be considered in the sequel). Then Eq. (1.1) has a unique solution. This solution is càdlàg (resp. continuous, resp. absolutely continuous) on \( \mathbb{R}_+ \) if and only if \( Z \) is càdlàg (resp. continuous, resp. absolutely continuous).

Given a measure \( a \), we call a function \( x_0 : [-r, \infty[ \to \mathbb{R} \) the fundamental solution of the homogeneous equation

\[
X(t) = \begin{cases} 
X(0) + \int_0^t \int_J X(s + u) a(du) \, ds, & t \geq 0, \\
X_0(t), & t \in J,
\end{cases}
\]

if it is the solution of (2.1) corresponding to the initial condition

\[
X_0(t) = \begin{cases} 
1, & t = 0, \\
0, & -r \leq t < 0.
\end{cases}
\]

In other words, a function \( x_0(t), t \geq -r \), is the fundamental solution of (2.1) if it is absolutely continuous on \( \mathbb{R}_+ \), \( x_0(t) = 0 \) for \( t < 0 \), \( x_0(0) = 1 \), and

\[
x_0(t) = \int_J x_0(t + u) a(du)
\]

for Lebesgue-almost all \( t > 0 \). To facilitate some notation in the sequel it is convenient to put \( x_0(t) = 0 \) for \( t < -r \).

The solution of (1.1) can be represented via the fundamental solution \( x_0 \) of (2.1):

\[
X(t) = \begin{cases} 
x_0(t)X_0(0) + \int_J \int_0^t X_0(s)x_0(t + u - s) \, ds \, a(du) \\
\quad + \int_{[0,t]} Z(t-s) \, dx_0(s), & t \geq 0, \\
X_0(t), & t \in J,
\end{cases}
\]

Remark. The domain of integration in the last integral in (2.3) includes zero:

\[
\int_{[0,t]} Z(t-s) \, dx_0(s) = Z(t) + \int_{[0,t]} Z(t-s) \, dx_0(s).
\]
The asymptotic behaviour of solutions of Eqs. (1.1) and (2.1) for $t \to \infty$ is connected with the set of complex solutions of the so-called characteristic equation

$$h(\lambda) = 0,$$  

(2.4)

where the function $h(\cdot)$ is defined in (1.5). Note that a complex number $\lambda$ solves (2.4) if and only if $(e^{it}, t \geq -r)$ solves (2.1) for the initial condition $X_0(t) = e^{it}, t \in J$.

The set $A := \{ \lambda \in \mathbb{C} | h(\lambda) = 0 \}$ is not empty; moreover, it is infinite except the case where $a$ is concentrated at 0. Since $h(\cdot)$ is an entire function, $A$ consists of isolated points only. It is easy to check that $\lambda_n \in A$ and $|\lambda_n| \to \infty$ imply $\text{Re} \lambda_n \to -\infty$, thus the set $\{ \lambda \in A | \text{Re} \lambda \geq c \}$ is finite for every $c \in \mathbb{R}$. In particular,

$$v_0 := \max \left\{ \text{Re} \lambda : \lambda \in A \right\} < \infty$$  

(2.5)

holds. Define

$$v_{i+1} := \max \left\{ \text{Re} \lambda : \lambda \in A, \text{Re} \lambda < v_i \right\}, \quad i \geq 0.$$  

For $\lambda \in A$ denote by $m(\lambda)$ the multiplicity of $\lambda$ as a solution of (2.4).

It is easy to check from (2.2) that $1/h(\lambda)$ is the Laplace transform of $(x_0(t), t \geq 0)$ at least if $\text{Re} \lambda$ is large enough. (In fact,

$$1/h(\lambda) = \int_0^\infty e^{-\lambda t} x_0(t) \, dt$$

if $\text{Re} \lambda > v_0$.) Applying a standard method based on the inverse Laplace transform and Cauchy’s residue theorem, we come to the following lemma which is essentially known and can be found in a slightly different form in Hale and Verduyn Lunel (1993) and Diekmann et al. (1995). The proof will be sketched in Section 4.

**Lemma 2.1.** For any $c \in \mathbb{R}$ we have

$$x_0(t) = \sum_{i, v_i \geq c} \left[ \sum_{\lambda \in A \atop \lambda = v_i} p_i(t)e^{\lambda t} + \sum_{\lambda \in A \atop \text{Re} \lambda = v_i, \text{Im} \lambda > 0} \left\{ q_1(t) \cos(t \text{Im} \lambda) + r_1(t) \sin(t \text{Im} \lambda) \right\} e^{\lambda t} \right]$$

$$+ o(e^{ct}),$$

t $\to \infty$, where $p_i(t)$ is a real-valued polynomial in $t$ of degree $m(\lambda) - 1$, $q_1(t)$ and $r_1(t)$ are real-valued polynomials in $t$ of degree less than or equal to $m(\lambda) - 1$, and the degree of either $q_1(t)$ or $r_1(t)$ is equal to $m(\lambda) - 1$.

This lemma and the following corollary describe properties of the fundamental solution $x_0(t)$, which are crucial for the proof of our main result.

**Corollary 2.2.** For some $\delta > 0$,

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t 1\{|x_0(s)| \geq \delta e^{as}\} \, ds > 0.$$
2.2. Lévy processes

Let \( Z = (Z(t), t \geq 0) \) be a Lévy process. Throughout the paper a continuous truncation function \( g \) is fixed, i.e. \( g: \mathbb{R} \to \mathbb{R} \) is a bounded continuous function with compact support satisfying \( g(y) = y \) in a neighbourhood of 0.

It is well known, see e.g. Jacod and Shiryaev (1987), that the distribution of \( Z \) is completely characterized by a triple \((b, c, F)\) of the Lévy–Khintchine characteristics, namely, a number \( b \in \mathbb{R} \) (the drift), a nonnegative number \( c \in \mathbb{R}_+ \) (the variance of the Gaussian part), and a nonnegative \( \sigma \)-finite measure \( F \) on \( \mathbb{R} \) that satisfies \( F(\{0\}) = 0 \) and

\[
\int_{\mathbb{R}} (y^2 \wedge 1) F(dy) < \infty
\]

(2.6)

(the Lévy measure of jumps). In particular,

\[
E \exp\{iu(Z(t) - Z(s))\} = \exp\{(t-s)\psi_{b,c,F}(u)\}, \quad u \in \mathbb{R}, \quad s < t,
\]

where

\[
\psi_{b,c,F}(u) := iub - \frac{1}{2} u^2 c + \int_{\mathbb{R}} (e^{iuy} - 1 - iug(y)) F(dy).
\]

(2.7)

Moreover, this triple \((b, c, F)\) is unique, and, for every triple \((b, c, F)\) satisfying the above assumptions, there is a Lévy process \( Z \) with the characteristics \((b, c, F)\).

In the following, we shall deal with integrals of the form

\[
I_f(t) := \int_0^t f(s) dZ(s),
\]

where \( f: \mathbb{R}_+ \to \mathbb{R} \) is a càdlàg function of locally bounded variation. In this simple case there is no need to use an advanced theory of stochastic integration (however, let us mention that the results stated below are valid for at least locally bounded measurable \( f \)). Indeed, the integral \( I_f(t) \) can be defined by formal integration by parts:

\[
I_f(t) = f(t)Z(t) - \int_{[0,t]} Z(s-) df(s),
\]

(2.8)

where \( Z(s-) = \lim_{s' \downarrow s} Z(s') \). Of course, this pathwise definition is equivalent to the usual definitions of stochastic integrals.

The next lemma is a simple exercise. The first equality in its statement can be found e.g. in Lukacs (1969).

**Lemma 2.3.** The integral \( I_f(t) \) has an infinitely divisible distribution:

\[
E \exp\{iuI_f(t)\} = \exp\left\{ \int_0^t \psi_{b,c,F}(uf(s)) ds \right\} = \exp\{\psi_{B(t), C(t), F(t)}(u)\},
\]

where

\[
B(t) := b \int_0^t f(s) ds + \int_{\mathbb{R}_+} \int_0^t (g(yf(s)) - f(s)g(y)) ds F(dy),
\]

\[
C(t) := c \int_0^t f^2(s) ds.
\]
\( F(t; \{0\}) = 0, \quad \int_{\mathbb{R}} \chi(y) F(t; dy) = \int_{\mathbb{R}} \int_{0}^{t} \chi(yf(s)) ds \) \( F(dy) \) \tag{2.11}

for any nonnegative measurable function \( \chi \) satisfying \( \chi(0) = 0 \).

**Lemma 2.4.** \( I_f(t) \) converges in distribution as \( t \to \infty \) if and only if there exist finite limits

\[
B(\infty) := \lim_{t \to \infty} B(t), \quad C(\infty) := \lim_{t \to \infty} C(t)
\]

and

\[
\sup_{t} \int_{\mathbb{R}} (y^2 + 1) F(t; dy) < \infty.
\]

Moreover, in that case the limit \( \lim_{t \to \infty} I_f(t) =: \int_{0}^{\infty} f(s) dZ(s) \) exists almost surely and

\[
E \exp \left\{ iu \int_{0}^{\infty} f(s) dZ(s) \right\} = \exp \left\{ \lim_{t \to \infty} \int_{0}^{t} \psi_{b,c,F}(uf(s)) ds \right\} = \exp \left\{ \psi_{b(\infty),C(\infty),F(\infty)}(u) \right\},
\]

where \( F(\infty) \) is a \( \sigma \)-finite measure on \( \mathbb{R} \) such that

\[
\int_{\mathbb{R}} \chi(y) F(\infty; dy) = \sup_{t} \int_{\mathbb{R}} \chi(y) F(t; dy)
\]

for any nonnegative measurable function \( \chi \).

**Remark.** The assumptions of Lemma 2.4 do not imply the integrability of \( \psi_{b,c,F}(uf(s)) \) on \([0, \infty[\). Of course, if the Lebesgue integral \( \int_{0}^{\infty} \psi_{b,c,F}(uf(s)) ds \) exists, then

\[
E \exp \left\{ iu \int_{0}^{\infty} f(s) dZ(s) \right\} = \exp \left\{ \int_{0}^{\infty} \psi_{b,c,F}(uf(s)) ds \right\}.
\]

### 3. The main result

In this section we assume that there are a fixed finite signed measure \( a \) on \( J \) and a triple \((b,c,F)\) of the Lévy–Khintchine characteristics such that either \( c > 0 \) or \( F \neq 0 \). We say that a process \( X = (X(t), t \geq -r) \) is a solution to Eq. (1.1) if there are a Lévy process \( Z = (Z(t), t \geq 0) \) with the characteristics \((b,c,F)\) and a process \( X_0 = (X_0(t), t \in J) \) with càdlàg trajectories such that (1.1) holds; moreover \( Z \) and \( X_0 \) are assumed to be independent. In other words, a càdlàg stochastic process \( X = (X(t), t \geq -r) \) is a solution to (1.1) if

1. \( Z(t) = X(t) - X(0) - \int_{0}^{t} \int_{J} X(s + u) a(du) ds, \quad t \geq 0, \) is a Lévy process with the characteristics \((b,c,F)\);
2. the processes \( X = (X(t), t \in J) \) and \( Z = (Z(t), t \geq 0) \) are independent.

We say that a solution \( X = (X(t), t \geq -r) \) is a stationary solution to (1.1) if

\[
(X(t_k), k \leq n) \overset{d}{=} (X(t + t_k), k \leq n)
\]

for all \( t > 0, n \geq 1, t_1, \ldots, t_n \geq -r \).

Recall that \( x_0(\cdot) \) is the fundamental solution of Eq. (2.1) and \( t_0 \) is defined by (2.5).
Theorem 3.1. There is equivalence between:

(i) Eq. (1.1) admits a stationary solution;
(ii) there is a solution \( X \) of (1.1) such that \( X(t) \) has a limit distribution as \( t \to \infty \);
(iii) for any solution \( X \) of (1.1), \( X(t) \) has a limit distribution as \( t \to \infty \);
(iv) \( v_0 < 0 \) and \( \int_{|y|>1} \log |y| F(dy) < \infty \).

Moreover, in that case for an arbitrary solution \( X(t) \) of (1.1)

(v) the distribution of \( (X(t+t_k), k \leq n) \), where \( n \geq 1 \), \( 0 \leq t_1 < t_2 < \cdots < t_n \) are fixed, weakly converges as \( t \to \infty \) to the distribution of the vector

\[
\left( \int_{t_n-t_0}^{\infty} x_0(s + t_k - t_n) dZ(s), \; k \leq n \right),
\tag{3.2}
\]

where \( Z = (Z(s), \; s \geq 0) \) is a Lévy process with the characteristics \((b,c,F)\);

(vi) the distribution of the process \( (X(t+s), \; s \geq 0) \) weakly converges in the Skorokhod topology as \( t \to \infty \) to the distribution of a stationary solution \((Y(s), \; s \geq 0)\), which is uniquely determined due to (v).

Remark. (1) The integrals in (3.2) are defined in Lemma 2.4. The correctness of their definition will be shown in Lemma 4.3.

(2) It follows from the proof of Theorem 3.1 that, given a Lévy process \( Z \) with the characteristics \((b,c,F)\) on a probability space \((\Omega, \mathcal{F}, P)\), one can construct, under the condition (iv), a stationary solution on the same probability space if it is large enough, in particular, if there is another Lévy process on \((\Omega, \mathcal{F}, P)\) with the same characteristics independent of \( Z \).

4. Proofs

Proof of Lemma 2.1. According to Lemma I.5.1 and Theorem I.5.4 in Diekmann et al. (1995),

\[
x_0(t) = \sum_{\lambda \in A, \; \text{Re} \lambda \geq c} \text{Res}_{z=\lambda} \frac{e^{zt}}{h(z)} + o(e^{ct}), \quad t \to \infty.
\tag{4.1}
\]

Let \( \lambda \in A, \; \text{Re} \lambda \geq c \), and \( m := m(\lambda) \). Write Laurent’s series of \( 1/h(z) \) at \( z = \lambda \) in the form

\[
1/h(z) = \sum_{k=-m}^{\infty} A_k(\lambda)(z-\lambda)^k, \quad A_{-m}(\lambda) \neq 0.
\]

Since

\[
e^{zt} = e^{-ct} \sum_{k=0}^{\infty} \frac{t^k}{k!}(z-\lambda)^k,
\]

the multiplication of the above series yields

\[
\text{Res}_{z=\lambda} \frac{e^{zt}}{h(z)} = e^{-ct} \sum_{k=-m}^{\infty} \frac{A_k(\lambda)}{(-1-k)!} t^{1-k}.
\]
Note that \( h(\mathring{z}) = \overline{h(z)} \) (where a bar means the complex conjugate). Therefore, we have \( \lambda \in A \) if and only if \( \mathring{\lambda} \in A \). Moreover, it holds \( A_k(\lambda) = A_k(\mathring{\lambda}) \). Hence, if \( \text{Im} \, \lambda = 0 \), then \( A_k(\lambda) \in \mathbb{R} \) and \( p_k(t) = \sum_{k=-m}^{l-1} [A_k(\lambda)/(-1-k)!] t^{-1-k} \). If \( \text{Im} \, \lambda \neq 0 \), we join two terms in (4.1) corresponding to \( \lambda \) and \( \mathring{\lambda} \). After simple calculations we obtain (for definiteness, we assume that \( \text{Im} \, \lambda > 0 \))

\[
\text{Res}_{z=\lambda} \frac{e^{zt}}{h(z)} + \text{Res}_{z=\lambda} \frac{e^{zt}}{\overline{h(z)}} = \{q_\lambda(t) \cos(t \text{ Im} \, \lambda) + r_\lambda(t) \sin(t \text{ Im} \, \lambda)\} e^{t \text{ Re} \, \lambda},
\]

where

\[
q_\lambda(t) = 2 \sum_{k=-m}^{l-1} \frac{\text{Re} A_k(\lambda)}{(-1-k)!} t^{-1-k}, \quad r_\lambda(t) = -2 \sum_{k=-m}^{l-1} \frac{\text{Im} A_k(\lambda)}{(-1-k)!} t^{-1-k}. \quad \square
\]

**Proof of Corollary 2.2.** According to Lemma 2.1, it is enough to check that, for some \( \delta > 0 \),

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t 1(|f(s)| \geq \delta) \, ds > 0
\]

for a continuous function \( f(t) \) satisfying

\[
f(t) = p(t) + \sum_{j=1}^n \{q_j(t) \cos(\xi_j t) + r_j(t) \sin(\xi_j t)\} + o(1), \quad t \to \infty,
\]

where \( p(t), q_j(t), r_j(t), i = 1, \ldots, n \), are polynomials, not all of them being equal to zero identically, \( 0 < \xi_1 < \cdots < \xi_n \). Thus,

\[
f(t) = t^m \hat{f}(t) + o(t^m), \quad t \to \infty
\]

for some \( m \geq 0 \) and

\[
\hat{f}(t) = A_0 + \sum_{j=1}^n \{A_j \cos(\xi_j t) + B_j \sin(\xi_j t)\}, \quad \text{with} \quad M := |A_0| + \sum_{j=1}^n (|A_j| + |B_j|) > 0.
\]

Then

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t 1(|f(s)| \geq \hat{\delta}) \, ds \geq \liminf_{t \to \infty} \frac{1}{t} \int_0^t 1(|f(s)| \geq \delta^m) \, ds \geq \liminf_{t \to \infty} \frac{1}{t} \int_0^t 1(|\hat{f}(s)| \geq \hat{\delta}) \, ds
\]

for any \( \hat{\delta} > \delta \). Since

\[
\int_0^t 1(|\hat{f}(s)| \geq \hat{\delta}) \, ds \geq \frac{1}{M^2} \int_0^t \hat{f}^2(s) 1(|\hat{f}(s)| \geq \hat{\delta}) \, ds \geq \frac{1}{M^2} \int_0^t \hat{f}^2(s) \, ds - \frac{\delta^2}{M^2} t,
\]

we obtain

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t 1(|f(s)| \geq \delta) \, ds \geq \frac{1}{M^2} \left( \liminf_{t \to \infty} \frac{1}{t} \int_0^t \hat{f}^2(s) \, ds - \delta^2 \right) = \frac{1}{M^2} \left\{ A_0^2 + \frac{1}{2} \sum_{j=1}^n (A_j^2 + B_j^2) - \delta^2 \right\} > 0
\]

for \( \hat{\delta} \) small enough. \( \square \)
Proof of Lemma 2.4. According to the well-known conditions for the weak convergence of infinitely divisible distributions (see e.g. Remark VII.2.10 in Jacod and Shiryaev, 1987), \( I_f(t) \) converges in distribution as \( t \to \infty \) if and only if there is a finite limit \( \lim_{t \to \infty} B(t) \) and the measures \( \mu_0(dv) + (v^2 + 1)F(t, dv) \) weakly converge to a measure \( \mu_0(dv) + (v^2 + 1)\tilde{F}(dv) \) with \( \tilde{F}([0,\infty)) = 0 \), the limit distribution being infinitely divisible with the characteristics \( (B(\infty), \tilde{C}, \tilde{F}) \) (here \( \mu_0(\cdot) \) is the Dirac measure at \( 0 \)). In our case \( F(t) - F(s) \) is a nonnegative measure for all \( t > s \) due to (2.11). Therefore, the conditions just mentioned take place if and only if the conditions of the lemma are satisfied; moreover, \( \tilde{C} = C(\infty) \) and \( \tilde{F} = F(\infty) \). It remains to note that \( I_f(t) \) is a càdlàg process with independent increments, hence the convergence in distribution of \( I_f(t) \) as \( t \to \infty \) implies the convergence of \( I_f(t) \) almost surely as \( t \to 0 \).

Before proving Theorem 3.1 we need a number of preliminary lemmas. We keep the notation and the conventions of Section 2.

Lemma 4.1. Assume that \( v_0 < 0 \) and \( X(t) \) is a solution of (deterministic) Eq. (2.1). Then \( \lim_{t \to \infty} X(t) = 0 \).

Proof. According to (2.3),
\[
X(t) = x_0(t)X_0(0) + \int_0^t \int_u X_0(s)x_0(t + u - s)ds \, a(du), \quad t \geq 0.
\]
By Lemma 2.1, \( |x_0(t)| \leq ce^{-\gamma t}, \ t \geq 0, \) for some \( c > 0 \) and \( \gamma \) such that \( 0 < \gamma < |v_0| \), from which the claim follows easily.

Lemma 4.2. Let \( z : [0, T] \to \mathbb{R}, \ T \geq 0, \) be a càdlàg function. Put
\[
X(t) = x_0(t + T)z(T) - \int_{[0,T]} z(s-) \, dx_0(t + s), \quad t \geq -r.
\]
Then \( X(t), \ t \geq -r \) is a càdlàg solution of the homogeneous equation (2.1).

Remark. If \( z \) has a bounded variation and we put \( z(t) = 0 \) for \( t < 0 \), integration by parts gives
\[
X(t) = \int_{[0,T]} x_0(t + s) \, dz(s), \quad t \geq -r,
\]
i.e. \( X(\cdot) \) is a mixture of \( x_0(\cdot + s), \ s \in [0, T] \). Thus, the statement of the lemma is not surprising since every \( x_0(\cdot + s) \) is a solution of (2.1).

Proof. If \( z \) is a piecewise constant function, the claim follows immediately from the previous remark. For the general case, use a uniform approximation of \( z \) by piecewise constant functions.

Lemma 4.3. Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be a function of locally bounded variation such that \( |f(t)| \leq ce^{-\gamma t} \) for some \( c > 0 \) and \( \gamma > 0 \). If (1.4) holds, then \( I_f(t) \) has a limit distribution as \( t \to \infty \).
Proof. We will check the conditions of Lemma 2.4. First, in view of (2.10),

$$\lim_{t \to \infty} C(t) = \int_0^\infty f^2(s) \, ds < \infty. \tag{4.3}$$

Let us show that

$$\sup_t \int \limits_{\mathbb{R}} (y^2 \wedge 1) F(t; \, dy) < \infty. \tag{4.4}$$

Indeed, by (2.11),

$$\int \limits_{\mathbb{R}} (y^2 \wedge 1) F(t; \, dy)$$

$$= \int \limits_{\mathbb{R}} \int_0^t (y^2 f^2(s) \wedge 1) \, ds \, F(dy) \leq \int \limits_{\mathbb{R}} \int_0^\infty (c^2 y^2 e^{-2y} \wedge 1) \, ds \, F(dy)$$

$$= c^2 \int \limits_{|y| \leq c^{-1}} \int_0^\infty y^2 e^{-2y} \, ds \, F(dy) + c^2 \int \limits_{|y| > c^{-1}} \int_0^\infty y^2 e^{-2y} \, ds \, F(dy)$$

$$+ \int \limits_{|y| > c^{-1}} \int_0^\infty y^{-1} \log(c|y|) \, ds \, F(dy)$$

$$= (2\gamma^{-1}) c^2 \int \limits_{|y| \leq c^{-1}} y^2 F(dy) + \gamma^{-1} \int \limits_{|y| > c^{-1}} (\log c + \log |y| + \frac{1}{2}) F(dy).$$

The right-hand side of the previous inequality is finite in view of (2.6) and (1.4). In view of (2.9), in order to show that

$$B(t) = b\int \limits_0^\infty f(s) \, ds + \int \limits_{\mathbb{R}} \int_0^\infty \left\{ g(yf(s)) - f(s)g(y) \right\} \, ds \, F(dy), \quad t \to \infty, \tag{4.5}$$

it is enough to check that

$$\int \limits_\mathbb{R} \int_0^\infty |g(yf(s)) - f(s)g(y)| \, ds \, F(dy) < \infty. \tag{4.6}$$

Choose a $\kappa > 0$ such that $g(y) = y$ if $|y| \leq \kappa$. Without loss of generality assume that $c \geq 1$.

Since $|f(s)| \leq c$,

$$\int_0^\infty |g(yf(s)) - f(s)g(y)| \, ds = 0 \quad \text{if} \quad |y| \leq \kappa c^{-1}. \tag{4.7}$$

Let $|y| > \kappa c^{-1}$ and put $L = \max\{\sup_{y \in \mathbb{R}} |g(y)|, \kappa\}$. Then

$$\int_0^\infty |g(yf(s)) - f(s)g(y)| \, ds$$

$$\leq L \int_0^\infty |f(s)| \, ds + \int_0^\infty (|yf(s)| \leq \kappa) + L (|yf(s)| > \kappa) \, ds$$

$$\leq \gamma^{-1} Lc + \int_0^\infty (c |y| e^{-\gamma^{-1} \log(\kappa)} 1(s \geq \gamma^{-1} \log(\kappa^{-1}|y|))) + L (s < \gamma^{-1} \log(\kappa^{-1}|y|)) \, ds$$

$$= \gamma^{-1} (Lc + \kappa + L \log(\kappa^{-1}) + \log |y|). \tag{4.8}$$
Now (4.6) follows from (4.7), (4.8), (2.6), and (1.4), and the statement follows from (4.3)–(4.5) and Lemma 2.4.

Lemma 4.4. Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) be a locally bounded measurable function such that
\[
\int_{\mathbb{R}} \int_{0}^{\infty} (y^2 f^2(s) \land 1) \, ds \, F(dy) < \infty
\]
and
\[
\liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} 1(|f(s)| \geq \delta e^{-\gamma s}) \, ds > 0
\]
for some \( \delta > 0 \) and \( \gamma > 0 \). Then (1.4) holds.

Proof. Put
\[
G(t) = \int_{0}^{t} 1(|f(s)| \geq \delta e^{-\gamma s}) \, ds.
\]
By the assumption, there are a \( T > 0 \) and an \( \varepsilon > 0 \) such that \( G(t) \geq \varepsilon t \) for all \( t \geq T \).

We have
\[
\int_{\mathbb{R}} \int_{0}^{\infty} (y^2 f^2(s) \land 1) \, ds \, F(dy) \\
\geq \int_{|y| \geq \delta^{-1} e^\gamma} \int_{0}^{y^{-1} \log(\delta |y|)} (y^2 f^2(s) \land 1) 1(|f(s)| \geq \delta e^{-\gamma s}) \, ds \, F(dy) \\
= \int_{|y| \geq \delta^{-1} e^\gamma} G(y^{-1} \log(\delta |y|)) \, F(dy) \\
\geq \varepsilon y^{-1} \int_{|y| \geq \delta^{-1} e^\gamma} \log(\delta |y|) \, F(dy).
\]
The left-hand side of the above inequality is finite by the assumptions, so we easily obtain (1.4).

Proof of Theorem 3.1. Let us first note that by (2.8),
\[
\int_{[0,t]} X(t-s) \, d\xi(s) = \int_{0}^{t} x_0(t-s) \, dZ(s).
\]
Thus, using (2.3), any solution of Eq. (1.1) can be written in the form
\[
X(t) = x_0(t)X_0(0) + \int_{0}^{t} \int_{u}^{t} x_0(s) x_0(t+u-s) \, ds \, a(du) \\
+ \int_{0}^{t} x_0(t-s) \, dZ(s), \quad t \geq 0.
\]  
(4.9)

Note also that, by Lemma 2.3,
\[
\int_{0}^{t} x_0(t-s) \, dZ(s) \overset{d}{=} \int_{0}^{t} x_0(s) \, dZ(s).
\]  
(4.10)

Implications (i) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (ii) are trivial.
Let us prove (iv) ⇒ (i). Let $Z = (Z(t), t \geq 0)$ and $\tilde{Z} = (\tilde{Z}(t), t \geq 0)$ be two independent Lévy processes with the same characteristics $(b, c, F)$. To make the idea more clear, let us define a two-sided Lévy process $(Z(t), t \in \mathbb{R})$ by

$$Z(t) = \begin{cases} Z(t), & t \geq 0, \\ -\tilde{Z}(-t - 0), & t < 0 \end{cases}$$

and put

$$X(t) = \int_{-\infty}^t x_0(t - s) dZ(s)$$

for $t > -r$. By Lemmas 2.3 and 2.4, $X(t)$ is stationary in the sense of (3.1). Note also that by Lemma 2.4 the characteristic function of vector (3.2) coincides with the right-hand side of (4.12).

Therefore, the process $X = (X(t), t \geq -r)$ is well defined up to a modification according to Lemmas 2.4, 4.3 and 2.1. Moreover, let $-r \leq t_1 < \cdots < t_n$. By Lemmas 2.3 and 2.4,

$$E \exp \left( i \sum_{k=1}^n u_k X(t_k) \right) = \exp \left( \int_0^\infty \psi_{b,c,F} \left( \sum_{k=1}^n u_k x_0(t_k + s) \right) ds \right)$$

$$+ \mathbf{1}(t_n > 0) \int_0^{t_n} \psi_{b,c,F} \left( \sum_{k,n > 0} u_k x_0(t_k) \right) ds$$

$$= \exp \left( \int_{t_n}^\infty \psi_{b,c,F} \left( \sum_{k=1}^n u_k x_0(s - t_n + t_k) \right) ds \right)$$

$$+ \mathbf{1}(t_n > 0) \int_0^{t_n} \psi_{b,c,F} \left( \sum_{k,n > 0} u_k x_0(s - t_n + t_k) \right) ds$$

$$= \exp \left( \int_0^\infty \psi_{b,c,F} \left( \sum_{k=1}^n u_k x_0(s - t_n + t_k) \right) ds \right).$$

Therefore, the process $X$ is stationary in the sense of (3.1). Note also that by Lemma 2.4 the characteristic function of vector (3.2) coincides with the right-hand side of (4.12).

We shall show that there exists a modification of $X(t)$ being càdlàg and solving Eq. (1.1). To this aim we shall construct a sequence $(X_N(t), t \geq -r)$ of càdlàg solutions to (1.1) such that $X_N(t)$ converges uniformly in $t \geq -r$ as $N \to \infty$ almost surely and

$$\lim_{N \to \infty} X_N(t) = X(t)$$

with probability one for every $t \geq -r$. 

(4.13)
For an integer $N > r$ define

$$X_N(t) = \begin{cases} 
\int_{[0,t]} Z(t - s) \, dx_0(s) + x_0(N + t) \tilde{Z}(N) \\
- \int_{[0,N]} \tilde{Z}(s-) \, dx_0(t + s), & t \geq 0, \\
x_0(N + t) \tilde{Z}(N) - \int_{[0,N]} \tilde{Z}(s-) \, dx_0(t + s), & -r \leq t < 0.
\end{cases} \quad (4.14)$$

Combining (2.3) and Lemma 4.2, we obtain that $(X_N(t), t \geq -r)$ is a càdlàg solution to Eq. (1.1). By (2.8), (4.14) can be rewritten in the form

$$X_N(t) = \begin{cases} 
\int_{0}^{t} x_0(t - s) \, dZ(s) + \int_{0}^{N} x_0(t + s) \, d\tilde{Z}(s), & t \geq 0, \\
\int_{0}^{N} x_0(t + s) \, d\tilde{Z}(s), & -r \leq t < 0.
\end{cases} \quad (4.15)$$

Comparing the last equality with (4.11), we deduce (4.13) from Lemmas 2.4, 4.3 and 2.1.

To show the uniform convergence of $X_N(t)$, we shall prove that

$$\sum_{N} \sup_{t \geq -r} |X_{N+1}(t) - X_N(t)| < \infty \quad (4.16)$$

for almost all $\omega$.

Since $v_0 < 0$ in our case, by Lemma 2.1 and (2.2),

$$|x(t)| \leq ce^{-\gamma t}, \quad t \geq -r, \quad |\dot{x}(t)| \leq ce^{-\gamma t}, \quad t \geq 0 \quad (4.17)$$

for some $\gamma \in ]0, -v_0[ \text{ and } c > 0$.

It follows from (4.14) that

$$X_{N+1}(t) - X_N(t) = x_0(N + 1 + t)(\tilde{Z}(N + 1) - \tilde{Z}(N))$$

$$- \int_{N}^{N+1} (\tilde{Z}(s-) - \tilde{Z}(N)) \, dx_0(t + s). \quad (4.18)$$

Putting $\xi_N = \sup_{s \in [N,N+1]} |\tilde{Z}(s) - \tilde{Z}(N)|$, we get from (4.18) and (4.17) that

$$|X_{N+1}(t) - X_N(t)| \leq (|x_0(N + 1 + t)| + \int_{N}^{N+1} |\dot{x}_0(t + s)| \, ds) \xi_N$$

$$\leq 2ce^\gamma e^{-\gamma N} \xi_N. \quad (4.19)$$

It is well known that the Lévy process $\tilde{Z}$ can be decomposed into the sum

$$\tilde{Z}(t) = bt + M(t) + \sum_{0 < s \leq t} \Delta \tilde{Z}(s) 1(\Delta \tilde{Z}(s) > 1),$$

where $\Delta \tilde{Z}(s) = \tilde{Z}(s) - \tilde{Z}(s-)$ and $M(t)$ is a square-integrable martingale with the quadratic characteristic $(c + \int_{|y| \leq 1} y^2 F(dy))t$, see e.g. Jacod and Shiryaev (1987, Chapter II). Therefore,

$$\xi_N \leq |b| + \xi_N^{(1)} + \xi_N^{(2)} + \xi_N^{(3)}, \quad (4.20)$$
where
\[ \zeta_N^{(1)} = \sup_{s \in [N, N+1]} |M(s) - M(N)|, \]
\[ \zeta_N^{(2)} = \sum_{N < s < N+1} \Delta \tilde{Z}(s) \mathbf{1}(\Delta \tilde{Z}(s) > 1), \]
\[ \zeta_N^{(3)} = - \sum_{N < s < N+1} \Delta \tilde{Z}(s) \mathbf{1}(\Delta \tilde{Z}(s) < -1). \]

Due to (4.19) and (4.20), to prove (4.16) it is enough to check that \( \sum_N e^{-yN} \zeta_N^{(i)} < \infty \) almost surely, \( i = 1, 2, 3. \) For \( i = 1, \) we use Doob’s inequality
\[ E(\zeta_N^{(1)})^2 \leq 4E(M(N + 1) - M(N))^2 = 4\left(c + \int_{|y| \leq 1} y^2 F(dy)\right) < \infty, \]
which implies \( \sum_N e^{-yN} E\zeta_N^{(1)} < \infty \) and hence \( \sum_N e^{-yN} \zeta_N^{(1)} < \infty \) almost surely. For \( i = 2 \) or 3, the desired convergence follows from Lemma 4.3 applied to \( f(t) = e^{-yt} \) and the processes
\[ \sum_{0 < s <} \Delta \tilde{Z}(s) \mathbf{1}(\Delta \tilde{Z}(s) > 1) \quad \text{and} \quad \sum_{0 < s <} \Delta \tilde{Z}(s) \mathbf{1}(\Delta \tilde{Z}(s) < -1), \]
which are Lévy processes with the Lévy measures \( \mathbf{1}(y > 1) F(dy) \) and \( \mathbf{1}(y < -1) F(dy) \) respectively.

Our next step is to prove that (iv) implies (iii), (v) and (vi). According to Lemma 4.1, the first two summands on the right-hand side of (4.9) converge to zero for all \( \omega. \) Now (iii) follows from (4.10), Lemmas 4.3 and 2.1. Moreover, due to the proof of the previous implication, we can construct (extending the probability space if necessary) a stationary process \( Y(t) \) such that \( X(t) \) and \( Y(t) \) solve Eq. (1.1) with the same Lévy process \( Z(t). \) Rewriting representation (4.9) for the process \( Y(t) \) and comparing it with (4.9) for \( X(t), \) we obtain that \( \lim_{t \to \infty} \{X(t) - Y(t)\} = 0 \) almost surely, which yields \( \sup_{s \geq 0} |X(t) - Y(t)| \to 0 \) as \( t \to \infty \) almost surely. This immediately implies (vi) and (v) since we have already shown that the distribution of the vector \( (Y(t_k), k \leq n), 0 \leq t_1 < t_2 < \cdots < t_n, \) coincides with the distribution of vector (3.2).

Our last step is to prove (ii) \( \Rightarrow \) (iv). Let \( X \) be a solution of (1.1) such that \( X(t) \) converges in distribution as \( t \to \infty. \) Let \( \varphi(u), u \in \mathbb{R}, \) be the characteristic function of \( X(t). \) Then there is an interval \((0, u_0), u_0 > 0\) and numbers \( \delta \in ]0, 1[ \) and \( t_0 \geq 0 \) such that \( |\varphi(u)| \geq \delta \) for all \( u \in [0, u_0] \) and \( t \geq t_0. \)

In view of (4.9), (4.10) and independence of \( X_0 \) and \( Z, \)
\[ \left| E \exp\left( iu \int_0^t x_0(s) dZ(s) \right) \right| \geq |\varphi(u)| \geq \delta, \quad u \in [0, u_0], \quad t \geq t_0. \] (4.21)

Let \( (B(t), C(t), F(t)) \) be the Lévy–Khintchine characteristics of the distribution of \( \int_0^t x_0(s) dZ(s), \) i.e.
\[ E \exp\left( iuB(t) - \frac{1}{2} u^2 C(t) + \int_{\mathbb{R}} (e^{iuy} - 1 - iuy) F(t, dy) \right). \] (4.22)
We obtain from (4.21) and (4.22) that
\[
\frac{u^2}{2} C(t) + \int_{\mathbb{R}} (1 - \cos(uy)) F(t; dy) \leq L := -\log \delta, \quad u \in [0, u_0].
\] (4.23)

Let \( F = 0 \). Then \( c > 0 \) by our assumptions and \( C(t) = c \int_0^t x^2_0(s) \, ds \) by (2.10). Hence, \( \int_0^\infty x^2_0(s) \, ds < \infty \) by (4.23) and \( v_0 < 0 \) by Corollary 2.2.

Let \( F \neq 0 \). Integrating (4.23) over \( u \) from 0 to \( u_0 \), we get
\[
\int_{\mathbb{R}} \left( u_0 - \frac{\sin(u_0y)}{y} \right) F(t; dy) \leq L u_0.
\]

Taking into account that
\[
y^2 \wedge 1 \leq \kappa \left( u_0 - \frac{\sin(u_0y)}{y} \right)
\]
for all \( y \neq 0 \), where \( \kappa \) is a positive constant (depending on \( u_0 \)), and using (2.11) and (4.23), we obtain
\[
\int_{\mathbb{R}} \int_0^\infty (y^2 x^2_0(s) \wedge 1) \, ds \, F(dy) = \lim_{t \to \infty} \int_{\mathbb{R}} \int_0^t (y^2 x^2_0(s) \wedge 1) \, ds \, F(dy)
\]
\[
= \lim_{t \to \infty} \int_{\mathbb{R}} (y^2 \wedge 1) \, ds \, F(t; dy) \leq \kappa L u_0 < \infty.
\]

By Corollary 2.2, if \( v_0 > 0 \) then \( \int_0^\infty (y^2 x^2_0(s) \wedge 1) \, ds = \infty \) for all \( y \neq 0 \). Thus, \( v_0 < 0 \) and because of Corollary 2.2 the function \( x_0(t) \) satisfies the assumptions of Lemma 4.4, which yields (iv).

Acknowledgements

The research on the topic of this paper was supported by the Deutsche Forschungsgemeinschaft, the Sonderforschungsbereich 373 at Humboldt University in Berlin, and the DFG/RFBR Grant 98-01-04108. The authors thank Jean Jacod for helpful comments on a first version of this paper.

References

Wolfe, S.J., 1982. On a continuous analogue of the stochastic difference equation \( X_n = \rho X_{n-1} + B_n \). Stochastic Process. Appl. 12, 301–312.