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Chaotic and predictable representations for Lévy processes

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Abstract

The only normal martingales which posses the chaotic representation property and the weaker predictable representation property and which are at the same time also Lévy processes, are in essence Brownian motion and the compensated Poisson process. For a general Lévy process (satisfying some moment conditions), we introduce the power jump processes and the related Teugels martingales. Furthermore, we orthogonalize the Teugels martingales and show how their orthogonalization is intrinsically related with classical orthogonal polynomials. We give a chaotic representation for every square integral random variable in terms of these orthogonalized Teugels martingales. The predictable representation with respect to the same set of orthogonalized martingales of square integrable random variables and of square integrable martingales is an easy consequence of the chaotic representation.

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1. Introduction

The *chaotic representation property* (CRP) has been studied by Emery (1989) for normal martingales, that is, for martingales X such that $\langle X,X\rangle_t = ct$, for some constant $c>0$. This property says that any square integrable random variable measurable with respect to X can be expressed as an orthogonal sum of multiple stochastic integrals with respect to X. It is known (see for example Dellacherie et al., 1992, p. 207 and Dermoune, 1990), that the only normal martingales X, with the CRP, or even the weaker *predictable representation property* (PRP), which are also Lévy processes are the Brownian motion and the compensated Poisson process.

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In this paper we study the chaotic representation property for Lévy processes, in terms of a suitable orthogonal sequence of martingales, assuming that the Lévy measure has a finite Laplace transform outside the origin. These martingales are obtained as the orthogonalization of the compensated power jump processes of our Lévy process. In Section 2, we introduce these compensated power jump processes and we transform them into an orthogonal sequence. Section 3 is devoted to prove the chaos representation property from which a predictable representation is deduced. Finally, in Section 4, we discuss some particular examples.

2. Lévy processes and their power jump processes

A real-valued stochastic process $X=\{X_t, t\geqslant 0\}$ defined in a complete probability space $(\Omega,\mathscr{F},P)$ is called *Lévy process* if X has stationary and independent increments and $X_0=0$. A Lévy process possesses a càdlàg modification (Protter, 1990, Theorem 30, p. 21) and we will always assume that we are using this càdlàg version. If we let $\mathscr{F}_t=\mathscr{G}_t\vee\mathscr{N}$, where $\mathscr{G}_t=\sigma\{X_s, 0\leqslant s\leqslant t\}$ is the natural filtration of X, and $\mathscr{N}$ are the P-null sets of $\mathscr{F}$, then $\{\mathscr{F}_t, t\geqslant 0\}$ is a right continuous family of σ-fields (Protter, 1990, Theorem 31, p. 22). We assume that $\mathscr{F}$ is generated by X. For an up-to-date and comprehensive account of Lévy processes we refer the reader to Bertoin (1996) and Sato (1999).

Let X be a Lévy process and denote by

$$X_{t-}=\lim_{s\to t,\, s<t} X_s, \quad t>0,$$

the left limit process and by $\Delta X_t=X_t-X_{t-}$ the jump size at time t. It is known that the law of X_t is *infinitely divisible* with characteristic function of the form

$$E[\exp(\mathrm{i}\theta X_t)]=(\phi(\theta))^t,$$

where $\phi(\theta)$ is the characteristic function of X_1. The function $\psi(\theta)=\log\phi(\theta)$ is called the *characteristic exponent* and it satisfies the following famous *Lévy–Khintchine formula* (Bertoin, 1996):

$$\psi(\theta)=\mathrm{i}a\theta-\frac{\sigma^2}{2}\theta^2+\int_{-\infty}^{+\infty}(\exp(\mathrm{i}\theta x)-1-\mathrm{i}\theta x 1_{\{|x|<1\}})\nu(\mathrm{d}x),$$

where $a\in\mathbb{R}$, $\sigma^2\geqslant 0$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ with $\int_{-\infty}^{+\infty}(1\wedge x^2)\nu(\mathrm{d}x)<\infty$. The measure ν is called the *Lévy measure* of X.

Hypothesis 1. *We will suppose in the remaining of the paper that the Lévy measure satisfies for some* $\varepsilon>0$, *and* $\lambda>0$,

$$\int_{(-\varepsilon,\varepsilon)^c}\exp(\lambda|x|)\nu(\mathrm{d}x)<\infty.$$

This implies that

$$\int_{-\infty}^{+\infty}|x|^i\nu(\mathrm{d}x)<\infty, \quad i\geqslant 2 \tag{1}$$

and that the characteristic function $E[\exp(iuX_t)]$ is analytic in a neighborhood of 0. As a consequence, X_t has moments of all orders and the polynomials are dense in $L^2(\mathbb{R},\ P \circ X_t^{-1})$ for all $t > 0$.

The following transformations of X will play an important role in our analysis. We set

$$X_t^{(i)} = \sum_{0<s\leqslant t} (\Delta X_s)^i, \quad i \geqslant 2$$

and for convenience we put $X_t^{(1)} = X_t$. Note that not necessarily $X_t = \sum_{0<s\leqslant t} \Delta X_s$ holds; it is only true in the bounded variation case with $\sigma^2 = 0$. If $\sigma^2 = 0$, clearly $[X,X]_t = X_t^{(2)}$. The processes $X^{(i)} = \{X_t^{(i)}, t \geqslant 0\}$, $i = 1, 2, \ldots$, are again Lévy processes and we call them the *power jump processes*. They jump at the same points as the original Lévy process.

We have $E[X_t] = E[X_t^{(1)}] = tm_1 < \infty$ and by Protter (1990, p. 29), that

$$E[X_t^{(i)}] = E\left[\sum_{0<s\leqslant t} (\Delta X_s)^i\right] = t\int_{-\infty}^{\infty} x^i \nu(\mathrm{d}x) = m_i t < \infty, \quad i \geqslant 2.$$

Therefore, we can denote by

$$Y_t^{(i)} := X_t^{(i)} - E[X_t^{(i)}] = X_t^{(i)} - m_i t, \quad i = 1, 2, 3, \ldots$$

the compensated power jump process of order i. $Y^{(i)}$ is a normal martingale, since for an integrable Lévy process Z, the process $\{Z_t - E[Z_t], t \geqslant 0\}$ is a martingale. The second author calls $Y^{(i)}$ after his scientific mentor the *Teugels martingale of order i*.

Remark 1. In the case of a *Poisson process*, all power jump processes will be the same, and equal to the original Poisson process. In the case of a *Brownian motion*, all power jump processes of order strictly greater than one will be equal to zero.

We denote by $\mathscr{M}^2$ the space of square integrable martingales M such that $\sup_t E(M_t^2) < \infty$, and $M_0 = 0$ a.s. Notice that if $M \in \mathscr{M}^2$, then $\lim_{t\to\infty} E(M_t^2) = E(M_\infty^2) < \infty$, and $M_t = E[M_\infty | \mathscr{F}_t]$. Thus, each $M \in \mathscr{M}^2$ can be identified with its terminal value M_∞. As in Protter (1990, p. 148), we say that two martingales $M, N \in \mathscr{M}^2$ are strongly orthogonal and we denote this by $M \times N$, if and only if their product MN is a uniformly integrable martingale. As noted in Protter (1990, p. 148), one can prove that $M \times N$ if and only if $[M,N]$ is a uniformly integrable martingale. We say that two random variables $X, Y \in L^2(\Omega, \mathscr{F})$ are weakly orthogonal, $X \perp Y$, if $E[XY] = 0$. Clearly, strong orthogonality implies weak orthogonality.

We are looking for a set of pairwise strongly orthogonal martingales $\{H^{(i)}, i \geqslant 1\}$ such that each $H^{(i)}$ is a linear combination of $Y^{(j)}$, $j = 1, 2, \ldots, i$, with the leading coefficient equal to 1. If we set

$$H^{(i)} = Y^{(i)} + a_{i,i-1}Y^{(i-1)} + \cdots + a_{i,1}Y^{(1)}, \quad i \geqslant 1,$$

we have that

$$[H^{(i)}, Y^{(j)}]_t = X_t^{(i+j)} + a_{i,i-1}X_t^{(i+j-1)} + \cdots + a_{i,1}X_t^{(1+j)} + a_{i,1}\sigma^2 t 1_{\{j=1\}}$$

and that

$$E[H^{(i)}, Y^{(j)}]_t = t(m_{i+j} + a_{i,i-1}m_{i+j-1} + \cdots + a_{i,1}m_{j+1} + a_{i,1}\sigma^2 1_{\{j=1\}}).$$

In conclusion, we have that, $[H^{(i)}, Y^{(j)}]$ is a martingale if and only if we have that $E[H^{(i)}, Y^{(j)}]_1 = 0$.

Consider two spaces: The first space S_1 is the space of all real polynomials on the positive real line endowed with the scalar product $\langle .,. \rangle_1$ given by

$$\langle P(x), Q(x)\rangle_1 = \int_{-\infty}^{+\infty} P(x)Q(x)x^2\nu(\mathrm{d}x) + \sigma^2 P(0)Q(0).$$

Note that

$$\langle x^{i-1}, x^{j-1}\rangle_1 = m_{i+j} \ + \sigma^2 1_{\{i=j=1\}}, \quad i,j \geqslant 1.$$

The other space S_2 is the space of all linear transformations of the Teugels martingales of the Lévy process, i.e.

$$S_2 = \{a_1Y^{(1)} + a_2Y^{(2)} + \cdots + a_nY^{(n)};\ n \in \{1,2,\ldots\},\ a_i \in \mathbb{R},\ i = 1,\ldots,n\}.$$

We endow this space with the scalar product $\langle .,. \rangle_2$, given by

$$\langle Y^{(i)}, Y^{(j)}\rangle_2 = E([Y^{(i)}, Y^{(j)}]_1) = m_{i+j} + \sigma^2 1_{\{i=j=1\}}, \quad i,j \geqslant 1.$$

So one clearly sees that $x^{i-1} \leftrightarrow Y^{(i)}$ is an isometry between S_1 and S_2. An orthogonalization of $\{1, x, x^2, \ldots\}$ in S_1 gives an orthogonalization of $\{Y^{(1)}, Y^{(2)}, Y^{(3)}, \ldots\}$.

In the remaining of the paper, $\{H^{(i)},\ i = 1,2,\ldots\}$ is a set of pairwise strongly orthogonal martingales given by the previous orthogonalization of $\{Y^{(i)},\ i = 1,2,\ldots\}$.

In the examples some well-known orthogonal polynomials, like the Laguerre, the Meixner and the Meixner–Pollaczek polynomials, will turn up in this context. Another martingale relation between orthogonal polynomials and Lévy processes can be found in Schoutens and Teugels (1998) and Schoutens (1999).

3. Representation properties

3.1. Representation of a power of a Lévy process

We will express $(X_{t+t_0} - X_{t_0})^k$, $t_0, t \geqslant 0$, $k = 1,2,3,\ldots$, as a sum of stochastic integrals with respect to the special processes $Y^{(j)}$, $j = 1,\ldots,k$. For $k = 1$, we have $(X_{t+t_0} - X_{t_0}) = \int_{t_0}^{t_0+t} \mathrm{d}X_s = \int_{t_0}^{t_0+t} \mathrm{d}Y_s^{(1)} + m_1 t$.

Using Ito's formula (Protter, 1990, p. 74, Theorem 33) we can write for $k \geqslant 2$,

$$\begin{aligned}(X_{t+t_0} - X_{t_0})^k = &\int_0^t k(X_{(s+t_0)-} - X_{t_0})^{k-1}\,\mathrm{d}(X_{s+t_0} - X_{t_0})\\ &+ \frac{\sigma^2}{2}\int_0^t k(k-1)(X_{(s+t_0)-} - X_{t_0})^{k-2}\,\mathrm{d}s\\ &+ \sum_{0<s\leqslant t} [(X_{s+t_0} - X_{t_0})^k - (X_{(s+t_0)-} - X_{t_0})^k\\ &- k(X_{(s+t_0)-} - X_{t_0})^{k-1}\,\Delta X_{s+t_0}]\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_0}^{t_0+t} k(X_{u-} - X_{t_0})^{k-1}\,\mathrm{d}X_u^{(1)} \\
&\quad + \frac{\sigma^2}{2}k(k-1)\left((X_{t+t_0} - X_{t_0})^{k-2}t - \int_0^t s\,\mathrm{d}(X_{s+t_0} - X_{t_0})^{k-2}\right) \\
&\quad + \sum_{0<s\leqslant t} [(X_{(s+t_0)-} + \Delta X_{s+t_0} - X_{t_0})^k - (X_{(s+t_0)-} - X_{t_0})^k \\
&\quad - k(X_{(s+t_0)-} - X_{t_0})^{k-1}\Delta X_{s+t_0}] \\
&= \int_{t_0}^{t_0+t} k(X_{u-} - X_{t_0})^{k-1}\,\mathrm{d}X_u^{(1)} \\
&\quad + \frac{\sigma^2}{2}k(k-1)\left((X_{t+t_0} - X_{t_0})^{k-2}t - \int_0^t s\,\mathrm{d}(X_{s+t_0} - X_{t_0})^{k-2}\right) \\
&\quad + \sum_{0<s\leqslant t}\sum_{j=2}^{k}\binom{k}{j}(X_{(s+t_0)-} - X_{t_0})^{k-j}(\Delta X_{s+t_0})^j \\
&= \int_{t_0}^{t_0+t} k(X_{u-} - X_{t_0})^{k-1}\,\mathrm{d}X_u^{(1)} \\
&\quad + \frac{\sigma^2}{2}k(k-1)\left((X_{t+t_0} - X_{t_0})^{k-2}t - \int_0^t s\,\mathrm{d}(X_{s+t_0} - X_{t_0})^{k-2}\right) \\
&\quad + \sum_{t_0<u\leqslant t+t_0}\sum_{j=2}^{k}\binom{k}{j}(X_{u-} - X_{t_0})^{k-j}(\Delta X_u)^j \\
&= \sum_{j=1}^{k}\binom{k}{j}\int_{t_0}^{t+t_0}(X_{u-} - X_{t_0})^{k-j}\,\mathrm{d}X_u^{(j)} \\
&\quad + \frac{\sigma^2}{2}k(k-1)\left((X_{t+t_0} - X_{t_0})^{k-2}t - \int_0^t s\,\mathrm{d}(X_{s+t_0} - X_{t_0})^{k-2}\right). \qquad (2)
\end{aligned}
$$

Lemma 1. *The power of an increment of a Lévy process, $(X_{t+t_0} - X_{t_0})^k$, has a representation of the form*

$$
(X_{t+t_0} - X_{t_0})^k = f^{(k)}(t,t_0) + \sum_{j=1}^{k}\sum_{\substack{(i_1,\dots,i_j)\\ \in\{1,\dots,k\}^j}}\int_{t_0}^{t+t_0}\int_{t_0}^{t_1-}\cdots
$$

$$
\int_{t_0}^{t_{j-1}-} f^{(k)}_{(i_1,\dots,i_j)}(t,t_0,t_1,\dots,t_j)\,\mathrm{d}Y_{t_j}^{(i_j)}\dots\mathrm{d}Y_{t_2}^{(i_2)}\,\mathrm{d}Y_{t_1}^{(i_1)}, \qquad (3)
$$

where the $f^{(k)}_{(i_1,\dots,i_j)}$ are deterministic functions in $L^2(\mathbb{R}_+^j)$.

Proof. Representation (3) follows from (2), where we bring in the right compensators, i.e. we can write

$$\sum_{j=1}^{k}\binom{k}{j}\int_{t_0}^{t+t_0}(X_{s-}-X_{t_0})^{k-j}\,\mathrm{d}X_s^{(j)}$$

$$=\sum_{j=1}^{k}\binom{k}{j}\int_{t_0}^{t+t_0}(X_{s-}-X_{t_0})^{k-j}\,\mathrm{d}Y_s^{(j)}+\sum_{j=1}^{k}\binom{k}{j}m_j\int_{t_0}^{t+t_0}(X_{s-}-X_{t_0})^{k-j}\,\mathrm{d}s$$

$$=\sum_{j=1}^{k}\binom{k}{j}\int_{t_0}^{t+t_0}(X_{s-}-X_{t_0})^{k-j}\,\mathrm{d}Y_s^{(j)}+\sum_{j=1}^{k-1}\binom{k}{j}m_j t(X_{t+t_0}-X_{t_0})^{k-j}$$

$$-\sum_{j=1}^{k-1}\binom{k}{j}m_j\int_{t_0}^{t+t_0}s\,\mathrm{d}(X_s-X_{t_0})^{k-j}+m_k t. \tag{4}$$

Combining (2) and (4) gives

$$(X_{t+t_0}-X_{t_0})^k=\frac{\sigma^2}{2}k(k-1)\left((X_{t+t_0}-X_{t_0})^{k-2}t-\int_0^t s\,\mathrm{d}(X_{s+t_0}-X_{t_0})^{k-2}\right)$$

$$+\sum_{j=1}^{k}\binom{k}{j}\int_{t_0}^{t+t_0}(X_{s-}-X_{t_0})^{k-j}\,\mathrm{d}Y_s^{(j)}+\sum_{j=1}^{k-1}\binom{k}{j}m_j t(X_{t+t_0}-X_{t_0})^{k-j}$$

$$-\sum_{j=1}^{k-1}\binom{k}{j}m_j\int_{t_0}^{t+t_0}s\,\mathrm{d}(X_s-X_{t_0})^{k-j}+m_k t. \tag{5}$$

The last equation is in terms of powers of increments of X which are strictly lower than k. So by induction representation (3) can be proved. □

Notice that taking the expectation in (3) yields

$$E[(X_{t+t_0}-X_{t_0})^k]=f^{(k)}(t,t_0)=f^{(k)}(t),\quad t,t_0\geqslant 0,$$

which is independent of t_0.

Moreover, it can easily be seen that $f^{(k)}_{(i_1,\dots,i_j)}$ are just real multivariate polynomials of degree less than k and that we have $f^{(k)}_{(i_1,\dots,i_j)}=0$, whenever $i_1+\cdots+i_j>k$.

Because we can switch by a linear transformation from the $Y^{(i)}$ to the $H^{(i)}$, it is clear that we also proved the next representation.

Lemma 2. *The power of an increment of a Lévy process, $(X_{t+t_0}-X_{t_0})^k$, has a representation of the form*

$$(X_{t+t_0}-X_{t_0})^k=f^{(k)}(t)+\sum_{j=1}^{k}\sum_{\substack{(i_1,\dots,i_j)\\ \in\{1,\dots,k\}^j}}\int_{t_0}^{t+t_0}\int_{t_0}^{t_1-}\cdots$$

$$\int_{t_0}^{t_{j-1}-}h^{(k)}_{(i_1,\dots,i_j)}(t,t_0,t_1,\dots,t_j)\,\mathrm{d}H_{t_j}^{(i_j)}\dots\mathrm{d}H_{t_2}^{(i_2)}\,\mathrm{d}H_{t_1}^{(i_1)}, \tag{6}$$

where the $h^{(k)}_{(i_1,\dots,i_j)}$ are deterministic functions in $L^2(\mathbb{R}_+^j)$.

As an illustration we will give $f^{(k)}_{(i_1,\dots,i_j)}$'s for $k=1,2$, $t_0=0$ and $\sigma^2=0$. We start with the trivial case $k=1$. Because $X^1_t = Y^{(1)}_t + m_1 t$, we clearly see that $f^{(1)}(t)=m_1 t$ and $f^{(1)}_{(1)}(t,0,t_1)=1$. The case $k=2$, is a little more complex. We start from (5):

$$\begin{aligned}
X_t^2 &= 2\int_0^t X_{t_1-}\,\mathrm{d}Y^{(1)}_{t_1} + \int_0^t \mathrm{d}Y^{(2)}_{t_1} + 2m_1\left(tX_t - \int_0^t t_1\,\mathrm{d}X_{t_1}\right) + m_2 t\\
&= 2\int_0^t \left(\int_0^{t_1-} \mathrm{d}Y^{(1)}_{t_2} + m_1 t_1\right)\mathrm{d}Y^{(1)}_{t_1} + \int_0^t \mathrm{d}Y^{(2)}_{t_1}\\
&\quad + 2m_1\left(tX_t - \int_0^t t_1\,\mathrm{d}X_{t_1}\right) + m_2 t\\
&= 2\int_0^t\int_0^{t_1-} \mathrm{d}Y^{(1)}_{t_2}\,\mathrm{d}Y^{(1)}_{t_1} + 2m_1\int_0^t t_1\,\mathrm{d}Y^{(1)}_{t_1} + \int_0^t \mathrm{d}Y^{(2)}_{t_1}\\
&\quad + 2m_1 t\int_0^t \mathrm{d}Y^{(1)}_{t_1} + 2m_1^2 t^2 - 2m_1\int_0^t t_1\,\mathrm{d}Y^{(1)}_{t_1} - 2m_1^2\int_0^t t_1\mathrm{d}t_1 + m_2 t\\
&= 2\int_0^t\int_0^{t_1-} \mathrm{d}Y^{(1)}_{t_2}\,\mathrm{d}Y^{(1)}_{t_1} + \int_0^t \mathrm{d}Y^{(2)}_{t_1} + 2m_1 t\int_0^t \mathrm{d}Y^{(1)}_{t_1} + m_1^2t^2 + m_2 t.
\end{aligned}$$

So that

$$\begin{aligned}
&f^{(2)}(t,0) = m_1^2 t^2 + m_2 t,\\
&f^{(2)}_{(1)}(t,0,t_1) = 2m_1 t, \quad f^{(2)}_{(2)}(t,0,t_1) = 1,\\
&f^{(2)}_{(1,1)}(t,0,t_1,t_2) = 2, \quad f^{(2)}_{(1,2)}(t,0,t_1,t_2) = f^{(2)}_{(2,2)}(t,0,t_1,t_2) = 0.
\end{aligned}$$

3.2. Representation of a square integrable random variable

We first recall that $\{H^{(i)},\ i=1,2,\dots\}$ is a set of pairwise strongly orthogonal martingales, obtained by the orthogonalization procedure described at the end of Section 2.

We denote by

$$\mathscr{H}^{(i_1,\dots,i_j)} = \Bigg\{ F\in L^2(\Omega):\quad F = \int_0^\infty\int_0^{t_1-}\cdots\int_0^{t_{j-1}-} f(t_1,\dots,t_j)\,\mathrm{d}H^{(i_j)}_{t_j}\dots\mathrm{d}H^{(i_2)}_{t_2}\,\mathrm{d}H^{(i_1)}_{t_1},\ f\in L^2(\mathbb{R}^j_+)\Bigg\}.$$

We say that two multi-indexes $(i_1,\dots,i_k)$ and $(j_1,\dots,j_l)$ are different if $k\neq l$ or when $k=l$, if there exists a subindex $1\leqslant n\leqslant k=l$, such that $i_n\neq j_n$, and denote this by

$$(i_1,\dots,i_k)\neq(j_1,\dots,j_l).$$

Proposition 1. *If* $(i_1,\dots,i_k)\neq(j_1,\dots,j_l)$, *then* $\mathscr{H}^{(i_1,\dots,i_k)}\perp\mathscr{H}^{(j_1,\dots,j_l)}$.

Proof. Suppose we have two random variables $K\in\mathscr{H}^{(i_1,\dots,i_k)}$ and $L\in\mathscr{H}^{(j_1,\dots,j_l)}$. We need to prove that if $(i_1,\dots,i_k)\neq(j_1,\dots,j_l)$, then $K\perp L$.

For the case $k=l$, we use induction on k. Take first $k=l=1$ and assume the following representations for K and L:

$$K=\int_0^\infty f(t_1)\,\mathrm{d}H_{t_1}^{(i_1)},\qquad L=\int_0^\infty g(t_1)\,\mathrm{d}H_{t_1}^{(j_1)},$$

where we must have $i_1\neq j_1$. By construction $H^{(i_1)}$ and $H^{(j_1)}$ are strongly orthogonal martingales. Using the fact that stochastic integrals with respect to strongly orthogonal martingales are again strongly orthogonal (Protter, 1990, Lemma 2 and Theorem 35, p. 149) and thus also weakly orthogonal, it immediately follows that $K\perp L$.

Suppose the theorem holds for all $1\leqslant k=l\leqslant n-1$. We are going to prove the theorem for $k=l=n$. Assume the following representations:

$$K=\int_0^\infty\int_0^{t_1-}\dots\int_0^{t_{n-1}-} f(t_1,\dots,t_n)\,\mathrm{d}H_{t_n}^{(i_n)}\dots\mathrm{d}H_{t_2}^{(i_2)}\,\mathrm{d}H_{t_1}^{(i_1)}=\int_0^\infty \alpha_{t_1}\,\mathrm{d}H_{t_1}^{(i_1)},$$

$$L=\int_0^\infty\int_0^{t_1-}\dots\int_0^{t_{n-1}-} g(t_1,\dots,t_n)\,\mathrm{d}H_{t_n}^{(j_n)}\dots\mathrm{d}H_{t_2}^{(j_2)}\,\mathrm{d}H_{t_1}^{(j_1)}=\int_0^\infty \beta_{t_1}\,\mathrm{d}H_{t_1}^{(j_1)}.$$

There are two possibilities: (1) $i=i_1=j_1$ and (2) $i_1\neq j_1$. In the former case we must have that $(i_2,\dots,i_n)\neq(j_2,\dots,j_n)$, and thus by induction $\alpha_{t_1}\perp\beta_{t_1}$, so that

$$\begin{aligned}E[KL]&=E\left[\int_0^\infty \alpha_s\beta_s\,\mathrm{d}\langle H^{(i)},H^{(i)}\rangle_s\right]\\&=\int_0^\infty E(\alpha_s\beta_s)\,\mathrm{d}\langle H^{(i)},H^{(i)}\rangle_s=0.\end{aligned}$$

In the latter case we use again the fact that stochastic integrals with respect to strongly orthogonal martingales are again strongly orthogonal (Protter, 1990, Lemma 2 and Theorem 35, p. 149) and thus also weakly orthogonal. So it immediately follows that $K\perp L$.

For the case $k\neq l$, a similar argument can be used together with the fact that all elements of every $\mathscr{H}^{(i_1,\dots,i_n)}$, $n\geqslant 1$, have mean zero and thus are orthogonal w.r.t. the constants. □

Proposition 2. *Let*

$$\mathscr{P}=\{X_{t_1}^{k_1}(X_{t_2}-X_{t_1})^{k_2}\dots(X_{t_n}-X_{t_{n-1}})^{k_n}\colon n\geqslant 0, 0\leqslant t_1<t_2<\cdots<t_n, k_1,\dots,k_n\geqslant 1\},$$

then we have that $\mathscr{P}$ *is a total family in* $L^2(\Omega,\mathscr{F})$, *i.e. the linear subspace spanned by* $\mathscr{P}$ *is dense in* $L^2(\Omega,\mathscr{F})$.

Proof. Let $Z\in L^2(\Omega,\mathscr{F})$ and $Z\perp\mathscr{P}$. For any given $\varepsilon>0$, there exists a finite set $\{0<s_1<\cdots<s_m\}$ and a square integrable random variable $Z_\varepsilon\in L^2(\Omega,\sigma(X_{s_1},X_{s_2},\dots,X_{s_m}))$ such that

$$E\left[(Z-Z_\varepsilon)^2\right]<\varepsilon.$$

So there exists a Borel function f such that

$$Z_\varepsilon = f_\varepsilon(X_{s_1}, X_{s_2} - X_{s_1}, \dots, X_{s_m} - X_{s_{m-1}}).$$

Because the polynomials are dense in $L^2(\mathbb{R}, P\circ X_t^{-1})$ for each $t>0$, we can approximate Z_ε by polynomials. Furthermore because $Z \perp \mathscr{P}$, we have $E[ZZ_\varepsilon]=0$. Then

$$E[Z^2] = E[Z(Z-Z_\varepsilon)] \leqslant \sqrt{E[Z^2]E[(Z-Z_\varepsilon)^2]} \leqslant \sqrt{\varepsilon E[Z^2]},$$

and letting $\varepsilon \to 0$ yields $Z=0$ a.s. Thus $\mathscr{P}$ is a total family in $L^2(\Omega, \mathscr{F})$. □

We are now in a position to prove our main theorem.

Theorem 1 (Chaotic representation property (CRP)). *Every random variable F in $L^2(\Omega,\mathscr{F})$ has a representation of the form*

$$F = E[F] + \sum_{j=1}^{\infty} \sum_{i_1,\dots,i_j \geqslant 1} \int_0^\infty \int_0^{t_1-} \cdots \int_0^{t_{j-1}-} f_{(i_1,\dots,i_j)}(t_1,\dots,t_j)\, \mathrm{d}H_{t_j}^{(i_j)} \dots \mathrm{d}H_{t_2}^{(i_2)}\, \mathrm{d}H_{t_1}^{(i_1)},$$

where the $f_{(i_1,\dots,i_j)}$'s are functions in $L^2(\mathbb{R}_+^j)$.

Proof. Because $\mathscr{P}$ is a total family in $L^2(\Omega,\mathscr{F})$, it is sufficient to prove that every element of $\mathscr{P}$ has a representation of the desired form. This follows from the fact that $\mathscr{P}$ is build up from terms of the from $X_{t_1}^{k_1}(X_{t_2}-X_{t_1})^{k_2}\dots(X_{t_n}-X_{t_{n-1}})^{k_n}$, wherein every term has on its turn a representation of the form (6), and we can nicely combine two terms in the desired representation. Indeed, we have for all $k,l \geqslant 1$, and $0 \leqslant t < s \leqslant u < v$, that the product of $(X_s-X_t)^k(X_v-X_u)^l$ is a sum of products of the form AB where

$$A = \int_t^s \int_t^{t_1-} \cdots \int_t^{t_{n-1}-} h^{(k)}_{(i_1,\dots,i_n)}(s,t,t_1,\dots,t_n)\, \mathrm{d}H_{t_n}^{(i_n)} \dots \mathrm{d}H_{t_2}^{(i_2)}\, \mathrm{d}H_{t_1}^{(i_1)}$$

and

$$B = \int_u^v \int_u^{u_1-} \cdots \int_u^{u_{m-1}-} h^{(l)}_{(j_1,\dots,j_m)}(v,u,u_1,\dots,u_m)\, \mathrm{d}H_{u_m}^{(j_m)} \dots \mathrm{d}H_{u_2}^{(j_2)}\, \mathrm{d}H_{u_1}^{(j_1)}.$$

We can write

$$\begin{aligned} AB &= \int_u^v \int_u^{u_1-} \cdots \int_u^{u_{m-1}-} \int_t^s \int_t^{t_1-} \cdots \int_t^{t_{n-1}-} h^{(l)}_{(j_1,\dots,j_m)}(v,u,u_1,\dots,u_m) \\ &\quad \times h^{(k)}_{(i_1,\dots,i_n)}(s,t,t_1,\dots,t_n)\, \mathrm{d}H_{t_n}^{(i_n)} \dots \mathrm{d}H_{t_2}^{(i_2)}\, \mathrm{d}H_{t_1}^{(i_1)}\, \mathrm{d}H_{u_m}^{(j_m)} \dots \mathrm{d}H_{u_2}^{(j_2)}\, \mathrm{d}H_{u_1}^{(j_1)} \\ &= \int_0^\infty \int_0^{u_1-} \cdots \int_0^{u_{m-1}-} \int_0^{u_m-} \int_0^{t_1-} \cdots \int_0^{t_{n-1}-} 1_{(u,v]}(u_1) \\ &\quad \times 1_{(u,u_1)}(u_2) \dots 1_{(u,u_{m-1})}(u_m) 1_{(t,s]}(t_1) 1_{(t,t_1)}(t_2) \dots 1_{(t,t_{n-1})}(t_n) \\ &\quad \times h^{(l)}_{(j_1,\dots,j_m)}(v,u,u_1,\dots,u_m) h^{(k)}_{(i_1,\dots,i_n)}(s,t,t_1,\dots,t_n) \\ &\quad \times \mathrm{d}H_{t_n}^{(i_n)} \dots \mathrm{d}H_{t_2}^{(i_2)}\, \mathrm{d}H_{t_1}^{(i_1)}\, \mathrm{d}H_{u_m}^{(j_m)} \dots \mathrm{d}H_{u_2}^{(j_2)}\, \mathrm{d}H_{u_1}^{(j_1)}, \end{aligned}$$

and the desired representation follows. □

Theorem 2 (Predictable representation property (PRP)). *Every random variable F in $L^2(\Omega,\mathscr{F})$ has a representation of the form*

$$F = E[F] + \sum_{i=1}^{\infty} \int_0^{\infty} \phi_s^{(i)}\, \mathrm{d}H_s^{(i)},$$

where $\phi_s^{(i)}$ is predictable.

Proof. From the above theorem, we know that F has a representation of the form

$$\begin{aligned}
F - E[F] &= \sum_{j=1}^{\infty} \sum_{i_1,\dots,i_j \geqslant 1} \int_0^{\infty} \int_0^{t_1-} \cdots \int_0^{t_{j-1}-} f_{(i_1,\dots,i_j)}(t_1,\dots,t_j)\, \mathrm{d}H_{t_j}^{(i_j)} \dots \mathrm{d}H_{t_2}^{(i_2)}\, \mathrm{d}H_{t_1}^{(i_1)} \\
&= \sum_{i_1=1}^{\infty} \int_0^{\infty} f_{(i_1)}(t_1)\, \mathrm{d}H_{t_1}^{(i_1)} + \sum_{i_1=1}^{\infty} \int_0^{\infty} \left[\sum_{j=2}^{\infty} \sum_{i_2,\dots,i_j \geqslant 1} \int_0^{t_1-} \cdots \right. \\
&\quad \left. \int_0^{t_{j-1}-} f_{(i_1,\dots,i_j)}(t_1,\dots,t_j)\, \mathrm{d}H_{t_j}^{(i_j)} \dots \mathrm{d}H_{t_2}^{(i_2)} \right] \mathrm{d}H_{t_1}^{(i_1)} \\
&= \sum_{i_1=1}^{\infty} \int_0^{\infty} \left[f_{(i_1)}(t_1) \right. \\
&\quad \left. + \sum_{j=2}^{\infty} \sum_{i_2,\dots,i_j \geqslant 1} \int_0^{t_1-} \cdots \int_0^{t_{j-1}-} f_{(i_1,\dots,i_j)}(t_1,\dots,t_j)\, \mathrm{d}H_{t_j}^{(i_j)} \dots \mathrm{d}H_{t_2}^{(i_2)} \right] \mathrm{d}H_{t_1}^{(i_1)} \\
&= \sum_{i=1}^{\infty} \int_0^{\infty} \phi_{t_1}^{(i)}\, \mathrm{d}H_{t_1}^{(i)},
\end{aligned}$$

which is exactly of the form we want. □

Remark 2. Because we can identify every martingale $M \in \mathscr{M}^2$ with its terminal value $M_\infty \in L^2(\Omega,\mathscr{F})$ and because $M_t = E[M_\infty|\mathscr{F}_t]$, we have the predictable representation

$$M_t = \sum_{i=1}^{\infty} \int_0^{t} \phi_s^{(i)}\, \mathrm{d}H_s^{(i)},$$

which is a sum of strongly orthogonal martingales.

Another consequence of the chaotic representation property, is the following theorem:

Theorem 3. *We have the following space decomposition*:

$$L^2(\Omega,\mathscr{F}) = R \oplus \left(\bigoplus_{j=1}^{\infty} \bigoplus_{i_1,\dots,i_j \geqslant 1} \mathscr{H}^{(i_1,\dots,i_j)} \right).$$

Remark 3. The Lévy–Khintchine formula has a simpler expression when the sample paths of the related Lévy process have bounded variation on every compact time interval a.s. It is well known (Bertoin, 1996, p. 15), that a Lévy process has bounded variation if and only if $\sigma^2=0$, and $\int_{-\infty}^{+\infty}(1\wedge|x|)\nu(\mathrm{d}x)<\infty$. In that case the characteristic exponent can be re-expressed as

$$\psi(\theta)=\mathrm{i}d\theta+\int_{-\infty}^{+\infty}(\exp(\mathrm{i}\theta x)-1)\nu(\mathrm{d}x).$$

Furthermore, we can write

$$X_t=\mathrm{d}t+\sum_{0<s\leqslant t}\Delta X_s,\quad t\geqslant 0, \tag{7}$$

and the calculations simplify somewhat because $\sigma^2=0$ and for $k\geqslant 1$,

$$\int_0^t k(X_{(s+t_0)-}-X_{t_0})^{k-1}\,\mathrm{d}(X_{s+t_0}-X_{t_0})=\sum_{0<s\leqslant t}k(X_{(s+t_0)-}-X_{t_0})^{k-1}\Delta X_{s+t_0}.$$

4. Examples

4.1. The gamma process

The *Gamma process* is the Lévy process (of bounded variation) $G=\{G_t,t\geqslant 0\}$ with Lévy measure given by

$$\nu(\mathrm{d}x)=1_{(x>0)}\frac{\mathrm{e}^{-x}}{x}\,\mathrm{d}x.$$

It is called Gamma process because the law of G_t is a Gamma distribution with mean t and scale parameter equal to one. It is used i.a. in insurance mathematics (Dickson and Waters, 1993, 1996; Dufresne and Gerber, 1993; Dufresne et al., 1991).

We denote by

$$G_t^{(i)}=\sum_{0<s\leqslant t}(\Delta G_s)^i,\quad i\geqslant 1$$

the power jump processes of G. Using the exponential formula (Bertoin, 1996, p. 8), and the change of variable $z=x^j$, we obtain for $j\geqslant 1$

$$\begin{aligned}E[\exp(\mathrm{i}\theta G_t^{(j)})]&=\exp\left(t\int_0^\infty\left(\exp(\mathrm{i}\theta x^j)-1\right)\frac{\mathrm{e}^{-x}}{x}\,\mathrm{d}x\right)\\&=\exp\left(t\int_0^\infty(\exp(\mathrm{i}\theta z)-1)\frac{\exp(-z^{1/j})}{jz}\,\mathrm{d}z\right),\end{aligned}$$

which means that the Lévy measure of $G^{(j)}$ is

$$\frac{\exp(-z^{1/j})}{jz}\,\mathrm{d}z.$$

Because

$$E\left[\sum_{0<s\leqslant t}(\Delta G_s)^i\right]=t\int_0^\infty x^i\frac{\mathrm{e}^{-x}}{x}\,\mathrm{d}x,\quad i\geqslant 1,$$

we clearly have

$$E[G_t^{(i)}] = \Gamma(i)t = (i-1)!t, \quad i \geqslant 1$$

and thus

$$\hat{G}_t^{(i)} = G_t^{(i)} - (i-1)!t, \quad i \geqslant 1$$

is the *Teugels martingale of order i of the Gamma Process.*

Next, we orthogonalize the set $\{\hat{G}^{(i)},\ i=1,2,\ldots\}$ of martingales. So we are looking for a set of martingales

$$\{H^{(i)} = \hat{G}^{(i)} + a_{i,i-1}\hat{G}^{(i-1)} + a_{i,i-2}\hat{G}^{(i-2)} + \cdots + a_{i,1}\hat{G}^{(1)},\ i \geqslant 1\},$$

such that $H^{(i)}$ is strongly orthogonal to $H^{(j)}$, for $i \neq j$.

The first space S_1 is in the gamma case the space of all real polynomials on the positive real line endowed with a scalar product $\langle .,. \rangle_1$, given by

$$\langle P(x), Q(x) \rangle_1 = \int_0^\infty P(x)Q(x)x\mathrm{e}^{-x}\,\mathrm{d}x.$$

Note that

$$\langle x^{i-1}, x^{j-1} \rangle_1 = \int_0^\infty x^{i+j-1}\mathrm{e}^{-x}\,\mathrm{d}x = (i+j-1)!, \quad i,j \geqslant 1.$$

The other space S_2 is the space of all linear transformations of the Teugels martingales of the Gamma process, i.e.

$$S_2 = \{a_1\hat{G}^{(1)} + a_2\hat{G}^{(2)} + \cdots + a_n\hat{G}^{(n)}; n \in \{1,2,\ldots\},\ a_i \in \mathbb{R},\ i=1,\ldots,n\}$$

endowed with the scalar product $\langle .,. \rangle_2$, given by

$$\langle \hat{G}^{(i)}, \hat{G}^{(j)} \rangle_2 = E[[\hat{G}^{(i)}, \hat{G}^{(j)}]_1] = E[G_1^{(i+j)}] = (i+j-1)!, \quad i,j \geqslant 1.$$

So one clearly sees that $x^{i-1} \leftrightarrow G^{(i)}$ is an isometry between S_1 and S_2. An orthogonalization of $\{1, x, x^2, \ldots\}$ in S_1 gives the *Laguerre polynomials* $L_n^{(1)}(x)$ (Koekoek and Swarttouw, 1998), so by isometry we also find an orthogonalization of $\{\hat{G}^{(1)}, \hat{G}^{(2)}, \hat{G}^{(3)}, \ldots\}$.

4.2. The negative binomial process

The next process of bounded variation we look at is the *negative binomial process*, sometimes also called *Pascal process.* It has a Lévy measure $\nu(\mathrm{d}u)$ given by

$$\nu(\mathrm{d}u) = d\sum_{x=1}^{\infty} \frac{q^x}{x} 1_{(u \geqslant x)},$$

where $0<q<1$. One can prove that a Lévy process, $P=\{P_t,\ t \geqslant 0\}$, with such a Lévy measure has a Negative Binomial distribution with characteristic function given by

$$E[\exp(\mathrm{i}\theta P_t)] = \left(\frac{p}{1-q\mathrm{e}^{\mathrm{i}\theta}}\right)^t,$$

where $p=1-q$.

Let us denote with $P^{(i)} = \{P_t^{(i)},\ t \geqslant 0\}$ the corresponding power jump processes and with $Q^{(i)} = \{Q_t^{(i)},\ t \geqslant 0\}$ the corresponding Teugels martingales.

We look for the orthogonalization of the set $\{Q^{(i)},\ i\geqslant 1\}$ of martingales. The space S_1 is the now the space of all real polynomials on the positive real line endowed with a scalar product $\langle .,.\rangle_1$, given by

$$\langle P(x),R(x)\rangle_1=\sum_{x=1}^{\infty}P(x)R(x)xq^x.$$

Note that

$$\langle x^{i-1},x^{j-1}\rangle_1=\sum_{x=1}^{\infty}q^x x^{i+j-1},\quad i,j\geqslant 1.$$

The space S_2 is now the space of all linear transformations of the Teugels martingales of the negative binomial process, i.e.

$$S_2=\{a_1Q^{(1)}+a_2Q^{(2)}+\cdots+a_nQ^{(n)};n\in\{1,2,\dots,n\},\ a_i\in\mathbb{R},\ i=1,\dots,n\},$$

and is endowed with the scalar product $\langle .,.\rangle_2$, given by

$$\langle Q^{(i)},Q^{(j)}\rangle_2=E[[Q^{(i)},Q^{(j)}]_1]=E[P_1^{(i+j)}]=\sum_{x=1}^{\infty}q^x x^{i+j-1},\quad i,j\geqslant 1.$$

By construction $x^{i-1}\leftrightarrow Q^{(i)}$ is an isometry between S_1 and S_2. An orthogonalization of $\{1,x,x^2,\dots\}$ in S_1 gives the *Meixner polynomials* $M_n(x-1;2,p)$ (Koekoek and Swarttouw, 1998), so by isometry we also find an orthogonalization of the set $\{Q^{(1)},Q^{(2)},Q^{(3)},\dots\}$.

4.3. The Meixner process

A *Meixner process* $M=\{M_t,\ t\geqslant 0\}$ is a bounded variation Lévy process based on the infinitely divisible distribution with density function given by

$$f(x;m,a)=\frac{(2\cos(a/2))^{2m}}{2\pi\Gamma(2m)}\exp(ax)|\Gamma(m+\mathrm{i}x)|^2,\quad x\in(-\infty,+\infty),$$

and where a is a real constant and $m>0$. The corresponding probability distribution is the measure of orthogonality of the *Meixner–Pollaczek polynomials* (Koekoek and Swarttouw, 1998). The Meixner process was introduced in Schoutens and Teugels (1998). In Grigelionis (1998), it is proposed for a model for risky assets and an analogue of the famous Black and Scholes formula in mathematical finance was established. The characteristic function of M_1 is given by

$$E[\exp(\mathrm{i}\theta M_1)]=\left(\frac{\cos(a/2)}{\cosh((\theta-\mathrm{i}a)/2)}\right)^{2m}.$$

In Schoutens and Teugels (1998) and Schoutens (1999) its Lévy measure is calculated:

$$\nu(\mathrm{d}x)=m\frac{\exp(ax)}{x\sinh(\pi x)}\,\mathrm{d}x=m|\Gamma(1+\mathrm{i}x)|^2\frac{\exp(ax)}{\pi x^2}\,\mathrm{d}x.$$

Note that

$$x^2\nu(\mathrm{d}x)=m|\Gamma(1+\mathrm{i}x)|^2\frac{\exp(ax)}{\pi}\mathrm{d}x$$

is up to a constant equal to $f(x;1,a)$. Being completely similar as in the above two examples, we can orthogonalize the Teugels martingales for the Meixner process by isometry. The orthogonal polynomials involved will now be the Meixner–Pollaczek polynomials $P_n(x; 1; a)$.

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