Suprema of compound Poisson processes with light tails

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Abstract

It is known that if the Lévy measure of a Lévy process $X(t)$, $0 \leq t \leq 1$, is “heavy tailed”, then
the right tails of $\sup_{0 \leq t \leq 1} X(t)$ and $X(1)$ are of the same rate of decay. One of the results of this
note is a description of a class of compound Poisson processes with negative drift and “light”
tails (which is a subclass of Lévy processes) such that these tails are incomparable. © 2000 Elsevier
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1. Introduction

Lévy’s classical result states that if $X(t)$ is the Brownian motion, $X(0) = 0$, then

$$P \left( \sup_{0 \leq t \leq 1} X(t) > x \right) = 2P(X(1) > x) \quad \text{for } x > 0$$

(see, for example, Gikhman and Skorokhod, 1996, Chapter 6). This result was a starting
point for later studies, where the tail of the supremum of a process was compared with
the tail of $X(1)$. More precisely, the relations of the type

$$P \left( \sup_{0 \leq t \leq 1} X(t) > x \right) \sim cP(X(1) > x) \quad \text{as } x \to \infty,$$

(1)

where $c$ is a constant, have been established for some classes of Lévy processes (see
Berman, 1986; Willekens, 1987; Marcus, 1987; Rosiński and Samorodnitsky, 1993;
Albin, 1993; Braverman and Samorodnitsky, 1995; Braverman, 1997). Recall that a
Lévy process is a process with stationary independent increments. Most of these results
were proved under additional conditions on the tail of the corresponding Lévy spectral
measure. The main request is that this tail is to be “heavy” (the only exclusion is

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the paper by Albin, 1993). It means that if \( \mu \) is the Le\'vy measure and \( F(x,\infty) = \min\{1, \mu(x,\infty)\} \), then

\[
(F + F)(x,\infty) \sim bF(x,\infty) \quad \text{as } x \to \infty,
\]

where \( b \) is a constant.

So, the natural question arises: does (1) hold without (2)? We consider it for a special class of Lévy processes, namely for compound Poisson processes with (and without) drift. Examples of processes, for which the tails of supremum and of \( X(1) \) are incomparable, are given. We also describe a class of processes such that (1) takes place without (2).

2. Preliminaries

Here we introduce the light tailed distributions and prove some of their properties.

**Definition 1.** The distribution of a random variable \( X \) has a light right tail if one of the following conditions holds:

1. \( P(X > 0) > 0 \) and \( X \leq C \) for some constant \( C \); \( (3) \)

2. \( P(X > x) > 0 \) for all positive \( x \) and \( \lim_{x \to \infty} \frac{P(X_1 > x)}{P(X_1 + X_2 > x)} = 0 \), \( (4) \)

where \( X_1 \) and \( X_2 \) are the independent copies of \( X \).

The following lemma describes a class of light tailed distributions satisfying (4).

**Lemma 1.** Suppose that for a random variable \( X \) and for all \( x > 0 \)

\[
P(X > x) = e^{-\phi(x)},
\]

where the function \( \phi \) satisfies the following condition: there is a constant \( c > 0 \) such that for \( x, y > c \)

\[
\phi(x + y) \geq \phi(x) + \phi(y).
\]

Then the distribution of \( X \) has a light right tail.

**Proof.** Let \( X_1 \) and \( X_2 \) be independent copies of \( X \). Then, denoting by \( F \) the distribution of \( X \), we get for \( x > 2c \)

\[
P(X_1 + X_2 > x) \geq \int_c^{x-c} P(X > x)F(dy) = e^{-\phi(x)} \int_c^{x-c} e^{\phi(x) - \phi(x+y)} d(-e^{-\phi(y)}).
\]

According to \( (6) \) \( \phi(x) - \phi(x+y) \geq \phi(y) \) for \( x > 2c \) and \( c < y < x - c \). From here and \( (5) \)

\[
P(X_1 + X_2 > x) \geq P(X_1 > x) \int_c^{x-c} e^{\phi(x) - \phi(y)} d(-e^{-\phi(y)}) = P(X_1 > x)(\phi(x) - \phi(c)).
\]

It follows from \( (5) \) that \( \phi(x) \to \infty \) as \( x \to \infty \), which gives us \( (4) \). \( \square \)
Applying this lemma, one can conclude that, for example, the normal distribution, Weibull distributions with the parameter \( p > 1 \) and Poisson distribution have a light right tail.

**Lemma 2.** A random variable \( X \) has the distribution with a light right tail if and only if \( X^+ \) has such a distribution.

**Proof.** We have

\[
P(X_1 + X_2 > x) \leq P(X_1^+ + X_2^+ > x)
\]

and

\[
P(X_1 + X_2 > x) \geq P(X_1 + X_2 > x, X_1 \geq 0, X_2 \geq 0)
\]

\[= P(X_1^+ + X_2^+ > x) - 2P(X_1 > x)P(X_2 < 0).\]

So, (4) for \( X \) is equivalent to this condition for \( X^+ \). It is obviously true for (3). \( \square \)

**Lemma 3.** Suppose \( P(X > x) > 0 \) for all \( x > 0 \) and

\[
\lim_{x \to \infty} \frac{P(X > x + a)}{P(X > x)} = 0
\]

for some positive constant \( a \). Then \( X \) has a light right tail.

The proof immediately follows from (7) and the estimate

\[
P(X_1 + X_2 > x) \geq P(X_1 > x - a)P(X_1 > a).
\]

The converse statement is not true, because if \( P(X > x) = 1 - e^{-x} \), then (7) does not hold, while \( X \) has the light right tail.

**Lemma 4.** Let \( X_k \) be iid random variables with the distribution \( F \) and \( S_n = \sum_{k=1}^{n} X_k \). If (4) holds, then

\[
\lim_{x \to \infty} \frac{P(S_n > x)}{P(S_{n+1} > x)} = 0
\]

for each \( n = 1, 2, \ldots \).

**Proof.** Fix \( a > 0 \). Then

\[
P(S_n > x) = \int_{-\infty}^{x-a} P(S_{n-1} > x - y)F(dy) + \int_{x-a}^{\infty} P(S_{n-1} > x - y)F(dy)
\]

\[=: I_{n,a}(x) + J_{n,a}(x).\]

Denoting

\[z_a(x) = \sup_{x \geq a} \frac{P(S_n > x)}{P(S_{n+1} > x)}\]

we get

\[I_{n,a}(x) \leq z_a(x)J_{n+1,a}(x) \leq z_a(x)P(S_{n+1} > x).\]
We may assume \( n \geq 2 \). Then
\[
P(S_{n+1} > x) \geq P(S_n > x - a)P(X_1 + \cdots + X_{n+1} > a),
\]
which together with (4), (9), (10) and the obvious estimate \( J_{n,a}(x) \leq P(X_1 > x - a) \) implies that
\[
\limsup_{x \to \infty} \frac{P(S_n > x)}{P(S_{n+1} > x)} \leq \alpha_n(a)
\]
for every \( a > 0 \). So, letting \( a \to \infty \), we conclude that if (8) holds for \( n - 1 \), then it also holds for \( n \). The induction completes the proof.

\textbf{Corollary 1.} If iid \( X_k \) have a light right tail and \( Z = \sum_{k=1}^N X_k \), where \( N \) is a Poisson random variable independent of \( X_k \) (i.e. if \( Z \) has the corresponding compound Poisson distribution), then for each \( n \) and every \( a > 0 \)
\[
\lim_{x \to \infty} \frac{P(S_n > x)}{P(Z > x + a)} = 0.
\]

The proof follows from Lemma 4 and the estimate
\[
P(Z > x + a) \geq e^{-\hat{\lambda}} \frac{\hat{\lambda}^{n+2}}{(n+2)!} P(X_1 > a)P(S_{n+1} > x),
\]
where \( \hat{\lambda} \) is the parameter of \( Z \).

\textbf{Lemma 5.} Let \( X_k \) be the same as in Lemma 4 and \( Z_1, Z_2 \) have the corresponding compound Poisson distributions with the parameters \( \hat{\lambda}_1 \) and \( \hat{\lambda}_2 \), respectively. If \( \hat{\lambda}_1 < \hat{\lambda}_2 \), then for every fixed \( a > 0 \)
\[
\lim_{x \to \infty} \frac{P(Z_1 > x)}{P(Z_2 > x + a)} = 0.
\]

\textbf{Proof.} Suppose first (4) holds. Then (12) implies
\[
\frac{P(Z_1 > x)}{P(Z_2 > x + a)} = e^{-\hat{\lambda}_1} \sum_{n=1}^{\infty} \frac{\hat{\lambda}_1^n P(S_n > x)}{n! P(Z_2 > x + a)} \\
\leq e^{-\hat{\lambda}_1} \frac{\hat{\lambda}_2^{n+2}}{\lambda_2^2 P(X_1 > a)} \sum_{n=1}^{\infty} \frac{P(S_n > x)}{P(S_{n+1} > x)} (n+1)(n+2) \left( \frac{\hat{\lambda}_1}{\hat{\lambda}_2} \right)^n .
\]

Since \( P(S_{n+1} > x) \geq P(S_n > x)P(X_1 > 0) \), we conclude that the \( n \)th summand in the last series is bounded from above by
\[
b_n := \frac{1}{P(X_1 > 0)} (n+1)(n+2) \left( \frac{\hat{\lambda}_1}{\hat{\lambda}_2} \right)^n .
\]
The condition \( \hat{\lambda}_1 < \hat{\lambda}_2 \) implies \( \sum_{n=1}^{\infty} b_n < \infty \). Now, using (14) and Lemma 4 we get (13). Suppose now (3) holds. Then
\[
P(Z_1 > x) = e^{-\hat{\lambda}_1} \sum_{n \geq x \in C} \frac{\hat{\lambda}_1^n P(S_n > x)}{n!}.
\]
Choose \( m \) under the condition \( P(S_m > a) > 0 \). Then, as above,

\[
\frac{P(Z_1 > x)}{P(Z_2 > x + a)} \leq \frac{e^{\lambda_1 - \lambda_1}}{2^m P(S_m > a)} \sum_{n \geq C} \prod_{j=1}^{m} (n + j) \left( \frac{\lambda_1}{\lambda_2} \right)^n
\]

\[
= \frac{e^{\lambda_1 - \lambda_1}}{2^m P(S_m > a)} \sum_{n \geq x/C} c_n.
\]

Since \( \sum_{n=1}^\infty c_n < \infty \), the last inequality implies (13). \( \square \)

3. The main result

**Theorem 1.** Let \( X(t) \) be a compound Poisson process, i.e.

\[
X(t) = \sum_{k=1}^{N(t)} X_k + bt,
\]

(15)

where \( b \) is a constant, \( N(t) \) is a standard Poisson process with rate \( \lambda \) and \( X_k \) are iid random variables independent of \( N(t) \). Let \( \Gamma_n \) be the arrival times of \( N(t) \) and

\[
T := \max \{ n : \Gamma_n \leq 1 \}.
\]

(16)

Suppose the distribution of \( X_k \) has a light right tail. Then

\[
P \left( \sup_{0 \leq t \leq 1} X(t) > x \right) \sim P(X(\Gamma_T) > x) \quad \text{as } x \to \infty
\]

(17)

if \( b \leq 0 \). If \( b \geq 0 \), then

\[
P \left( \sup_{0 \leq t \leq 1} X(t) > x \right) \sim P(X(1) > x) \quad \text{as } x \to \infty.
\]

(18)

**Remark 1.** If \( b = 0 \), then \( X(\Gamma_T) = X(1) \) and (17) and (18) coincide.

**Remark 2.** Under the additional assumptions \( EX_k^2 < \infty \) and \( EX_k = 0 \), and by a different method, this result was proved in Braverman (1999).

The proof is based on the following statement.

**Lemma 6.** Let \( X(t) \) be defined by (15) and \( b \leq 0 \). Then for all \( x > 0 \)

\[
P \left( \sup_{0 \leq t \leq 1} X(t) > x \right) \leq P(X(\Gamma_T) > x) + \lambda \delta^{-j} \sum_{k=1}^\infty \int_0^1 (e^{\lambda(t-1)} - 1) (\lambda t)^{k-1} \frac{b}{(k-1)!} P(S_k > x - bt) \, dt.
\]

(19)

**Proof.** Since \( b \leq 0 \), we have for \( x > 0 \)

\[
P \left( \sup_{0 \leq t \leq 1} X(t) > x \right) = \sum_{n=1}^\infty P \left( \sup_{1 \leq k \leq n} (S_k + b\Gamma_k) > x, \Gamma_n \leq 1 < \Gamma_{n+1} \right)
\]
\[ \sum_{n=1}^{\infty} \sum_{k=1}^{n} P(S_k + b \Gamma_k > x, \Gamma_n \leq 1 < \Gamma_{n+1}) \]
\[ = \sum_{n=1}^{\infty} P(S_n + b \Gamma_n > x, \Gamma_n \leq 1 < \Gamma_{n+1}) \]
\[ + \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} P(S_k + b \Gamma_k > x, \Gamma_n \leq 1 < \Gamma_{n+1}). \]

The first sum on the right-hand side is \( P(X(\Gamma_T) > x) \). So, changing the order of summation, one get the estimate
\[ P \left( \sup_{0 \leq t \leq 1} X(t) > x \right) \leq P(X(\Gamma_T) > x) \]
\[ + \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} P(S_k + b \Gamma_k > x, \Gamma_n \leq 1 < \Gamma_{n+1}). \quad (20) \]

The well-known formula for the density of \( \Gamma_k \) and elementary calculations give us for \( n > k \)
\[ P(S_k + b \Gamma_k > x, \Gamma_n \leq 1 < \Gamma_{n+1}) = \lambda^n e^{-\lambda} \int_0^1 P(S_k > x - bt) t^{k-1}(1-t)^{n-k} \frac{1}{(k-1)!(n-k)!} dt. \]

Hence,
\[ \sum_{n=k+1}^{\infty} (S_k > x - bt, \Gamma_n \leq 1 < \Gamma_{n+1}) \]
\[ = \lambda^n e^{-\lambda} \int_0^1 \left( \sum_{n=k+1}^{\infty} \frac{(\lambda(1-t))^{n-k}}{(n-k)!} \right) \frac{t^{k-1}}{(k-1)!} P(S_k > x - bt) dt \]
\[ = \lambda^n e^{-\lambda} \int_0^1 (e^{\lambda(1-t)} - 1) \frac{\lambda t^{k-1}}{(k-1)!} P(S_k > x - bt) dt. \quad (21) \]

From here and (20) the lemma follows. \( \square \)

**Proof of Theorem.** \( b \leq 0 \): It is enough to show that the sum on the right-hand side of (19) is \( o(P(X(\Gamma_T) > x)) \) as \( x \to \infty \). To this end we need the following relation, which can be easily verified:
\[ P(X(\Gamma_T) > x) = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \int_0^1 \frac{\lambda t^{k-1}}{(k-1)!} P(S_k > x - bt) dt. \quad (22) \]

Fix \( a \in (0,1) \) and represent the integrals in (19) as the sum of the integrals over \((0,a)\) and \((a,1)\). Denote these integrals by \( U_{k,a}(x) \) and \( V_{k,a}(x) \), respectively, and let \( W(x) \) be the considered sum. Then
\[ W(x) = \lambda e^{-\lambda} \left( \sum_{k=1}^{a} U_{k,a}(x) + \sum_{k=1}^{\infty} V_{k,a}(x) \right). \quad (23) \]
Since $b \leq 0$, then
\[
U_{k,a}(x) \leq (e^i - 1) \int_0^a \frac{(\lambda t)^{k-1}}{(k-1)!} P(S_k > x) \, dt \\
= (e^i - 1) \frac{\lambda^{k-1} a^k}{k!} P(S_k > x).
\]
So,
\[
\sum_{k=1}^{\infty} U_{k,a}(x) \leq \lambda^{-1} (e^i - 1) \sum_{k=1}^{\infty} \frac{(\lambda a)^k}{k!} P(S_k > x) \\
= \lambda^{-1} e^{\lambda a} (e^i - 1) P(Z_{\lambda a} > x),
\]
where $Z_{\lambda a}$ is a compound Poisson random variable with the parameter $\lambda a$.

On the other hand, the formula (22) and the condition $b \leq 0$ imply
\[
P(X(\Gamma_T) > x) \geq \lambda e^{-\lambda} \sum_{k=1}^{\infty} \int_0^1 \frac{(\lambda t)^{k-1}}{(k-1)!} P(S_k > x - bt) \, dt \\
= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} P(S_k > x - bt) = P(Z_{\lambda} > x - b).
\]

From here, (24) and Lemma 5
\[
\sum_{k=1}^{\infty} U_{k,a}(x) = o(P(X(\Gamma_T) > x)).
\]

Turn to the second sum in (23). We have
\[
V_{a,k}(x) = \int_a^1 (e^{(1-t)} - 1) \frac{(\lambda t)^{k-1}}{(k-1)!} P(S_k > x - bt) \, dt \\
\leq (e^{(1-a)} - 1) \int_0^1 \frac{(\lambda t)^{k-1}}{(k-1)!} P(S_k > x - bt) \, dt.
\]

From here and (22)
\[
\sum_{k=1}^{\infty} V_{a,k}(x) \leq \lambda^{-1} e^{\lambda a} (e^i - 1) P(X(\Gamma_T) > x),
\]
which, together with (25) allows us to conclude that
\[
\limsup_{x \to \infty} \frac{W(x)}{P(X(\Gamma_T) > x)} \leq e^{(1-a)} - 1.
\]

Letting $a \to 1$, we get
\[
W(x) = o(P(X(\Gamma_T) > x)),
\]
which proves the theorem for $b \leq 0$.

$b > 0$: Denote
\[
X_{i}(t) = \sum_{k=1}^{N(t)} X_k.
\]
Then \( X(t) = X_1(t) + bt \) and
\[
P(X_1(t) + b > x) = P(X(1) > x) \leq P \left( \sup_{0 \leq t \leq 1} X(t) > x \right)
\]
\[
\leq P \left( \sup_{0 \leq t \leq 1} X_1(t) + b > x \right).
\]

But according to the case \( b = 0 \),
\[
P \left( \sup_{0 \leq t \leq 1} X_1(t) > x \right) \sim P(X_1(t) > x).
\]

Now (18) follows from here and the previous estimate. \( \square \)

4. Examples

We begin with a description of compound Poisson processes with negative drift such that
\[
\limsup_{x \to \infty} \frac{P(\sup_{0 \leq t \leq 1} X(t) > x)}{P(X(1) > x)} = \infty
\]
and
\[
\liminf_{x \to \infty} \frac{P(\sup_{0 \leq t \leq 1} X(t) > x)}{P(X(1) > x)} = 1.
\]

**Theorem 2.** Let \( X_k \) be iid random variables with a lattice distribution, \( X_k \leq C \) for some positive constant \( C \) and \( P(X_k > 0) > 0 \). Let the process \( X(t) \) be defined by (15). Then for every \( b < 0 \) (26) and (27) hold.

**Proof.** We use (17), which allows us to consider \( P(X(t) > x) \) instead of the numerators in (26) and (27). Denote by \( h \) the minimal step of the lattice distribution and put
\[
x_n = nh + b,
\]
where \( b < 0 \). Suppose first that
\[
-b \leq h.
\]
Then \((n - 1)h < x_n - bt < nh\) for every \( 0 < t \leq 1 \), and because the values of the sums \( S_k \) are of the form \( mh, m \in \mathbb{Z} \), we get \( P(S_k > x_n - bt) = P(S_k > (n - 1)h) \). Therefore, according to (22),
\[
P(X(t) > x_n) = P(Z > (n - 1)h),
\]
where
\[
Z = \sum_{k=1}^{N(1)} X_k.
\]
On the other hand,
\[
P(X(1) > x_n) = P(Z > nh),
\]
and
\[
\limsup_{x \to \infty} \frac{P(\sup_{0 \leq t \leq 1} X(t) > x)}{P(X(1) > x)} = \infty
\]
and
\[
\liminf_{x \to \infty} \frac{P(\sup_{0 \leq t \leq 1} X(t) > x)}{P(X(1) > x)} = 1.
\]
and (26) will be proved if we will show that
\[
\limsup_{n \to \infty} \frac{P(Z > (n-1)h)}{P(Z > nh)} = \infty.
\] (32)
Suppose (32) does not hold. Then \(P(Z > (n-1)h) \leq D(Z > nh)\) for all \(n\) and some constant \(D\). Iterating, we get from here
\[
P(Z > nh) \geq D^{-(n-1)} P(Z > h).
\]
Now, using the condition \(X_k \leq C\) and applying Stirling’s formula, we conclude that
\[
P(Z > nh) = e^{-\lambda} \sum_{k=[nh/C]+1}^{\infty} \frac{\lambda^k P(S_k > nh)}{k!} \leq \exp(-c_1 n \log n),
\]
where \(c_1\) is a positive constant. Here, as usual \([x]\) is the integer part of \(x\). The last estimate contradicts the previous one. Hence, (32) is proved.

Turn now to (27). Put
\[
y_n = nh + b + r,
\] (33)
where \(0 < r < -b\). Then, \((n-1)h < y_n - bt < nh\) if \(0 < t < 1 + r/b\) and \(nh < y_n - bt < (n+1)h\) for \(1 + r/b < t < 1\). Denoting \(v = 1 + r/b\), we get from here and (22)
\[
P(X(I_T) > y_n) = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k P(S_k > nh)}{k!} \leq P(Z > (n-1)h)
\]
\[
+ P(Z > nh) - \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} (S_k > nh),
\] (34)
where \(Z\) is the same as above. We have also
\[
P(X(1) > y_n) = P(Z > y_n - b) = P(Z > nh + r) = P(Z > nh),
\]
because \(0 < r < h\). According to Lemma 5 each sum on the right-hand side of (34) is \(o(P(Z > nh))\) and (27) follows.

Suppose now \(-b > h\). Then \(-b = mh + s\), where \(0 < s \leq h\) and \(m \in \mathbb{N}\). Put
\[
a_0 = 0, \quad a_1 = \frac{s}{mh + s}, \quad a_i = a_1 + \frac{(i-1)h}{mh + s}, \quad 2 \leq i \leq m + 1.
\]
Then (22) yields
\[
P(X(I_T) > x) = \lambda e^{-\lambda} \sum_{i=1}^{m+1} \sum_{k=1}^{\infty} \left( \int_{a_{i-1}}^{a_i} (\lambda t)^{k-1} (k-1)! \, dt \right) P(S_k > x_n - ba_{i-1}).
\]
We have \(x_n - ba_m = (n-1)h\) and once more applying Lemma 5 we conclude that
\[
P(X(I_T) > x_n) \sim P(Z > (n-1)h).
\]
Now (26) follows from here, (31) and (32).

Reasoning as above, one can verify that if the sequence \(y_n\) is defined by (33), where \(r > 0\) and small enough, then \(P(X(I_T) > y_n) \sim P(X(1) > y_n)\). So, (27) follows. \(\square\)

Now we show that (1) can hold even in the case of the negative drift and light tail.
Theorem 3. Suppose $X_k$ are iid random variables such that for all $x > 0$
\[ P(X_i > x) = e^{-x + \psi(x)}, \quad (35) \]
where the function $\psi(x)$ is increasing on $(a, \infty)$ for some constant $a > 0$. Let the process $X(t)$ be determined by (15), where $b < 0$. Then
\[ P \left( \sup_{0 \leq t \leq 1} X(t) > x \right) \sim P(X(1) > x). \]

Putting $\phi(x) = x - \psi(x)$, we see that (6) is equivalent to $\psi(x+y) \leq \psi(x) + \psi(y)$ for $x, y$ large enough. So, if the last condition holds, then, according to Lemma 1, the right tail of $X_k$ is light. For example, if $\psi(x) = \beta x^\gamma$ where $0 < \alpha < 1$, $\beta \geq 0$ and $x > c = c(\alpha, \beta)$, then we get a light tailed distribution satisfying the conditions of Theorem 3.

The proof is based on the following auxiliary statement.

Lemma 7. Suppose (35) holds and the function $\psi$ is increasing on $(\bar{a}, \infty)$ for some $a > 0$. Then for every $n > 1$ and all $x > 0$
\[ P(S_n > x) = e^{-x + \psi_n(x)}, \quad (36) \]
where the function $\psi_n(x)$ is increasing on the interval $(na, \infty)$.

Proof. We have
\[ P(S_n > x) = \int_{-\infty}^{\infty} P(S_{n-1} > x - y) F(dy), \quad (37) \]
where $F$ is the distribution of $X_k$. Suppose (36) is proved for $n-1$. Split $(-\infty, \infty)$ into the intervals $(-\infty, x-(n-1)a]$ and $[x-(n-1)a, \infty)$ and denote the corresponding integrals by $I_1(x)$ and $I_2(x)$, respectively. Then, according to (35)
\[ I_1(x) = e^{-x} \int_{-\infty}^{x-(n-1)a} e^{x+y+\psi_{n-1}(x-y)} F(dy). \quad (38) \]
If $y < x-(n-1)a$, then $x-y > (n-1)a$ and, therefore, $\psi_{n-1}(x-y)$ is increasing with respect to $x$ for every such $y$. From here and (38)
\[ I_1(x) = e^{-x+y}\psi_{n,1}(x) \quad (39) \]
for $x > 0$, where the function $\psi_{n,1}(x)$ is increasing on $(0, \infty)$.

Further, according to (35)
\[ I_2(x) = -\int_{x-(n-1)a}^{\infty} P(S_{n-1} > x - y) dy e^{-y+\psi(y)}. \]

Putting $z = x - y$, we get
\[ I_2(x) = e^{-x} \int_{-\infty}^{x-(n-1)a} P(S_{n-1} > z) dz e^{x+\psi(x-z)} := e^{-x} J_2(x). \quad (40) \]

Integrating by parts we obtain
\[ J_2(x) = P(S_{n-1} > (n-1)a) e^{(n-1)a+\psi((n-1)a)} + \int_{-\infty}^{(n-1)a} e^{x+\psi(x-z)} F_{n-1}(dz), \quad (41) \]
where $F_{n-1}$ is the distribution of $S_{n-1}$. The first term on the right-hand side is increasing for $x > na$. Moreover, $\psi(x-z)$ is increasing with respect to $x > na$ for every $z <
\((n-1)a\), because if \(z < (n-1)a\), then \(x - z > na - (n-1)a = a\). So, the last integral is also increasing with respect to \(x > na\). Hence, according to (40) and (41),
\[
I_2(x) = e^{-x + \psi_{n,2}(x)},
\]
where \(\psi_{n,2}(x)\) is increasing on \((na, \infty)\). From here and (39) the lemma follows. \(\square\)

**Proof of Theorem 3.** We have according to (22) and (36)
\[
P(X(G_T) > x) = i e^{-\lambda - x} \int_0^1 e^{bt} \left( \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{\psi_k(x-b)} \right) dt
\]
and
\[
P(X(1) > x) = e^{-\lambda - x + b} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} e^{\psi_k(x-b)}.
\]

It is enough to show that
\[
\limsup_{x \to \infty} \frac{P(X(G_T) > x)}{P(X(1) > x)} \leq 1,
\]
because \(b < 0\) implies \(X(G_T) \geq X(1)\). According to (22),
\[
P(X(G_T) > x)
\]
\[
= \left( \sum_{k=1}^{\lfloor x/a \rfloor} + \sum_{k=\lfloor x/a \rfloor + 1}^{\infty} \right) \left( i e^{-\lambda} \int_0^1 \frac{\lambda^{k-1}}{(k-1)!} P(S_k > x - bt) dt \right) := W_1(x) + W_2(x).
\]

**Estimate for \(W_2(x)\):** Using Stirling’s formula, one gets
\[
W_2(x) \leq e^{-\frac{x}{a}} \sum_{k=\lfloor x/a \rfloor + 1}^{\infty} \frac{\lambda^k}{k!} \leq \frac{\lambda^{|x/a|+1}}{([x/a]+1)!} \leq \exp\left(-c_1 \frac{x}{a} \log \frac{x}{a}\right),
\]
where \(c_1\) is an absolute constant. On the other hand, because \(\psi(x)\) is increasing on \((a, \infty)\),
\[
P(X(1) > x) \geq e^{-\lambda} P(X_1 > x - b) = e^{-\lambda} e^{-x-b+\psi(x-b)} \geq e^{-x-\lambda-b+\psi(a)}
\]
for \(x - b > a\). These estimates imply that
\[
W_2(x) = o(P(X(1) > x)).
\]

**Estimate for \(W_1(x)\):** Fix \(c \in (0, 1)\) and split the interval \((0, 1)\) into \((0, c)\) and \((c, 1)\). Denote
\[
U_c(x) = i e^{-\lambda} \sum_{k=1}^{\lfloor x/a \rfloor} \int_0^c \frac{\lambda^{k-1}}{(k-1)!} P(S_k > x - bt) dt
\]
and
\[
V_c(x) = W_1(x) - U_c(x).
\]

We show first that
\[
U_c(x) = o(X(1) > x).
\]
Since \( b < 0 \), then
\[
U_c(x) \leq e^{-\lambda} \sum_{k=1}^{\infty} \frac{(\lambda c)^k}{k!} P(S_k > x) = e^{-\lambda c} P(Z_c > x),
\]
where \( Z_c \) is a compound Poisson random variable with the parameter \( \lambda c \), corresponding to the sequence \( X_k \). Now (49) follows from here, the relation \( P(X(1) > x) = P(Z_c > x - b) \) and Lemma 5.

Turn now to \( V_c(x) \). Since in this sum \( k \leq x/a \), then Lemma 7 and the condition \( b < 0 \) allow us to conclude that
\[
P(S_k > x - bt) = e^{-x + bt + \psi_b(x - bt)},
\]
where \( \psi_b(x - bt) \) is increasing with respect to \( x - bt \). Hence,
\[
P(S_k > x - bt) \leq e^{-x + bt + \psi_b(x - b)}.
\]

From here, (47) and (48)
\[
V_c(x) \leq \lambda e^{-\lambda} \sum_{k=1}^{\infty} e^{-x + \psi_b(x - b)} \int_c^1 e^{kt} (\lambda t)^{k-1} \frac{1}{(k-1)!} dt.
\]

Since \( b < 0 \), then \( bt < bc \) for \( t > c \) and we get
\[
V_c(x) \leq e^{-\lambda - x + bc} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} e^{\psi_n(x - b)} = e^{bc - b} P(X(1) > x).
\] (50)

The last equality follows from (43).

Now, according to (45)–(50)
\[
\limsup_{x \to \infty} \frac{P(X(I_T) > x)}{P(X(1) > x)} \leq e^{bc - b}
\]
for every \( c \in (0, 1) \). Putting \( c \to 1 \), we get (44). ∎

References


