Convergence of weighted sums of random variables with long-range dependence

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Abstract

Suppose that $f$ is a deterministic function, $\{\xi_n\}_{n \in \mathbb{Z}}$ is a sequence of random variables with long-range dependence and $B_H$ is a fractional Brownian motion (fBm) with index $H \in (\frac{1}{2}, 1)$. In this work, we provide sufficient conditions for the convergence

$$\frac{1}{m^H} \sum_{n=-\infty}^{\infty} f\left( \frac{n}{m} \right) \xi_n \to \int_{\mathbb{R}} f(u) dB_H(u)$$

in distribution, as $m \to \infty$. We also consider two examples. In contrast to the case when the $\xi_n$'s are i.i.d. with finite variance, the limit is not fBm if $f$ is the kernel of the Weierstrass–Mandelbrot process. If however, $f$ is the kernel function from the “moving average” representation of a fBm with index $H'$, then the limit is a fBm with index $H + H' - \frac{1}{2}$. © 2000 Published by Elsevier Science B.V.

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1. Introduction

This paper provides conditions for the series

$$\sum_{n=-\infty}^{\infty} f\left( \frac{n}{m} \right) \xi_n,$$  \hfill (1.1)

normalized by $m^H$, to converge in distribution, as $m$ tends to infinity, to the limit

$$\int_{\mathbb{R}} f(u) dB_H(u),$$ \hfill (1.2)

where $H \in (\frac{1}{2}, 1)$, $f$ is a deterministic function, $\{\xi_n\}_{n \in \mathbb{Z}}$ is a sequence of random variables and $B_H$ is a fractional Brownian motion (fBm) with index $H$. The fBm

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\{B_H(u)\}_{u \in \mathbb{R}}\) with index \(H \in (0, 1)\) is a Gaussian mean-zero process with \(B_H(0) = 0\), stationary increments and self-similar with index \(H(H\text{-ss})\), that is, for \(a > 0\),

\[
\{B_H(au)\}_{u \in \mathbb{R}} \overset{d}{=} \{a^H B_H(u)\}_{u \in \mathbb{R}},
\]

where \(\overset{d}{=}\) means the equality in the sense of the finite-dimensional distributions. If \(EB_H^2(1) = 1\), the fBm \(B_H\) is called standard. We will also say that a complex-valued process is a complex fBm with index \(H\) if its real and imaginary parts are two, possibly dependent, fBm’s with index \(H\) each. It follows from the stationarity of the increments and the self-similarity of a standard fBm \(B_H\) that its covariance function is given by

\[
\Gamma_H(u, v) = EB_H(u)B_H(v) = \frac{1}{4} \left\{ |u|^{2H} + |v|^{2H} - |u - v|^{2H} \right\}, \quad u, v \in \mathbb{R}. \tag{1.4}
\]

FBm with the index \(H = \frac{1}{2}\) is the usual Brownian motion (Bm) which has independent increments. When \(H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)\), however, the increments of fBm are no longer independent: they are positively correlated for \(H \in (\frac{1}{2}, 1)\) and negatively correlated for \(H \in (0, \frac{1}{2})\). Moreover, when \(H \in (\frac{1}{2}, 1)\) the dependence of fBm at long lags is so strong that the series \(\sum_{k=1}^{\infty} |\Gamma_H(1, k)|\) diverges (this follows from the asymptotic relation \(\Gamma_H(1, k) \sim H(2H - 1)|k|^{2H-2}\), as \(k \to \infty\)). It is this range \(H \in (\frac{1}{2}, 1)\) that we focus on in this work.

To make the connection between the series (1.1) and the integral (1.2) more apparent, observe that if \(B_H^m, m \geq 1\), is a family of processes defined by

\[
B_H^m(u) = \begin{cases} 
\frac{1}{m^H} \sum_{j=1}^{[mu]} \xi_j, & u \geq 0, \\
- \frac{1}{m^H} \sum_{j=[mu]+1}^{0} \xi_j, & u < 0,
\end{cases}
\]

then the series (1.1), normalized by \(m^H\), can be expressed as

\[
\int_{\mathbb{R}} f(u) \, dB_H^m(u). \tag{1.6}
\]

We prove, in this paper, that if the finite-dimensional distributions of the processes \(B_H^m\) converge to those of fBm with index \(H \in (\frac{1}{2}, 1)\), then, for suitable functions \(f\), the limit of the normalized series (1.1) (or the integral (1.6)) is (1.2).

When \(\{\xi_n\}_{n \in \mathbb{Z}}\) are i.i.d. random variables in the domain of attraction of an \(\alpha\)-stable random variable with \(\alpha \in (0, 2]\), their properly normalized sums converge to stable Lévy motion (\(\alpha = 2\) corresponds to the Gaussian case and Brownian motion in the limit). Then, for suitable functions \(f\), properly normalized sums (1.1) converge to integrals of functions \(f\) with respect to stable Lévy motion. This was shown in a slightly more general setting by Kasahara and Maejima (1986, 1988) (see also Avram and Taqqu, 1986; Pipiras and Taqqu, 2000a, 2000c). In this work we extend the previous results on convergence of sums (1.1) to the case when random variables \(\{\xi_n\}_{n \in \mathbb{Z}}\) are no longer independent (or weakly dependent) but have instead long-range dependence. As a consequence, the random measure which appears in the limit integral of the sums (1.1) will no longer have independent increments as stable Lévy motion but will have instead increments which are correlated.
One can define long-range dependence either through the “time domain” or the “spectral domain”. An $L^2(\Omega)$-stationary centered sequence $\{\xi_n\}_{n \in \mathbb{Z}}$ is said to be long-range dependent (with index $H$) if its covariance function $r(k) = E\xi_0\xi_k$ satisfies

$$r(k) \sim c_1|k|^{2H-2}, \quad k \to \infty,$$

(1.7)

where $H \in (\frac{1}{2}, 1)$ and $c_1$ is a nonzero constant. This is the “time domain” perspective. In the “spectral domain”, we let $\hat{r}(\lambda) = (\frac{1}{2\pi}) \sum_{k=-\infty}^{\infty} e^{-ik\lambda} r(k)$ denote the corresponding spectral density. Then, under some conditions on $r(k)$ (like monotonicity, see Bingham et al. 1987, p. 240), (1.7) is equivalent to

$$\hat{r}(\lambda) \sim c_2|\lambda|^{-2H+1}, \quad \lambda \to 0,$$

(1.8)

where $c_2$ is a nonzero constant. (In order not to obscure the arguments, we decided not to include slowly varying functions in (1.7) and (1.8). Our results extend easily to this slightly more general case.) It is useful for the sequel to keep in mind both the “time domain” and the “spectral domain” perspectives. Finally, to motivate the normalization $m^H$ of the sum (1.1) and fBm $B_H$ in the limit integral (1.2), we note that $H$ is the appropriate normalization exponent in the Central Limit Theorem for the sequences $\{\xi_n\}_{n \in \mathbb{Z}}$ which have long-range dependence and that the limit of the normalized sums (1.5) is then necessarily a fBm. This is consistent with the case when $\xi_n$ are i.i.d. random variables in the domain of attraction of an $\alpha$-stable random variable: the normalization of their sums is $m^{1/\alpha}$ (times a slowly varying function) and the limit is stable Lévy motion.

The paper is organized as follows. In Section 2, we recall classes of deterministic functions $f$ for which the integral (1.2) with respect to fBm $B_H$, $H \in (\frac{1}{2}, 1)$, is well defined in the $L^2(\Omega)$-sense. These classes then appear in Section 3 where we state the conditions on the function $f$ and the sequence $\{\xi_n\}_{n \in \mathbb{Z}}$ for the normalized sums (1.1) to converge to the integral (1.2). The results on the convergence are proved in Section 4. In Section 5, we consider the first application, which motivated this study. It concerns the asymptotic behavior of Weierstrass–Mandelbrot processes. In the second application, developed in Section 6, we take $f$ to be the kernel function from the “moving average” representation of fBm with another index $H'$.

2. Integration with respect to fBm

The integral (1.2) with respect to fBm $B_H$, $H \in (0, 1)$, is defined for various classes of deterministic functions $f$ in Pipiras and Taqqu (2000b, see also references therein). In this section, for convenience of the reader, we give a quick review of the integration when $H \in (\frac{1}{2}, 1)$, introduce the relevant classes of integrands and state some results which will be used in the following sections.

For notational ease, it is best to define the integral (1.2) by changing the usual $H$ parametrization of fBm $B_H$ to a parametrization $\kappa$, where

$$\kappa = H - \frac{1}{2}.$$  

(2.1)

The range $H \in (0, 1)$ then corresponds to $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ and, in particular, $H \in (\frac{1}{2}, 1)$ corresponds to $\kappa \in (0, \frac{1}{2})$. We will denote fBm $B_H$ in terms of a new parameter $\kappa$ by $B^\kappa$. 


The fractional integral \( I \) function

We extend the definition (2.3) to the class of functions for

\[
\int_{\mathbb{R}} f(u) \, dB^\kappa(u) = \sum_{k=1}^{n} f_k(B^\kappa(u_{k+1}) - B^\kappa(u_k)).
\]

(2.3)

Secondly, one extends this definition to a larger class of integrands. For this, recall that a standard fBm \( \{B^\kappa(t)\}_{t \in \mathbb{R}} \) with parameter \( \kappa \in (-\frac{1}{2}, \frac{1}{2}) \) has the moving average representation

\[
\{B^\kappa(t)\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ \frac{1}{c_1(\kappa)} \int_{\mathbb{R}} ((t - s)^\kappa - (-s)^\kappa) \, dB^p(s) \right\}_{t \in \mathbb{R}},
\]

where \( B^p \) is the usual Bm and

\[
c_1(\kappa)^2 = \int_{0}^{\infty} ((1 + s)^\kappa - s^\kappa)^2 \, ds + \frac{1}{2\kappa + 1}
\]

(see Samorodnitsky and Taqqu, 1994, p. 320). Observe now that for \( \kappa \in (0, \frac{1}{2}) \)

\[
(t - s)^\kappa - (-s)^\kappa = \kappa \int_{0}^{1} (f(u) - s)^{\kappa - 1} \, du = \Gamma(\kappa + 1)(I^\kappa 1_{[0,1]})(s),
\]

(2.5)

where \( \Gamma(p) = \int_{0}^{\infty} e^{-v}v^{p-1} \, dv, \ p > 0 \), is the gamma function and \( I^\kappa_+ \) is the fractional integral operator of order \( \kappa \) on the real line. (Fractional integrals of order \( \alpha > 0 \) of a function \( f \) on the real line are defined by \( (I^\alpha_+ f)(s) = (\Gamma(\alpha))^{-1} \int_{\mathbb{R}} f(u)(u-s)^{\alpha-1} \, ds, s \in \mathbb{R} \). The fractional integral \( I^\kappa_+ \) is called left-sided and the integral \( I^\kappa_- \) is called right-sided. An exhaustive source on fractional integrals and derivatives is the monograph by Samko et al. (1993).) It follows from (2.4) and (2.5) that, for any \( \kappa \in (0, \frac{1}{2}) \) and elementary functions \( f, g \in \mathcal{E} \),

\[
\int_{\mathbb{R}} f(u) \, dB^\kappa(u) \leq \frac{\Gamma(\kappa + 1)}{c_1(\kappa)} \int_{\mathbb{R}} (I^\kappa_+ f)(s) \, dB^p(s)
\]

(2.6)

and hence

\[
E \left( \int_{\mathbb{R}} f(u) \, dB^\kappa(u) \right) \leq \frac{\Gamma(\kappa + 1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} (I^\kappa_+ f)(s)(I^\kappa_+ g)(s) \, ds.
\]

(2.7)

We extend the definition (2.3) to the class of functions

\[
A^\kappa = \left\{ f : \int_{\mathbb{R}} [(I^\kappa_+ f)(s)]^2 \, ds < \infty \right\} = \left\{ f : \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} f(u)(u-s)^{\kappa-1} \, du \right]^2 \, ds < \infty \right\},
\]

(2.8)

for \( \kappa \in (0, \frac{1}{2}) \). It can be shown (see Theorem 3.2 in Pipiras and Taqqu, 2000b) that \( A^\kappa \) is a linear space with the inner product

\[
(f, g)_{A^\kappa} = \frac{\Gamma(\kappa + 1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} (I^\kappa_+ f)(s)(I^\kappa_+ g)(s) \, ds
\]

(2.9)
and that the set of elementary functions $\mathcal{E}$ is dense in the space $\mathcal{A}^\kappa$. The definition (2.3) can then be extended to the class of functions $\mathcal{A}_f^\kappa$ in the usual way as follows.

Let $\mathfrak{F}^\kappa(B^\kappa)$ be the closed linear subspace of $L^2(\Omega)$ spanned by linear combinations of the increments of fBm $B^\kappa$. Then the map $f \mapsto \int_{\mathbb{R}} f(u) dB^\kappa(u)$ from $\mathcal{E}$ into $\mathfrak{F}^\kappa(B^\kappa)$ is linear and preserves inner products. Since $\mathfrak{F}^\kappa(B^\kappa)$ is a Hilbert space and $\mathcal{E}$ is dense in $\mathcal{A}^\kappa$, this map can be extended (so that it exists and is well defined) to the map from $\mathcal{A}^\kappa$ into $\mathfrak{F}^\kappa(B^\kappa)$, which is also linear and preserves inner products. Denote this map by $\mathcal{J}_\kappa$. One now defines the integral of $f \in \mathcal{A}^\kappa$ with respect to fBm $B^\kappa$ by $\mathcal{J}_\kappa(f)$ and uses the notation $\mathcal{J}_\kappa(f) = \int_{\mathbb{R}} f(u) dB^\kappa(u)$.

The space of integrands $\mathcal{A}^\kappa$ has two subspaces which we will use in the sequel. The first subspace is obtained as follows. Let $f \in \mathcal{A}^\kappa$ be such that $|f| \in \mathcal{A}^\kappa$ as well. Then, by using the Fubini’s theorem and the change of variables $s = \min(u,v) - |v-u|(z^{-1} - 1)$ below, we get

\[
\begin{align*}
\Gamma(\kappa)^2 \int_{\mathbb{R}} [(I_\kappa^f)(s)]^2 ds &= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} f(u)(u-s)^{\kappa-1} du \right]^2 ds \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)(u-s)^{\kappa-1} \left[ \int_{\mathbb{R}} (u-s)^{\kappa-1} (v-s)^{-\kappa-1} ds \right] du dv \\
&= \mathcal{B}(\kappa, 1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)(u-v)^{2\kappa-1} du dv,
\end{align*}
\]

where $\mathcal{B}(p,q) = \int_0^1 z^{p-1}(1-z)^{q-1} dz$, $p, q > 0$, is the beta function. Hence, the function space

\[
\mathcal{A}^\kappa = \left\{ f: \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\kappa-1} du dv < \infty \right\},
\]

for $\kappa \in (0, \frac{1}{2})$, is a subspace of $\mathcal{A}^\kappa$. In fact, by Proposition 4.1 in Pipiras and Taqqu (2000b), $|\mathcal{A}^\kappa|$ is a strict subspace of $\mathcal{A}^\kappa$. One can define the norm $\|f\|_{\mathcal{A}^\kappa}$ on $|\mathcal{A}^\kappa|$ by

\[
\|f\|^2_{\mathcal{A}^\kappa} = \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\kappa-1} du dv
\]

(see Pipiras and Taqqu, 2000b), which we will use in the sequel.

The second subspace is obtained as follows. Suppose that $f \in L^2(\mathbb{R})$ and its $L^2(\mathbb{R})$-Fourier transform $\hat{f}(x) = \int_{\mathbb{R}} \frac{x^{ix}}{i^\kappa f(u) du$ satisfies $\int_{\mathbb{R}} |\hat{f}(x)|^2 |x|^{-2\kappa} dx < \infty$. Then, by Proposition 3.3 in Pipiras and Taqqu (2000b), the fractional integral $I_\kappa^x f$ is well defined, belongs to $L^2(\mathbb{R})$ and

\[
I_\kappa^x f(x) = (ix)^{-\kappa} \hat{f}(x).
\]

In particular, $f \in \mathcal{A}^\kappa$. Hence, the function space

\[
\mathcal{A}_f^\kappa = \left\{ f: f \in L^2(\mathbb{R}), \int_{\mathbb{R}} |\hat{f}(x)|^2 |x|^{-2\kappa} dx < \infty \right\},
\]

for $\kappa \in (0, \frac{1}{2})$, is a subspace of $\mathcal{A}^\kappa$. By Proposition 3.4 in Pipiras and Taqqu (2000b), $\mathcal{A}_f^\kappa$ is a strict subspace of $\mathcal{A}^\kappa$. We will use in the sequel the norm $\|f\|_{\mathcal{A}_f^\kappa}$ on $\mathcal{A}_f^\kappa$.
defined by
\[ \| f \|_{H^k}^2 = \int_{\mathbb{R}} \hat{f}(x)^2 |x|^{-2k} \, dx. \]
(2.15)

Note also that, by (2.13) and the Parseval’s equality,
\[ \int_{\mathbb{R}} |\hat{f}(x)|^2 |x|^{-2k} \, dx = \int_{\mathbb{R}} [I^{-k}_f(x)]^2 \, dx = 2\pi \int_{\mathbb{R}} [(I^{-k}_f(x))]^2 \, dx. \]
(2.16)

**Remark.**

1. When \(2(0; \frac{1}{2})\), the following inclusions hold:

\[ L^{2/2k+1}(\mathbb{R}) \subset |A|^k \subset A^k \quad \text{and} \quad L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \subset \tilde{A}^k \subset A^k. \]

Moreover, for some constant \(c_\kappa\), \(\| f \|_{|A|^k} \leq c_\kappa \| f \|_{L^{2/2k+1}(\mathbb{R})}\) for all \(f \in L^{2/2k+1}(\mathbb{R})\) and \(\| f \|_{L^1} \leq c_\kappa \| f \|_{L^2(\mathbb{R})} + c_\kappa \| f \|_{L^1(\mathbb{R})}\) for all \(f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})\). For more details see Sections 3 and Proposition 4.2 in Pipiras and Taqqu (2000b).

2. There are examples of functions \(f\) belonging to \(|A|^\kappa\) that do not belong to \(\tilde{A}^\kappa\) and vice versa. For an example of the first case see Proposition 4.1 in Pipiras and Taqqu (2000b). For an example of the second case, take any function \(f\) which is in \(L^{2/2k+1}(\mathbb{R})\), thus in \(|A|^\kappa\) as well, but not in \(L^2(\mathbb{R})\), and hence not in \(\tilde{A}^\kappa\) either.

3. **Results**

We shall continue to work with the parameterization \(\kappa\) of fBm given by (2.1). The following two theorems provide sufficient conditions on the function \(f\) and on the sequence \(\{\xi_n\}_{n \in \mathbb{Z}}\) for the weak convergence
\[ \frac{1}{m^{\kappa+1/2}} \sum_{n = -\infty}^{n = \infty} f \left( \frac{n}{m} \right) \xi_n \xrightarrow{d} \int_{\mathbb{R}} f(u) \, dB^\kappa(u) \]
(3.1)
to hold for \(\kappa \in (0, \frac{1}{2})\), as \(m \to \infty\). They involve the following notation. \(B^\kappa_m\) is the sequence of processes defined similarly to (1.5), by

\[ B^\kappa_m(u) = \begin{cases} 
\frac{1}{m^{\kappa+1/2}} \sum_{j=1}^{[mu]} \xi_j, & u \geq 0, \\
-\frac{1}{m^{\kappa+1/2}} \sum_{j=[mu]+1}^{0} \xi_j, & u < 0, 
\end{cases} \]

\(B^\kappa\) is a standard fBm with parameter \(\kappa \in (0, \frac{1}{2})\) and \(f\) is a deterministic function on the real line. The integral \(\int_{\mathbb{R}} f(u) \, dB^\kappa(u)\) is then defined as in Section 2. In particular, recall the classes of integrands \(|A|^\kappa\) and \(\tilde{A}^\kappa\), defined by (2.11) and (2.14), respectively, with the norms \(\| f \|_{|A|^\kappa}\) and \(\| f \|_{\tilde{A}^\kappa}\), defined by (2.12) and (2.15), respectively. The space \(|A|^\kappa\) is used when working in the “time domain” and the space \(\tilde{A}^\kappa\) is used in...
Remark. The “spectral domain”. For \( k \in \mathbb{N} \cup \{\infty\} \), we also define the approximations
\[
    f^+_{m,k} = \sum_{n=0}^{k} f\left(\frac{n}{m}\right) 1_{[n/(m+n),1/m)}, \quad f^-_{m,k} = \sum_{n=-k}^{-1} f\left(\frac{n}{m}\right) 1_{[n/(m+n),1/m)},
\]
and in particular, Propositions 5.1 and 6.1 therein. The first theorem involves \( f \).

Theorem 3.1. Let \( \kappa \in (0, \frac{1}{2}) \). Suppose that the following conditions are satisfied:

(i) \( f, f^+_{m} \in \hat{A}^\kappa \), \( \|f^+_{m} - f^{-}_{m}\|_{\hat{A}^\kappa} \to 0 \), as \( k \to \infty \), \( \|f - f_{m}\|_{\hat{A}^\kappa} \to 0 \), as \( m \to \infty \),

(ii) \( \{\xi_n\}_{n \in \mathbb{Z}} \) is an \( L^2(\Omega) \)-stationary sequence of centered random variables such that \( |E\xi_n|^2 \leq c|k|^{2\kappa-1} \), \( k \in \mathbb{N} \), and is such that the sequence of processes \( B^\kappa_{m} \) converges to \( B^\kappa \) in the sense of the finite-dimensional distributions.

Then the series in (3.1) is well defined in the \( L^2(\Omega) \)-sense and the convergence (3.1) holds.

The next theorem involves \( A^\kappa \).

Theorem 3.2. Let \( \kappa \in (0, \frac{1}{2}) \). Suppose that the following conditions are satisfied:

(i) \( f, f^+_{m} \in \hat{A}^\kappa \), \( \|f^+_{m} - f^{-}_{m}\|_{\hat{A}^\kappa} \to 0 \), as \( k \to \infty \), \( \|f - f_{m}\|_{\hat{A}^\kappa} \to 0 \), as \( m \to \infty \),

(ii) \( \{\xi_n\}_{n \in \mathbb{Z}} \) is an \( L^2(\Omega) \)-stationary sequence of centered random variables with spectral density \( \hat{f} \) satisfying \( |\hat{f}(\lambda)| \leq c|\lambda|^{-2\kappa} \), \( \lambda \in [-\pi,\pi] \backslash \{0\} \), and is such that the sequence of processes \( B^\kappa_{m} \) converges to \( B^\kappa \) in the sense of the finite-dimensional distributions.

Then the series in (3.1) is well defined in the \( L^2(\Omega) \)-sense and the convergence (3.1) holds.

Remark.

1. Sequences \( \{\xi_n\}_{n \in \mathbb{Z}} \) satisfying condition (ii) of Theorem 3.1 or Theorem 3.2 can be found, for example, in Davydov (1970) and Taqqu (1975).

2. As stated in Remark 2 at the end of Section 2, there are functions belonging to \( |A|^\kappa \) that do not belong to \( A^\kappa \) and vice versa.

3. The following conditions are often useful in practice. If \( f, f^+_{m} \in L^2(\mathbb{R}) \) and
\[
\|f^+_{m} - f^{-}_{m}\|_{L^2(\mathbb{R})} \to 0, \quad \|f - f_{m}\|_{L^2(\mathbb{R})} \to 0,
\]
then condition (i) of Theorem 3.1 is satisfied by Remark 1 at the end of Section 2. For example, this is the case when \( f \) is a continuous function such that \( \|f(u)\| = O(|u|^{-\lambda}) \) with \( \kappa + 1/2 < \lambda \), as \( |u| \to \infty \) (convergence (3.2) follows from the dominated convergence theorem).
4. Similarly, by Remark 1 at the end of Section 2, if \( f, f_m \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and
\[
\| f_m - f_m^{k} \|_{L^1(\mathbb{R})} \xrightarrow{k \to \infty} 0, \quad \| f - f_m \|_{L^1(\mathbb{R})} \xrightarrow{m \to \infty} 0, \quad i = 1, 2,
\]
then condition (i) of Theorem 3.2 is satisfied. If \( f, f_m^{k} \in \mathcal{A} \), then, by (2.16) and (2.10), the conditions (3.2) of Remark 3 are also sufficient for the condition (i) of Theorem 3.2.

5. If \( \kappa \in (-\frac{1}{2}, 0] \), the space \( \hat{\mathcal{A}}^\kappa \) of functions defined by (2.14) is still a class of integrands for fBm \( \mathcal{B}^{\kappa} \) (see Pipiras and Taqqu, 2000b). It is then easy to verify that Theorem 3.2 and its proof remain valid for \( \kappa \in (-\frac{1}{2}, 0] \).

4. The proofs of Theorems 3.1 and 3.2

For the proofs of Theorems 3.1 and 3.2, we set
\[
X_m = \frac{1}{m^{k+1/2}} \sum_{n = -\infty}^{\infty} f \left( \frac{n}{m} \right) \xi_n, \quad X = \int_{\mathbb{R}} f(u) \, dB^\kappa(u).
\]

4.1. The proof of Theorem 3.1

The integral \( X \) is well defined, since, by the assumption (i), \( f \in |A|^{\kappa} \). Let us show that the series \( X_m \) is defined in the \( L^2(\Omega) \)-sense. We consider only the right tail of the series \( X_m \), since the arguments for the left tail are similar. We have by assumption (ii) that, for some constants \( c \) (which may change from line to line),
\[
E \left| \frac{1}{m^{k+1/2}} \sum_{n = k_1+1}^{k_2} f \left( \frac{n}{m} \right) \xi_n \right|^2 \leq \frac{1}{m^{2k+1}} \sum_{n_1, n_2 = k_1+1}^{k_2} \left| f \left( \frac{n_1}{m} \right) \right| \left| f \left( \frac{n_2}{m} \right) \right| |E\xi_{n_1} \xi_{n_2}|
\]
\[
\leq c \sum_{n_1, n_2 = k_1+1}^{k_2} \left| f \left( \frac{n_1}{m} \right) \right| \left| f \left( \frac{n_2}{m} \right) \right| \left| \frac{n_1}{m} - \frac{n_2}{m} \right|^{2\kappa - 1} \frac{1}{m^2}
\]
\[
\leq c \sum_{n_1, n_2 = k_1+1}^{k_2} \left| f \left( \frac{n_1}{m} \right) \right| \left| f \left( \frac{n_2}{m} \right) \right|
\]
\[
\times \int_{\frac{n_1}{m}}^{\frac{n_1+1}{m}} \int_{\frac{n_2}{m}}^{\frac{n_2+1}{m}} |t - s|^{2\kappa - 1} \, ds \, dt
\]
\[
= c \| f_{m,k_2}^{+} - f_{m,k_1}^{+} \|^2_{\mathcal{A}^{\kappa}}, \quad (4.1)
\]
as \( k_1, k_2 \to \infty \) (when \( n_1 = n_2 \) we suppose in the sum above that 0\(^{2\kappa - 1} = 1 \)).

Let us now prove that \( X_m \) converges to \( X \) in distribution. Since elementary functions are dense in \( \mathcal{A}^{\kappa} \) and since \( |A|^{\kappa} \) is a subspace of \( \mathcal{A}^{\kappa} \), there exists a sequence of elementary functions \( f^j \) such that \( \| f - f^j \|_{\mathcal{A}^{\kappa}} \to 0 \), as \( j \to \infty \) (see Section 2). Set
\[
X_m^j = \frac{1}{m^{k+1/2}} \sum_{n = -\infty}^{\infty} f^j \left( \frac{n}{m} \right) \xi_n, \quad X^j = \int_{\mathbb{R}} f^j(u) \, dB^\kappa(u), \quad (4.2)
\]
which are well defined since the series $X_m^d$ has a finite number of elements and the elementary function $f^j$ is always in $|A|^v$. By Theorem 4.2 in Billingsley (1968), the series $X_m^d$ converges in distribution to $X$ if

**Step 1:** $X^j \xrightarrow{d} X$, as $j \to \infty$.

**Step 2:** for all $j \in \mathbb{N}$, $X_m^d \xrightarrow{d} X^j$, as $m \to \infty$.

**Step 3:** $\limsup_j \limsup_m E|X_m^d - X_m|^2 = 0$.

**Step 1:** The random variables $X^j$ and $X$ are normally distributed with mean zero and variances $\|f^j\|_{A^v}$ and $\|f\|_{A^v}$, respectively (see Section 2 where $A^v$ is defined). As indicated in Section 2, $A^v$ is an inner product space and $|A|^v \subset A^v$. This step follows, since $\|f^j\|_{A^v} - \|f\|_{A^v} \leq \|f^j - f\|_{A^v} \leq c \|f^j - f\|_{|A|^v} \to 0$, as $j \to \infty$.

**Step 2:** Observe that $X_m^d = \int \mathbb{R} f^j(u) \, dB_m^u$. Since $f^j$ is an elementary function, the integral $X_m^d$ depends on the process $B_m^u$ through a finite number of time points only. It then converges in distribution to $X^j$ by assumption (ii) of the theorem.

**Step 3:** As in the inequalities (4.1), we obtain that $E|X_m^d - X_m|^2 \leq c \|f_m^d - f_m\|_{|A|^v}^2$, where

\[ f_m^j(u) = \sum_n f^j \left( \frac{n}{m} \right) 1_{(n/m, (n+1)/m]}/m(u). \]  

(4.3)

Since $f^j$ is an elementary function, for fixed $j$, $f_m^j$ converges to $f^j$ almost everywhere and $|f_m^j - f^j|$ is bounded uniformly in $m$ by $c 1_{(-N, N]}$, for some constants $c$ and $N$. Since $\|1_{(-N, N]}\|_{|A|^v} < \infty$, $\|f_m^j - f^j\|_{|A|^v} \to 0$, as $m \to \infty$, by the dominated convergence theorem. Hence, by assumption (i), $\limsup_m E|X_m^d - X_m|^2 \leq c \|f^j - f\|_{|A|^v}^2$, which tends to 0 as $j \to \infty$.

### 4.2. The Proof of Theorem 3.2

We proceed as in Theorem 3.1. The integral $X$ is well defined, since $f \in A_\infty^v$. To show that the series $X_m$ is defined in the $L^2(\Omega)$-sense, we consider again the right tail of the series $X_m$ only. Since the sequence $\{\xi_n\}_{n \in \mathbb{Z}}$ has the spectral representation

\[ \xi_n = \int_{-\pi}^\pi e^{inx} \, dZ(\lambda), \]

where $Z$ is its spectral random measure, we obtain

\[
\begin{align*}
\frac{1}{m^{k+1/2}} \sum_{n=0}^k f \left( \frac{n}{m} \right) \xi_n &= \frac{1}{m^{k+1/2}} \int_{-\pi}^\pi \left( \sum_{n=0}^k f \left( \frac{n}{m} \right) e^{inx} \right) \, dZ(\lambda) \\
&= \frac{1}{m^{k+1/2}} \int \left( \sum_{n=0}^k f \left( \frac{n}{m} \right) e^{inx/m} \right)^2 1_{\left[-nm, nm\right]}(x) \, dZ \left( \frac{x}{m} \right) \\
&= \frac{1}{m^{k-1/2}} \int \left( \sum_{n=0}^k f \left( \frac{n}{m} \right) e^{ix/m} - e^{-ix/m} \right) \\
&\times \frac{ix/m}{e^{ix/m} - 1} 1_{\left[-nm, nm\right]}(x) \, dZ \left( \frac{x}{m} \right) \\
&= \frac{1}{m^{k+1/2}} \int_{-\pi}^{\pi} f \left( \frac{\lambda}{m} \right) e^{ix/m} - e^{-ix/m} \right) \\
&\times \frac{ix/m}{e^{ix/m} - 1} 1_{\left[-nm, nm\right]}(x) \, dZ \left( \frac{x}{m} \right)
\end{align*}
\]
\( (f^+_{m,k}(x) = \sum_{n=0}^{k} f(n/m) \int_{\mathbb{R}} e^{iux} 1_{[n/m,(n+1)/m]}(u) \, du \) denotes the Fourier transform of \( f^+_{m,k}(u) \)) and, hence by assumption (ii),

\[
E \left| \frac{1}{m^{k+1/2}} \sum_{n=0}^{k} f \left( \frac{n}{m} \right) \varepsilon_n \right|^2 = \int_{\mathbb{R}} \left| \hat{f}^+_{m,k}(x) \right|^2 \left| \frac{ix/m}{\sin x/m} - 1 \right|^2 1_{\left[ -\pi/m, \pi/m \right]}(x) \frac{1}{m^{2k}} \hat{f} \left( \frac{x}{m} \right) \, dx \\
\leq c \int_{\mathbb{R}} \left| \hat{f}^+_{m,k}(x) \right|^2 |x|^{-2k} \, dx \\
= c\|f^+_{m,k}\|_{L^2}^2.
\]

Then, by using assumption (i),

\[
E \left| \frac{1}{m^{k+1/2}} \sum_{n=-k_2+1}^{k_2} f \left( \frac{n}{m} \right) \varepsilon_n \right|^2 \leq c\|f^+_{m,k_2} - f^+_{m,k_1}\|_{L^2}^2 
\]

as \( k_1, k_2 \to \infty \).

We now prove that \( X_m \) converges to \( X \) in distribution. For the same reasons as in the proof of Theorem 3.1, there exists a sequence of elementary functions \( f^j \) such that \( \|f - f^j\|_{L^2} \to 0 \), as \( j \to \infty \). Now let \( X'_m \) and \( X' \) be defined by (4.2). The proof is in three steps as in Section 4.1.

**Step 1:** This step follows as in the proof of Theorem 3.1 by using the convergence \( \|f - f^j\|_{L^2} \to 0 \).

**Step 2:** This step is identical to that of the proof of Theorem 3.1.

**Step 3:** We obtain as above that \( E|X'_m - X_m|^2 \leq c\|f'_m - f_m\|_{L^2}^2 \), where \( f'_m \) is defined by (4.3). Since \( f^j \) is an elementary function, for fixed \( j \), \( f'_m \) converges to \( f^j \) at every point and \( |f'_m(x) - f^j(x)| \leq \hat{g}^j(x) \) uniformly in \( m \), for some function \( \hat{g}^j(x) \) which is bounded by \( c_1 \) and \( c_2|x|^{-1} \), for all \( x \in \mathbb{R} \) and some constants \( c_1 \) and \( c_2 \). To see this, suppose \( f^j = 1_{[a,b)} \). Then, for large enough \( m \), \( f'_m(x) - f^j(x) = (ix)^{-1} \left\{ e^{ix(m+1)} - e^{ixm} \right\} + (ix)^{-1} \left\{ e^{ix(m+1)} - e^{ixb} \right\} \) and these functions have indeed the above uniform bound. If \( \lambda^2 \) is the measure on the real line defined by \( \lambda^2(dx) = |x|^{-2k} \, dx \), then \( \hat{g}^j \in L^2(\mathbb{R}, \lambda^2) \). It follows from the dominated convergence theorem that \( \|f'_m - f^j\|_{L^2}^2 = \|f'_m - f^j\|_{L^2(\mathbb{R}, \lambda^2)}^2 \to 0 \), as \( m \to \infty \). Hence, by the assumption (i), \( \limsup_m E|X'_m - X_m|^2 \leq c\|f' - f\|_{L^2}^2 \) which tends to 0 as \( j \to \infty \).

5. Weierstrass–Mandelbrot-type processes

As mentioned in Section 1, our motivation for this study was understanding the asymptotic behavior of the Weierstrass–Mandelbrot (W–M) process

\[
\sum_{n=1}^{\infty} (e^{inr} - 1)r^{-\left(k'+1/2\right)}r^r e^{i\varepsilon_n + i\eta_n},
\]

(5.1)

indexed by the time parameter \( t \in \mathbb{R} \). Here, \( \{\varepsilon_n\}_{n \in \mathbb{Z}}, \{\eta_n\}_{n \in \mathbb{Z}} \) are two sequences of random variables, and \( r > 1, k' \in (-\frac{1}{2}, \frac{1}{2}) \) are real numbers. In Pipiras and Taqqu (2000a, 2000c) we studied its convergence when the parameter \( r \) tends to 1. If \( \varepsilon_n, \eta_n, \)}
$n \in \mathbb{Z}$, are i.i.d. or, more generally, some weakly dependent random variables with a finite second moment, we showed in Pipiras and Taqqu (2000a) that the W–M process, when normalized by $(1/\log r)^{1/2}$, converges to the complex fBm with parameter $\kappa' \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. When the i.i.d. random variables $\xi_n, \eta_n$ are in the domain of attraction of an $\alpha$-stable random variable, the limit is the complex harmonizable fractional stable motion (Pipiras and Taqqu, 2000c), which is one of many different extensions of the complex fBm to the stable case (see Section 7.8 in Samorodnitsky and Taqqu (1994)). The normalization of (5.1) used in Pipiras and Taqqu (2000c) is $(1/\log r)^{1/2}$.

The process (5.1) is named after Weierstrass and Mandelbrot for the following reasons. First, the series (5.1) without the random components $\xi_n + i\eta_n \in \mathbb{Z}$ is related to the deterministic nowhere differentiable Weierstrass function

$$W^{(0)}(t) = \sum_{n=0}^{\infty} a^n e^{ib^n t}$$

with $a = r^{-\kappa'-1/2}$ and $b = r$. Second, it was Mandelbrot who pointed out the similarities between the Weierstrass function (5.2) and fBm. (For a more detailed discussion on the Weierstrass function and its random modification (5.1) see Mandelbrot (1982, pp. 388–390) or the introduction in Pipiras and Taqqu (2000a).)

In the case considered in Pipiras and Taqqu (2000a), $\xi_n, \eta_n$ are weakly dependent with finite variance, their normalized sums converge to Bm and the limit of the process (5.1), normalized by $(1/\log r)^{1/2}$, is the complex fBm (Pipiras and Taqqu, 2000a). Suppose now that $\xi_n$ and $\eta_n$ are no longer independent or weakly dependent but are long-range dependent (see the introduction for a definition) so that their normalized sums, instead of converging to Bm, converge to fBm. In this case, what does (5.1), adequately normalized, converge to? To complex fBm? What would be its index? These are some of the questions that we want to explore.

We shall study the limit behavior of the normalized W–M process

$$W_{t} = (\log r)^{\kappa+1/2} \sum_{n=-\infty}^{\infty} (e^{i\xi_n t} - 1) r^{-\kappa'+1/2} u^n (\xi_n + i\eta_n),$$

where the sequences $\{\xi_n\}_{n \in \mathbb{Z}}$ and $\{\eta_n\}_{n \in \mathbb{Z}}$ are independent but are each long-range dependent with index $H = \alpha + 1/2 \in (1, 1)$ (as in (1.7) or (1.8)). To motivate the normalization $(1/\log r)^{\kappa+1/2}$, we note again that the exponent $H = \kappa + \frac{1}{2}$ is the appropriate normalization exponent in the Central Limit Theorem for the sequences $\{\xi_n\}_{n \in \mathbb{Z}}$ and $\{\eta_n\}_{n \in \mathbb{Z}}$. This is consistent with the cases considered earlier: if $\xi_n, \eta_n$ are i.i.d. random variables, the normalization is $(1/\log r)^{1/2}$ in the finite-variance case of Pipiras and Taqqu (2000a) and $(1/\log r)^{1/2}$ in the stable case of Pipiras and Taqqu (2000c).

To apply Theorem 3.1 or 3.2 to the process $W_r$, we write $r^n = e^{\alpha \log r}$ in (5.3) and set $\log r = 1/a$. The process $W_{r}$ then becomes

$$W^a_{t} = \frac{1}{a^{\kappa+1/2}} \sum_{n=-\infty}^{\infty} (e^{i\xi_n t} - 1) e^{-(\kappa'+1/2) u^n} (\xi_n + i\eta_n).$$

Observe that as $r$ tends to 1, $a$ tends to infinity. Observe also that the kernel function

$$f_\alpha(u) = (e^{i\alpha u} - 1) e^{-(\kappa'+1/2) u}, \quad u \in \mathbb{R},$$
in (5.4) is such that its real and imaginary parts \( R f_r \) and \( I f_r \), respectively, satisfy conditions of Remarks 3 and 4 in Section 3 for each fixed \( t \in \mathbb{R} \). It follows that, if \( \{ z_n \}_{n \in \mathbb{Z}} \) and \( \{ \eta_n \}_{n \in \mathbb{Z}} \) are two independent sequences of random variables satisfying conditions (ii) of either Theorem 3.1 or 3.2, then \( W^n(t) \) converges in distribution to

\[
W(t) = \int_{\mathbb{R}} \left( e^{i \omega t} - 1 \right) e^{-\left( \kappa + \frac{1}{2} \right) u} \left( u dB_1^f(u) + i dB_2^f(u) \right)
\]  

(5.5)

if \( a \in \mathbb{N} \) tends to infinity, where \( B_1^f \) and \( B_2^f \) are two independent fBm’s with parameter \( \kappa \in (0, \frac{1}{2}) \). The convergence holds also in the sense of the finite-dimensional distributions by considering linear combinations of \( W^n(t) \) at different times \( t \). When \( a \notin \mathbb{N} \) tends to infinity, to show that \( W^n \) converges in the sense of the finite-dimensional distributions to \( W \), it is enough to establish that \( E \left| W^n(t) - W^{[u]}(t) \right|^2 \to 0 \), for each \( t \in \mathbb{R} \) ([ \( \cdot \) ] is the integer part function). This can be done by using the arguments in the proofs of Theorems 3.1 and 3.2. We now conclude that the limit, as \( r \to 1 \), of the normalized \( W-M \) process (5.3) is the process \( W \) in (5.5). Hence, we have

**Proposition 5.1.** Let \( \kappa \in (0, \frac{1}{2}) \) and \( \kappa' \in (-\frac{1}{2}, \frac{1}{2}) \). If \( \{ z_n \}_{n \in \mathbb{Z}} \) and \( \{ \eta_n \}_{n \in \mathbb{Z}} \) are two independent sequences of random variables each satisfying conditions (ii) of either Theorem 3.1 or Theorem 3.2, then

\[
W_r(t) \to W(t), \text{ as } r \to 1,
\]

in the sense of the finite-dimensional distributions, where \( W_r \) and \( W \) are processes defined by (5.3) and (5.5), respectively.

Is the process \( W \) a complex fBm, namely of the form \( B_1^{\kappa''} + i B_2^{\kappa''} \) where \( B_1^{\kappa''} \) and \( B_2^{\kappa''} \) are two real-valued fBm’s with some parameter \( \kappa'' \)? It is easy to verify (see also Pipiras and Taqqu, 2000a) that, if \( \kappa = 0 \), then the process \( W \) in (5.5) is indeed a complex fBm with parameter \( \kappa' \). When, however, as it is the case here, the parameter \( \kappa \neq 0 \), then \( W \) is not a complex fBm. It is still self-similar. Indeed, by replacing \( t \) by \( at \) and making a change of variables, it is easy to see that the process \( W \) is \( (\kappa' + \frac{1}{2}) \)-ss. For \( W \) to be a complex fBm, self-similarity is not enough. It is also necessary that the processes \( \Re W \) and \( \Im W \) have stationary increments. This is not the case and to see why this is so, consider the following heuristic argument. Saying that the processes \( \Re W \) and \( \Im W \) have stationary increments is equivalent to saying that the processes of their derivatives \( \Re W' \) and \( \Im W' \) are stationary. (The derivative \( W' \) is not defined here. To make the arguments rigorous, one can either view it as a generalized process or use an approximation.) When formally computed, the derivative of \( W \) is

\[
W'(t) = \int_{\mathbb{R}} e^{i \omega t} \left( i e^\omega e^{-\left( \kappa' + \frac{1}{2} \right) u} \right) \left( u dB_1^f(u) + i dB_2^f(u) \right),
\]

or, by making a change of variables \( x = e^\omega \),

\[
W'(t) = \int_{0}^{\infty} e^{iu} dZ(x),
\]

(5.6)

where, for \( x > 0 \),

\[
dZ(x) = i x^{-\kappa' - \frac{1}{2}} \left( u dB_1^f(ln x) + i dB_2^f(ln x) \right).
\]

(5.7)
Consider now an approximation of $W_0$ (we use the same notation)

$$W'(t) = \sum_{l=1}^{k} e^{i\omega lt} Z_l;$$

(5.8)

where $0 < x_0 < \cdots < x_k$ and $Z_l = Z(x_l) - Z(x_{l-1})$, for $l = 1, \ldots, k$. We show in Appendix A that for the processes $RW'$ and $IW'$ (with $W'$ as in (5.8)) to be stationary it is necessary that $E R Z_p R Z_q = E I Z_p I Z_q = 0$, for $p \neq q$. The latter conditions are not satisfied if $\kappa \neq 0$ because $R Z$ and $I Z$ do not have orthogonal increments.

Note also that, if $Z$ has orthogonal increments (the case $\kappa = 0$), the process $W$ has an important physical interpretation in the "spectral domain". Its "derivative" (5.6) is an $L^2(\Omega)$-stationary (generalized) process which could be viewed as a linear combination of sinusoids of random but uncorrelated amplitudes. The variable $x$ has then the interpretation of a frequency and $E j d Z(x) j^2$ is known as the spectral measure of the process (5.6). If the random measure $Z$ does not have orthogonal increments (the case $\kappa \neq 0$), the process (5.6) is not $L^2(\Omega)$-stationary and, hence, it does not have a simple "spectral domain" interpretation. In this context, the function $e^{ixt} - 1$, which is used in the kernel of the W–M process, does not characterize anymore stationarity of the increments. In order to have a framework which encompasses both the case where the innovations are i.i.d. and the case where they exhibit long-range dependence, we must switch from the "spectral domain" to the "time domain".

6. "Time domain" approximations

Set $\kappa' \in (-\frac{1}{2}, \frac{1}{2})$. FBm $B^{\kappa'}$ can be represented (in distribution) as

$$B^{\kappa'}(t) = R \int_0^\infty (e^{ixt} - 1) x^{-\kappa' - 1} (dB_1^0(x) + i dB_2^0(x)), $$

where $B_1^0$ and $B_2^0$ are two independent Brownian motions. This is the "spectral domain" representation of fBm. The corresponding "time domain" representation for $B^{\kappa'}$ is

$$\int_{\mathbb{R}} ((t - u)^{\kappa'} - (-u)^{\kappa'}) dB(u),$$

(6.1)

where $B^0$ is Bm. The process (6.1) is fBm with parameter $\kappa'$ given by its moving average representation (see (2.4)). While the real part of the W–M process provides a "spectral domain" approximation of fBm, the corresponding "time domain" approximation is

$$\sum_{n = -\infty}^{\infty} \left( \left( t - \frac{n}{m} \right)^{\kappa'} - \left( -\frac{n}{m} \right)^{\kappa'} \right) \xi_n.$$ 

(6.2)

If $\xi_n$, $n \in \mathbb{Z}$, are i.i.d. random variables with a finite second moment, the convergence scheme described in the introduction by (1.1), (1.5) and (1.6) can be used to show that the process (6.2), normalized by $m^{1/2}$, converges as $m \to \infty$ to the limit (6.1).

We can now ask what happens if the sequence $\xi_n$ in (6.2) exhibits long-range dependence with index $H = \kappa + \frac{1}{2} \in (\frac{1}{2}, 1)$. The following proposition provides the answer.
Proposition 6.1. Let \( \kappa \in (0, \frac{1}{2}) \) and \( \kappa' \in (-\frac{1}{2}, \frac{1}{2}) \) \((\kappa' \neq 0)\) be such that \( \kappa + \kappa' \in (-\frac{1}{2}, \frac{1}{2}) \). Suppose that the sequence \( \{\zeta_n\}_{n \in \mathbb{Z}} \) satisfies condition (ii) of either Theorem 3.1 or 3.2. Then, the processes

\[
\frac{1}{m^{\kappa+1/2}} \sum_{n=-\infty}^{\infty} \left( \left( t - \frac{n}{m} \right)^{\kappa'} - \left( \frac{n}{m} \right)^{\kappa'} \right) \zeta_n, \quad t \in \mathbb{R},
\]

are well defined in the \( L^2(\Omega) \)-sense and converge to fBm with parameter \( \kappa + \kappa' \) in the sense of the finite-dimensional distributions.

Proof. Let us denote the process in (6.3) by \( Y_m(t), \ t \in \mathbb{R} \). We will give the proof of the proposition in two cases: (1) \( \kappa' \in (0, \frac{1}{2}) \), and (2) \( \kappa' \in (-\frac{1}{2}, 0) \).

Case 1. Let \( \kappa' \in (0, \frac{1}{2}) \). It is assumed that \( \kappa + \kappa' < \frac{1}{2} \). Observe that the kernel function

\[
f_i(u) = (t-u)^{\kappa'} - (-u)^{\kappa'}, \quad u \in \mathbb{R},
\]

in (6.3) satisfies the conditions of Remark 3 in Section 3 for all \( t \in \mathbb{R} \). Indeed, the function \( f_i \) is continuous for \( \kappa' > 0 \), \( |f_i(u)| = O(|u|^{\kappa'-1}) \) as \( |u| \to \infty \) and \( \kappa + \frac{1}{2} < 1 - \kappa' \), since \( \kappa + \kappa' < 1/2 \). Hence, \( f_i \) satisfies the condition (i) of Theorem 3.1. If the sequence \( \{\zeta_n\}_{n \in \mathbb{Z}} \) satisfies condition (ii) of Theorem 3.1, then \( Y_m(t) \) converges in distribution to

\[
Y(t) = \int_{\mathbb{R}} f_i(u) \, dB^\kappa(u) = \int_{\mathbb{R}} ((t-u)^{\kappa'} - (-u)^{\kappa'}) \, dB^\kappa(u),
\]

for each \( t \in \mathbb{R} \). \( Y_m \) converges to \( Y \) in the sense of the finite-dimensional distributions as well by considering linear combinations at different times. Let us now identify the process \( Y \). It follows from (2.6) that the process \( Y \) has the same finite-dimensional distributions (up to a constant) as the process \( \int_{\mathbb{R}} (I^{-1}_- f_i)(s) \, dB^\kappa(u) \). Since, for \( \kappa > 0 \) and \( s \in \mathbb{R} \),

\[
\Gamma(\kappa)(I^{-1}_- f_i)(s) = \int_{\mathbb{R}} ((t-u)^{\kappa'} - (-u)^{\kappa'}) (u-s)^{\kappa-1} \, du
\]

\[
= \int_{\mathbb{R}} (t-u)^{\kappa'} (u-s)^{\kappa-1} \, du - \int_{\mathbb{R}} (-u)^{\kappa'} (u-s)^{\kappa-1} \, du
\]

\[
= \mathcal{B}(\kappa'+1, \kappa)(t^{-\kappa'} - s^{-\kappa'}),
\]

the process \( Y \) is, in fact, a fBm with parameter \( \kappa + \kappa' \) (see (2.4)).

Assume now that the sequence \( \{\zeta_n\}_{n \in \mathbb{Z}} \) satisfies the condition (ii) of Theorem 3.2. To prove that the processes \( Y_m \) also converge in the finite-dimensional distributions to a fBm with parameter \( \kappa + \kappa' \), it is sufficient to prove that the kernel function \( f_i \) in (6.4) satisfies the condition (i) of Theorem 3.2. Since \( f_i \) was already shown to satisfy the condition (i) of Theorem 3.1 and since \( \|f\|_{\mathcal{H}} = c \|f\|_{\mathcal{A}} \leq c \|f\|_{\mathcal{A}'} \) for all \( f \in \mathcal{A}^\kappa \), it is enough to show that the functions \( f_i, (f_i)_m^+ \) and \( (f_i)_m^- \) belong to \( \mathcal{A}^{\kappa'} \) for every \( t \in \mathbb{R} \).

We can assume without loss of generality that \( t = 1 \) and denote \( f_1 \) by \( f \). By (2.5), \( f = \Gamma(\kappa+1)I^{-1}_0 1_{[0,1)} \) and hence, by Proposition 3.3 in Pipiras and Taqqu (2000b),

\[
\hat{f}(x) = \Gamma(\kappa'+1)(ix)^{-\kappa'} 1_{[0,1)}(x) = \Gamma(\kappa'+1)(ix)^{-\kappa'} e^{ix} - \frac{1}{ix}.
\]
Since $\int_{\mathbb{R}} |(e^{ix}-1)ix|^2 |x|^{-2\kappa} |x|^{-2\kappa} \, dx < \infty$ when $\kappa \in (0, \frac{1}{2})$ and $\kappa + \kappa' < \frac{1}{2}$, the function $f$ is in $L^\kappa$. The fact that $f_m^+$ belongs to $L^\kappa$ is obvious since $f_m^+$ is an elementary function and an elementary function is always in $L^\kappa$. The case of $f_m^-$ is more delicate because it does not have a compact support. Let us show that $f_m^+ \in L^\kappa$ for $m = 1$ only. When $m \geq 2$, the proof is similar. Observe that

$$f_1^-(u) = \sum_{n=-\infty}^{-1} ((1-n)^{\kappa'} - (-n)^{\kappa'}) \delta_1(u) = \sum_{n=1}^{\infty} ((1+n)^{\kappa'} - n^{\kappa'}) \delta_{n-1}(u).$$

To compute the Fourier transform of $f_1^-$, we use its approximations $f_{1,k}^-$, where

$$f_{1,k}^-(u) = \sum_{n=-k}^{-1} ((1-n)^{\kappa'} - (-n)^{\kappa'}) \delta_1(u) = \sum_{n=1}^{k} ((1+n)^{\kappa'} - n^{\kappa'}) \delta_{n-1}(u).$$

Since $f_{1,k}^- \to f_1^-$ in $L^2(\mathbb{R})$, by the Parseval’s equality, $f_{1,k}^- \to \hat{f}_1^-$ in $L^2(\mathbb{R})$ as well. We have that

$$\hat{f}_{1,k}(x) = \sum_{n=1}^{k} ((1+n)^{\kappa'} - n^{\kappa'}) \frac{e^{ixn} - e^{-ix(n-1)}}{ix} = \frac{e^{ix} - 1}{ix} \sum_{n=1}^{k} ((1+n)^{\kappa'} - n^{\kappa'}) e^{-inx}.$$

By Theorem 2.6 in Zygmund (1979, p. 4), $\hat{f}_{1,k}^-$ converges (except at points $x = 2\pi l$, $l \in \mathbb{Z}$) to

$$\frac{e^{ix} - 1}{ix} \sum_{n=1}^{\infty} ((1+n)^{\kappa'} - n^{\kappa'}) e^{-inx},$$

which necessarily is $\hat{f}_1^-$ a.e. We want to show that $\int_{\mathbb{R}} |\hat{f}_1^-(x)|^2 |x|^{-2\kappa} \, dx < \infty$. Write the limit $\hat{f}_1^-$ as

$$\hat{f}_1^-(x) = \frac{e^{ix} - 1}{ix} \sum_{n=1}^{\infty} n^{\kappa'-1} b_n e^{-inx},$$

where $b_n = b(n)$ and $b(u) = ((1+u)^{\kappa'} - u^{\kappa'})/u^{\kappa'-1}, u > 0$. We first analyze the behavior of the left-hand side as $x \to 0$. One can verify that the function $b$ is slowly varying in the sense of Zygmund (1979, p. 186). For this, it is easiest to use Theorem 1.5.5 in Bingham et al. (1987) (by (1.3.6) of Bingham et al. (1987), this amounts to verifying that the function $\varepsilon(u) = (\ln b(u))'u$ tends to 0 as $u$ tends to infinity). Theorem 2.6 in Zygmund (1979, p. 185), now applies that $|\hat{f}_1^-(x)|$ behaves (up to a constant) like $|x|^{-\kappa'}$, as $x \to 0$. Moreover, observe that the function $\hat{h}(x) = ix \hat{f}_1^-(x) = (e^{ix} - 1) \sum_{n=1}^{\infty} n^{\kappa'-1} b_n e^{-inx}$ is bounded on $\mathbb{R}$. Indeed, since $\hat{h}$ is periodic on $\mathbb{R}$, it is enough to consider it on an interval $[0,2\pi]$. When $x \in [\varepsilon, 2\pi - \varepsilon]$ with arbitrary small $\varepsilon > 0$, the function $\hat{h}(x)$ is a uniform limit of continuous functions (Theorem 2.6 in Zygmund (1979, p. 4)). Hence, $\hat{h}(x)$ is continuous on $(0,2\pi)$. As $x \to 0$, $|\hat{h}(x)|$ behaves like $|x| |x|^{-\kappa'}$ which tends to 0. It tends to 0 at the same rate as $x \to 2\pi$. Hence, $\int_{\mathbb{R}} |\hat{f}_1^-(x)|^2 |x|^{-2\kappa} \, dx < \infty$ when $\kappa \in (0, \frac{1}{2})$ and $\kappa + \kappa' < \frac{1}{2}$. In other words, $f_1^- \in L^\kappa$. Case 2. Let $\kappa' \in (-\frac{1}{2}, 0)$ and recall that $\kappa + \kappa' \in (-\frac{1}{2}, \frac{1}{2})$. One can easily verify that the kernel function $f_t^-$ in (6.4) belongs to $L^{2\kappa+1}(\mathbb{R})$ for each $t \in \mathbb{R}$. Observe,
however, that $f_t(u)$ explodes around $u = t$ and $0$ (for the shape of $f_t$ see Fig. 7.3 in Samorodnitsky and Taqqu (1994, p. 324)). The divergence around $u = t$ causes a problem. Since the approximation $(f_t)_m$ involves the interval $[[mt]/m, ([mt] + 1)/m)$ which includes the point $t$, we may not be able to apply the dominated convergence theorem to show that $(f_t)_m \to f_t$ in $L^{2/2k+1}(\mathbb{R})$ for all $t \in \mathbb{R}$. The way out is to redefine the approximation $(f_t)_m$ as follows:

$$(f_t)_m(u) = \sum_{n \neq [mt]} f_t \left( \frac{n}{m} \right) 1_{[n/(n+1)/m)}(u) + f_t \left( \frac{[mt]}{m} \right) 1_{[[mt],m)}(u)$$

and

$$(f_t)_{m,k}^+ = (f_t)_m 1_{[0,\infty)}, \quad (f_t)_{m,k}^- = (f_t)_m 1_{[0,(k+1)/m)},$$

$$(f_t)_{m,k}^- = (f_t)_m 1_{(-\infty,0)}, \quad (f_t)_{m,k}^- = (f_t)_m 1_{(-k/m,0)}.$$

It is clear that this modification does not affect the proofs of Theorems 3.1 and 3.2 and hence, if the functions $f_t$, $t \in \mathbb{R}$, and the above approximations satisfy condition (i) of either of the two theorems, and, if the sequence $\{ \xi_n \}_{n \in \mathbb{Z}}$ satisfies condition (ii) of either of the two theorems, then the convergence (3.1) holds with $f = f_t$ in the sense of the finite-dimensional distributions. Now, by the dominated convergence theorem, $(f_t)_m^+ \to (f_t)_m^-$ as $k \to \infty$, and $(f_t)_m \to f_t$, as $m \to \infty$, in $L^{2/2k+1}(\mathbb{R})$ for all $t \in (\mathbb{R})$.

It follows from Remark 3 in Section 3 that, if $\{ \xi_n \}_{n \in \mathbb{Z}}$ satisfies condition (ii) of Theorem 3.1, then the process $Y_m$ in (6.3) converges in the sense of the finite-dimensional distributions to the process $Y$ in (6.5). The process $Y$ is again a fBm with parameter $\kappa + \kappa'$.

Suppose now that the sequence $\{ \xi_n \}_{n \in \mathbb{Z}}$ satisfies the condition (ii) of Theorem 3.2. To be able to apply this theorem when $\kappa' \in (-\frac{1}{2}, 0)$, we have to verify again that $f_t$, $(f_t)_m^+$ and $(f_t)_m^-$ belong to $\mathcal{A}_\kappa$ for every $t \in \mathbb{R}$. By Lemma 3.1 in Pipiras and Taqqu (2000b), $f_t = \Gamma(\kappa' + 1) \mathbf{D}^\kappa \mathbf{e}^{ix}$, where $\mathbf{D}^\kappa$ is the so-called fractional Marchaud derivative on the real line of order $(-\kappa')$. (For more information on fractional Marchaud derivatives, see Samko et al., 1993). By Lemma 5.2 in Pipiras and Taqqu (2000b), $1_{[0,t)} = \mathbf{L}^{-\kappa'} f_{0,t}$, where $f_{0,t}(u) = (\Gamma(1 + \kappa'))^{-1}((t - u)^{\kappa'} - (-u)^{\kappa'})$ belongs to $L^2(\mathbb{R})$ when $\kappa' \in (-\frac{1}{2}, 0)$. Using Proposition 3.3 in Pipiras and Taqqu (2000b), we have

$$(f_t)_m(x) = \Gamma(\kappa' + 1)(ix)^{-\kappa'} 1_{[0,t)}(x) = \Gamma(\kappa' + 1)(ix)^{-\kappa'} e^{ixt} - \frac{1}{ix}$$

and hence $\int_{\mathbb{R}} |(f_t)_m(x)|^2 dx < \infty$ when $\kappa + \kappa' \in (-\frac{1}{2}, 1)$. The fact that $(f_t)_m^+$ belongs to $\mathcal{A}_\kappa$ follows since $(f_t)_m^+$ is an elementary function. The function $(f_t)_m^-$, on the other hand, does not have a bounded support. Nevertheless, $(f_t)_m^- \in \mathcal{A}_\kappa$ by Remark 2 in Section 2, since $(f_t)_m^- \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ when $\kappa' \in (-\frac{1}{2}, 0)$. \Box

**Remark.** If $\kappa = H - \frac{1}{2}$ and $\kappa' = H' - \frac{1}{2}$, then the limiting fBm in Proposition 6.1 is self-similar with exponent $\kappa + \kappa' + \frac{1}{2} = H + H' - \frac{1}{2}$, as stated in the abstract.
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Appendix A \(L^2\)-stationarity of a complex-valued process

The following lemma, which we referred to in Section 5, provides necessary and sufficient conditions on complex-valued random variables \(Z_l; l = 1, \ldots, k\), for the \(L^2(\Omega)\)-stationarity of \(X_t = \sum_{l=1}^{k} e^{itl}Z_l\), that is, for the \(L^2(\Omega)\)-stationarity of its real and imaginary parts \(\Re X_t\) and \(\Im X_t\), respectively.

**Lemma A.1.** Let \(0 < x_1 < \cdots < x_k\) be real numbers. Assume that \(Z_l; l = 1, \ldots, k\), are complex-valued random variables with zero mean and finite second moments \(E|Z_l|^2\). For \(t \in \mathbb{R}\), set \(X_t = \sum_{l=1}^{k} e^{itl}Z_l\). Then the processes \(\Re X\) and \(\Im X\) are \(L^2(\Omega)\)-stationary if and only if

\[
EZ_pZ_q = 0, \text{ for all } p, q, \text{ and } EZ_p\hat{Z}_q = 0, \text{ for all } p \neq q, \quad (A.1)
\]

or equivalently, if and only if

\[
E\Re Z_p\Re Z_q = E\Im Z_p\Im Z_q = 0 \quad \text{for } p \neq q, \quad E\Re Z_p\Im Z_q = E\Im Z_p\Re Z_q \quad \text{for all } p, q. \quad (A.2)
\]

**Proof.** Assume first that the conditions (A.1) of the lemma are satisfied. Let us show, for example, that the process \(\Re X\) is \(L^2(\Omega)\)-stationary. Since \(E\Re X_t\Re X_{t+h}\) does not depend on \(t \in \mathbb{R}\). Since \(\Re X = (X + \hat{X})/2\), we obtain

\[
E\Re X_t\Re X_{t+h} = \frac{1}{4}(EX_tX_{t+h} + EX_t\hat{X}_{t+h} + E\hat{X}_tX_{t+h} + E\hat{X}_t\hat{X}_{t+h})
\]

\[
= \frac{1}{4} \left( \sum_{p,q} e^{ix_pit} e^{ix_q(t+h)} EZ_pZ_q + \sum_{p,q} e^{ix_pit} e^{-ix_q(t+h)} EZ_p\hat{Z}_q \right. 
\]

\[
+ \sum_{p,q} e^{-ix_pit} e^{ix_q(t+h)} E\hat{Z}_pZ_q + \sum_{p,q} e^{-ix_pit} e^{-ix_q(t+h)} E\hat{Z}_p\hat{Z}_q \right) \quad (A.3)
\]

Then, by (A.1), \(E\Re X_t\Re X_{t+h} = (1/4) \sum_p (e^{ix_pit} + e^{-ix_pit}) EZ_p\hat{Z}_p\), which is independent of \(t\).

To prove the converse implication, assume, for example, that the process \(\Re X\) is \(L^2(\Omega)\)-stationary. Since \(E\Re X_t\Re X_{t+h}\) is independent of \(t\), its derivative with respect to \(t\) is zero. We then obtain from (A.3) that, for all \(t, h \in \mathbb{R}\),

\[
\sum_{p,q} (x_p + x_q)e^{it(x_p+x_q)} e^{ix_pit} EZ_pZ_q + \sum_{p,q} (x_p - x_q)e^{it(x_p-x_q)} e^{-ix_pit} EZ_p\hat{Z}_q 
\]

\[
+ \sum_{p \neq q} (x_p + x_q)e^{it(x_p+x_q)} e^{ix_pit} E\hat{Z}_pZ_q 
\]

\[
+ \sum_{p \neq q} (x_p - x_q)e^{it(x_p-x_q)} e^{-ix_pit} E\hat{Z}_p\hat{Z}_q = 0.
\]
It is an easy exercise to show that the complex-valued functions $e^{iu_1t}, \ldots, e^{iu_kt}$ of $t \in \mathbb{R}$ are linearly independent for real numbers $u_1 < u_2 < \cdots < u_k$. This fact implies that, for all $q$ and $t \in \mathbb{R}$,

$$
\sum_{p} (x_p + x_q) E Z_p Z_q e^{i(x_p + x_q) t} + \sum_{p \neq q} (-x_p + x_q) E Z_p Z_q e^{i(-x_p + x_q) t} = 0,
$$

$$
\sum_{p \neq q} (x_p - x_q) E Z_p \tilde{Z}_q e^{i(x_p - x_q) t} + \sum_{p} (-x_p - x_q) E Z_p \tilde{Z}_q e^{i(-x_p - x_q) t} = 0.
$$

Since $0 < x_1 < \cdots < x_k$, the exponents $\pm x_p + x_q$, $p = 1, \ldots, k$, are all different. By using the above fact on linear independence, we obtain the conditions (A.1).

The equivalence of the conditions (A.1) and (A.2) follows from

$$
E Z_p Z_q = E \Re Z_p \Re Z_q - E \Im Z_p \Im Z_q + i (E \Re Z_p \Im Z_q + E \Im Z_p \Re Z_q),
$$

$$
E Z_p \tilde{Z}_p = E \Re Z_p \Re Z_q + E \Im Z_p \Im Z_q + i (-E \Re Z_p \Im Z_q + E \Im Z_p \Re Z_q).
$$

References


