The logarithmic average of sample extremes is asymptotically normal

István Berkes\textsuperscript{a,1}, Lajos Horváth\textsuperscript{b,*}

\textsuperscript{a}Mathematical Institute of the Hungarian Academy of Sciences, P.O. Box 127, H-1364 Budapest, Hungary
\textsuperscript{b}Department of Mathematics, University of Utah, 155 South 1440 East, Salt Lake City, UT 84112-0090, USA

Received 15 November 1999; received in revised form 18 May 2000; accepted 22 May 2000

Abstract

We obtain a strong approximation for the logarithmic average of sample extremes. The central limit theorem and laws of the iterated logarithm are immediate consequences. © 2001 Elsevier Science B.V. All rights reserved.

MSC: primary 60F17; secondary 60F05

Keywords: Extreme value; Domain of attraction; Wiener process; Strong approximation; Laws of the iterated logarithm

1. Introduction and results

Let $X_1, X_2, \ldots$ be independent identically distributed random variables with distribution function $F$. Several authors studied the asymptotic properties of

$$ T_n = \sum_{1 \leq i \leq n} \frac{1}{i} h \left( \left( \sum_{1 \leq j \leq n} X_j - a^* (i) \right) / b^* (i) \right). $$

Brosamler (1988), Schatte (1990) and Lacey and Philipp (1990) obtained the first laws of large numbers for $T_n / \log n$ in the case when the $X_i$'s have finite second moments. For extensions for the non-i.i.d. case and unbounded $h$, see Berkes and Dehling (1993), Berkes et al. (1998a) and Ibragimov and Lifshits (1998, 1999). If $h$ is the indicator of $(-\infty, x]$, Weigl (1989) and Csörgő and Horváth (1992) established the asymptotic normality of $(T_n - ET_n) / (\log n)^{1/2}$. For refinements we refer to Horváth and Khoshnevisan (1996), Berkes et al. (1998b) and Berkes and Horváth (1997). A detailed survey on almost sure limit theorems can be found in Berkes (1998).

* Corresponding author. Tel.: +1-801-581-8159; fax: +1-801-581-4148.
E-mail address: horvath@math.utah.edu (L. Horváth).
1 Research supported by Hungarian National Foundation for Scientific Research Grant T 19346.
In this paper we investigate the asymptotic properties of logarithmic averages of maxima. Let
\[
\xi_k = \left( \max_{1 \leq i \leq k} X_i - a(k) \right) / b(k)
\]
and define
\[
S(n) = \sum_{1 \leq i \leq n} \frac{1}{i} h(\xi_i).
\]
Throughout this paper we assume that there is a non-degenerate distribution function \( H \) such that
\[
\lim_{n \to \infty} P\{\xi_n \leq t\} = H(t).
\] (1.1)
Fisher and Tippett (1928) and Gnedenko (1943) showed that all distribution functions \( H \) arising as limits in (1.1) must be one of the following types:
\[
\phi_2(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
\exp(-x^z) & \text{if } x > 0,
\end{cases}
\]
and
\[
\psi_2(x) = \begin{cases} 
\exp(-(x)^y) & \text{if } x \leq 0, \\
1 & \text{if } x > 0
\end{cases}
\]
and
\[
A(x) = \exp(-e^{-x}), \quad -\infty < x < \infty,
\]
where \( z > 0 \). We say that \( F \) is in the domain of attraction of \( H \) \((F \in \mathcal{D}(H))\), if (1.1) holds. If \( F \in \mathcal{D}(\phi_2) \) then we may choose
\[
a(k) = 0, \quad b(k) = \inf \{ x : 1 - F(x) \leq 1/k \}.
\] (1.2)
If \( F \in \mathcal{D}(\psi_2) \), then
\[
a(k) = a_+ = \sup \{ x : F(x) < 1 \} < \infty,
\]
\[
b(k) = \sup \{ x : 1 - F(a_+ - x) \leq 1/k \}
\] (1.3)
and if \( F \in \mathcal{D}(A) \), then we use
\[
a(k) = U(\log k), \quad b(k) = U(1 + \log k) - U(\log k),
\] (1.4)
where \( U(t) \) denotes the (generalized) inverse of \(-\log(1 - F(x))\). For further properties and applications of extreme values we refer to de Haan (1970) and Galambos (1978).

The first almost sure max-limit theorems were established by Fahrner and Stadtmüller (1998) and Cheng et al. (1998). They proved that if \( h \) is an almost everywhere continuous, bounded function and (1.1) holds, then
\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{1 \leq i \leq n} \frac{1}{i} h(\xi_i) = \int_{-\infty}^{\infty} h(t) dH(t) \quad \text{a.s.}
\]
In this paper we provide an almost sure approximation for \( S(n) \). The asymptotic variance of \( S(n) \) will depend on the type of the limit distribution function in (1.1). Let
\[
\sigma^2 = 2 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} h(s) h(t) \phi_2(s) \phi_2(t) (1 - e^{-x})^{-1/2} dH(t) \quad \text{a.s.}
\]
\[
\times \varphi_s(s(1 - e^{-x})^{-1/2} - \varphi_x(s)) \, dt \, ds \, dx
\]
\[+ 2 \int_0^\infty \int_0^0 h(s)h(s e^{x^2})\Phi_x(s) \left( s^2 - 1 \right) \varphi_x(s) \, ds \, dx,
\] (1.5)

if \( F \in \mathcal{D}(\Phi_x), \) where \( \varphi_x = \Phi_x' , \)

\[
\sigma^2 = 2 \int_0^\infty \int_0^0 h(s)h(s e^{x^2})\Psi_x(s) \left( s^2 - 1 \right) \varphi_x(s) \, ds \, dx,
\] (1.6)

if \( F \in \mathcal{D}(\Psi_x), \) where \( \psi_x = \Psi_x' , \) and

\[
\sigma^2 = 2 \int_0^\infty \int_0^\infty \int_0^\infty h(s)h(s + x)\lambda(s) \left( I \{ t \leq s + x \} \lambda(s + x) - \lambda(s) \right) \, ds \, dx,
\] (1.7)

if \( F \in \mathcal{D}(A), \) where \( \lambda = A'. \)

Our main result is the following strong approximation:

**Theorem 1.1.** We assume that \( h \) is of bounded variation and has compact support. If (1.1) holds, then there is a Wiener process \( \{ \hat{W}(t), \, 0 \leq t \leq \infty \} \) and a positive numerical sequence \( \tau_n \) such that

\[
S(n) - ES(n) - W(\tau_n) = o(\tau_n^{\frac{1}{2} - \varepsilon}) \quad \text{a.s.}
\] (1.8)

with some \( \varepsilon > 0 \) and

\[
\lim_{n \to \infty} \frac{\tau_n}{\log n} = \sigma^2,
\] (1.9)

where \( \sigma^2 \) is defined by (1.5)–(1.7).

The weak convergence of \( S(n'), \, 0 \leq t \leq 1 \) and the laws of the iterated logarithm are immediate consequences of Theorem 1.1. Namely, under the conditions of Theorem 1.1 we have

\[
(\log n)^{-1/2} \{ S(n') - ES(n') \} \overset{\mathcal{D}(\mathbb{R}^1)}{\to} \sigma W(t),
\]

\[
\limsup_{n \to \infty} (2 \log n \log \log \log n)^{-1/2} \{ S(n) - ES(n) \} = \sigma \quad \text{a.s.}
\]

and

\[
\liminf_{n \to \infty} \left( \frac{\log \log \log n}{\log n} \right)^{1/2} \max_{1 \leq k \leq n} |S(k) - ES(k)| = \left( \frac{\sigma^2 \pi^2}{8} \right)^{1/2} \quad \text{a.s.}
\]

**Remark 1.1.** In the proof of Theorem 1.1 we will show that (1.8) holds with any \( 0 < \varepsilon < 1/24. \)
Remark 1.2. The norming and centering sequences in (1.2)–(1.4) are used in the proof of Theorem 1.1. However, these special choices of \(a(k)\) and \(b(k)\) are unimportant. If (1.1) holds then the norming and centering sequences are equivalent with \(a(k)\) and \(b(k)\) in (1.2)–(1.4) (cf. Bingham et al. (1987)), so Theorem 1.1 remains true for any choices of \(a(k)\) and \(b(k)\) as long as (1.1) holds.

2. Computation of the variance

In this section we show that

\[
\lim_{n \to \infty} \frac{1}{\log n} \text{var} S(n) = \sigma^2,
\]

where \(\sigma^2\) is defined by (1.5), (1.6), or (1.7) depending on which domain of attraction \(F\) belongs to. The proof of (2.1) will be done in two steps. First, we establish (2.1) in three special cases and then we show that (2.1) holds if \(F\) is in the domain of attraction of an extreme value distribution.

Since \(h\) is bounded, by the Cauchy–Schwartz inequality we have

\[
\text{var} S(n) = \text{var} \left( \sum_{R \leq i \leq n} \frac{1}{i} h(\xi_i) \right) + O((\log \log n)^{1/2}) \left( \text{var} \left( \sum_{R \leq i \leq n} \frac{1}{i} h(\xi_i) \right) \right)^{1/2},
\]

(2.2)

where \(R = R(n) = (\log n)^2\). Also,

\[
\text{var} \left( \sum_{R \leq i \leq n} \frac{1}{i} h(\xi_i) \right) = 2 \sum_{R \leq i < j \leq n} \frac{1}{ij} \{ Eh(\xi_i) h(\xi_j) - Eh(\xi_i) Eh(\xi_j) \} + O(1). \quad (2.3)
\]

Elementary arguments give that

\[
Eh(\xi_i) = \int_{-\infty}^{\infty} h(t) dH_i(t),
\]

(2.4)

where \(H_i(t)\) is the distribution function of \(\xi_i\). Let \(M_i = \max_{1 \leq k \leq i} X_k\), \(M_{i,j}^* = \max_{i < \ell \leq j} X_\ell\) and

\[
A_1 = \{(t,s): a(i) + tb(i) < a(j) + sb(j)\},
\]

\[
A_2 = \{(t,s): a(i) + tb(i) = a(j) + sb(j)\}.
\]

Then for any \(i < j\) we have

\[
P\{\xi_i \leq t, \xi_j \leq s\} = P\{M_i \leq (a(i) + tb(i)) \wedge (a(j) + sb(j)), M_{i,j}^* \leq a(j) + sb(j)\}
\]

\[
= H_i \left( t \wedge \left( s \frac{b(j)}{b(i)} + \frac{a(j) - a(i)}{b(i)} \right) \right) H_{j-i} \left( s \frac{b(j)}{b(j-i)} + \frac{a(j) - a(j-i)}{b(j-i)} \right).
\]

Let \(\mu_{i,j}\) be the measure generated by the distribution of \((\xi_i, \xi_j)\). Then for any \(\Delta s > 0\)

\[
P\{\xi_i \in [s, s + \Delta s) H_{j-i} \left( s \frac{b(i)}{b(j-i)} + \frac{a(i) - a(j-i)}{b(j-i)} \right) \}
\]
\[ \begin{align*}
&\leq P\{ (\xi_i, \xi_j) \in A_2, \xi_i \in [s, s + \Delta s) \} \\
&= P\{ M_i = M_j, \xi_i \in [s, s + \Delta s) \} \\
&= P\{ M_{ij}^* \leq M_i, \xi_i \in [s, s + \Delta s) \} \\
&\leq P\{ \xi_i \in [s, s + \Delta s) \} H_{j-i}(s + \Delta s) \left( \frac{b(i)}{b(j-i)} + \frac{a(i) - a(j-i)}{b(j-i)} \right).
\end{align*} \]

Since \( H_k \) is right continuous, it follows that

\[ \int \int_{A_2} h(s) h(t) d\mu_{i,j} \]
\[ = \int_{-\infty}^{\infty} h(s) h(t) \left( \int_{-\infty}^{\infty} \left( \int_{s}^{t} \frac{b(j)}{b(i)} + \frac{a(j) - a(i)}{b(i)} \right) dH_{j-i}(s) \right) dH_{i}(s). \]

On the other hand, the formula for \( P\{ \xi_i \leq t, \xi_j \leq s \} \) shows that in the open half-plane \((A_1 \cup A_2)^c\), \( P\{ \xi_i \leq t, \xi_j \leq s \} \) depends only on \( s \) and thus

\[ \int \int_{(A_1 \cup A_2)^c} h(s) h(t) d\mu_{i,j} = 0. \]

Therefore,

\[ E h(\xi_i) h(\xi_j) = \int \int_{A_1 \cup A_2} h(s) h(t) d\mu_{i,j} \]
\[ = \int_{-\infty}^{\infty} h(s) \left( \int_{-\infty}^{\infty} h(t) \left\{ t < s \frac{b(j)}{b(i)} + \frac{a(j) - a(i)}{b(i)} \right\} dH_{i}(t) \right) \]
\[ \times dH_{j-i}(s) \left( s \frac{b(j)}{b(j-i)} + \frac{a(j) - a(j-i)}{b(j-i)} \right) \]
\[ + \int_{-\infty}^{\infty} h(s) h(t) \left( s \frac{b(j)}{b(i)} + \frac{a(j) - a(i)}{b(i)} \right) \]
\[ \times H_{j-i}(s) \left( s \frac{b(i)}{b(j-i)} + \frac{a(i) - a(j-i)}{b(j-i)} \right) dH_{i}(s). \quad (2.5) \]

if \( i < j \).

**Lemma 2.1.** If the conditions of Theorem 1.1 are satisfied

\[ F(x) = \begin{cases} 
0 & \text{if } x \leq 1 \\
1 - x^{-2} & \text{if } x > 1,
\end{cases} \]

and \( a(i) = 0 \) and \( b(i) = i^{1/2} \) \((x > 0)\), then (2.1) holds where \( \sigma^2 \) is defined in (1.5).
Proof. Elementary calculations give

$$\sup_{-\infty < x < \infty} |H_i(x) - \Phi_2(x)| = O\left(\frac{1}{i}\right) \quad \text{as} \quad i \to \infty. \quad (2.6)$$

Next, we show that

$$\sum_{2 \leq i < j \leq n, j - i \leq R} \frac{1}{ij} = O((\log \log n)^2). \quad (2.7)$$

Observing that

$$\sum_{2 \leq i < j \leq n, j - i \leq R} \frac{1}{ij} \leq \sum_{1 \leq k \leq R} \sum_{2 \leq i \leq n} \frac{1}{i(i + k)} \leq \sum_{1 \leq k \leq R} \int_1^n \frac{1}{x(x + k)} \, dx \leq \sum_{1 \leq k \leq R} \frac{2}{k} \log k,$$

we obtain immediately (2.7). By (2.7) we have that

$$\left| \sum_{R \leq i < j \leq n, j - i \leq R} \frac{1}{ij} \text{cov}(h(\xi_i), h(\xi_j)) \right| = O((\log \log n)^2). \quad (2.8)$$

Let

$$d = d(\Phi_2) = \int_0^\infty h(t) \, d\Phi_2(t)$$

and

$$c_{ij} = c_{ij}(\Phi_2)$$

$$= \int_0^\infty h(s) \left( \int_0^\infty h(t) I\{t < s(j/i)^{1/2}\} \, d\Phi_2(t) \right) \, d\Phi_2(s(j/(j - i))^{1/2})$$

$$+ \int_0^\infty h(s) (s(j/i)^{1/2}) \Phi_2(s(i/(j - i))^{1/2}) \, d\Phi_2(s). \quad (2.9)$$

By (2.6) we conclude

$$|Eh(\xi_i) - d| \leq \frac{C}{i} \quad (2.10)$$

and

$$|Eh(\xi_i)h(\xi_j) - c_{ij}| \leq C \left( \frac{1}{i} + \frac{1}{j - i} \right) \quad (2.11)$$

if \( i < j \). Putting together (2.10) and (2.11) we get

$$\left| \sum_{R \leq i < j \leq n, j - i \geq R} \frac{1}{ij} \text{cov}(h(\xi_i), h(\xi_j)) - \sum_{R \leq i < j \leq n, j - i \geq R} \frac{1}{ij}(c_{ij} - d^2) \right|$$

$$\leq C \left\{ \sum_{R \leq i < j \leq n, j - i \geq R} \frac{1}{ij} \left( \frac{1}{i} + \frac{1}{j - i} \right) + \sum_{R \leq i < j \leq n, j - i \geq R} \frac{1}{ij} \left( \frac{1}{i} + \frac{1}{j} \right) \right\}. $$
Since
\[ \sum_{R \leq i < j \leq n} \left( \frac{1}{j^2} + \frac{1}{ij^2} \right) = O(1) \]
and
\[ \sum_{R \leq i < j \leq n, j-i \geq R} \frac{1}{ij(j-i)} = O\left( \frac{1}{R} (\log n)^2 \right) = O(1), \]
we get that
\[ \sum_{R \leq i < j \leq n, j-i \geq R} \frac{1}{ij} \left[ \text{cov}(h(\xi_i), h(\xi_j)) - (c_{ij} - d^2) \right] = O(1). \] (2.12)
The boundedness of \( h \) gives that \( c_{ij} \) and \( d_i \) are uniformly bounded and thus by (2.7) we have
\[ \left| \sum_{R \leq i < j \leq n, j-i \geq R} \frac{1}{ij} (c_{ij} - d^2) \right| = O((\log \log n)^2). \] (2.13)
Combining (2.3), (2.7), (2.8), (2.12) and (2.13) we obtain that
\[ \left| \text{var} \left( \sum_{R \leq i < j \leq n} \frac{1}{i} h(\xi_i) \right) - 2 \sum_{R \leq i < j \leq n} \frac{1}{ij} (c_{ij} - d^2) \right| = O((\log \log n)^2). \] (2.14)
Using the definitions of \( c_{ij} \) and \( d_i \) one can easily verify that
\[ \sum_{R \leq i < j \leq n} \frac{1}{ij} (c_{ij} - d^2) = I_{n,1} + o(\log n), \] (2.15)
where
\[
I_{n,1} = \int_{\log R}^{\log n} \int_{u}^{\log n} \frac{1}{x y} \left\{ \int_{0}^{\infty} \int_{0}^{\infty} h(s) h(t) I\{ t < s(x/y)^{1/2} \} \, ds \, d\Phi_x(t) \right\} \times d\Phi_x(s(y/(y-x))^{1/2}) - \left( \int_{0}^{\infty} h(t) \, d\Phi_x(t) \right)^2 \\
+ \int_{0}^{\infty} h(s) h(s(y/(y-x))^{1/2}) \Phi_x(s(x/(y-x))^{1/2}) \, d\Phi_x(s) \right\} \, dy \, dx + o(\log n). \]
\[+ \int_0^\infty h(s)h(se^{t-u})\Phi_2(s(e^{t-u})-1)^{-1/2})\,\Phi_2(s)\,d\Phi_2(s)\right) \,dv \,du\]

\[= \int_{\log R}^{\log n} \int_{u}^{\log n} G(v,u)dv \,du, \]

where

\[G(v,u) = \int_0^\infty \int_0^\infty h(s)h(t)\left( I\{t < se^{(t-u)}\} \varphi_2(t)(1-e^{-(t-u)})^{-1/2} \times \varphi_2(s(1-e^{-(t-u)})^{-1/2}) - \varphi_2(s)\varphi_2(t) \right) dt \,ds \]

\[+ \int_0^\infty h(s)h(se^{(t-u)})\Phi_2(s(e^{t-u})-1)^{-1/2})\varphi_2(s)\,ds \]

\[= \overline{G}_1(v-u) + \overline{G}_2(v-u) = \overline{G}(v-u) \]

and \( \varphi_2(t) = \Phi_2'(t) \).

Next, we prove that

\[\overline{G}(y) \leq c_1 \exp(-c_2 y) \quad (2.16)\]

with some \( c_1 > 0 \) and \( c_2 > 0 \). We can assume, without loss of generality, that the support of \( h \) is in \([0,A]\) with some \( A > 0 \). We write

\[\overline{G}_1'(y) = \int_0^A \exp(-y/2) \int_0^{\infty} h(s)h(t)\varphi_2(t)\left( I\{t < se^{(t-u)}\}(1-e^{-(t-u)})^{-1/2} \times \varphi_2(s(1-e^{-(t-u)})^{-1/2}) - \varphi_2(s)\varphi_2(t) \right) dt \,ds \]

\[+ \int_0^A \exp(-y/2) \int_0^{\infty} h(s)h(t)\varphi_2(t)\left( I\{t < se^{(t-u)}\}(1-e^{-(t-u)})^{-1/2} \times \varphi_2(s(1-e^{-(t-u)})^{-1/2}) - \varphi_2(s)\varphi_2(t) \right) dt \,ds \]

\[= \overline{G}_1(y) + \overline{G}_2(y). \]

Since

\[\left| \int_0^\infty h(s)h(t)\varphi_2(t)\left( I\{t < se^{(t-u)}\}(1-e^{-(t-u)})^{-1/2} \varphi_2(s(1-e^{-(t-u)})^{-1/2}) - \varphi_2(s)\varphi_2(t) \right) dt \right| \leq 2^{1/2+1} \sup_s \varphi_2(s) \sup_s \left| h(s) \right| \int_0^\infty |h(t)|\varphi_2(t) \,dt, \]

we get that

\[\overline{G}_1(y) \leq c_3 \exp(-y/2) \]

with some \( c_3 \). For all \( 0 \leq t \leq A \) and \( A \exp(-y/2) \leq s \leq A \) we have that \( I\{t < s \exp(y/2)\} = 1 \). Since \( \varphi_2 \) and \( \varphi_2' \) are bounded, Taylor expansion yields

\[\left| (1-e^{-y})^{-1/2} \varphi_2(s(1-e^{-y})^{-1/2}) - \varphi_2(s) \right| \leq c_4 |1 - (1-e^{-y})^{-1/2}| + |\varphi_2(s(1-e^{-y})^{-1/2}) - \varphi_2(s)| \]
If $c^*$ is so large that $h(t) = 0$ if $t \geq c^*$, then

$$|G_2(y)| = \left| \int_0^{c^*e^{-y/2}} h(s) h(se^{y/2}) \Phi_s(s(e^y - 1)^{-1/\alpha}) ds \right| \leq \sup_t h^2(t) \Phi_s(c^*e^{-y/2}),$$

completing the proof of (2.16).

Let $r = r(n) = (\log n)^{-1/2}$. Using (2.16) we obtain that

$$\sup_{\log R \leq u \leq (1-r)\log n} \int_0^{\log n-u} \mathcal{G}(y) dy - \int_0^\infty \mathcal{G}(y) dy \leq (c_2/2) \exp(-c_2(\log n)^{1/2})$$

and

$$\int_{(1-r)\log n}^{\log n} \int_0^{\log n-u} \mathcal{G}(y) du dy \leq \int_{(1-r)\log n}^{\log n} \int_0^\infty \mathcal{G}(y) dy du$$

$$= O((\log n)^{1/2}).$$

Thus, we have

$$\int_{\log R}^{\log n} \int_0^{\log n-u} \mathcal{G}(y) du dy = \log n \int_0^\infty \mathcal{G}(y) dy + O((\log n)^{1/2}),$$

which also completes the proof of Lemma 2.1. \qed

In the following lemma, we take a specially chosen function in the domain of attraction of $\Psi_2$ and show that (2.1) holds again.

**Lemma 2.2.** If the conditions of Theorem 1.1 are satisfied,

$$F(x) = \begin{cases} 
0 & \text{if } x \leq -1, \\
1 - (-x)^2 & \text{if } -1 < x \leq 0, \\
1 & \text{if } 0 < x
\end{cases}$$

and $a(i) = 0$ and $b(i) = i^{-1/\alpha}$ ($\alpha > 0$), then (2.1) holds where $\sigma^2$ is defined in (1.6).

**Proof.** Similarly to (2.6) we have

$$\sup_{-\infty < x < \infty} |H_i(x) - \Psi_2(x)| = O\left(\frac{1}{i}\right) \quad \text{as} \quad i \to \infty.$$

Let

$$d = d(\Psi_2) = \int_{-\infty}^0 h(t) d\Psi_2(t)$$
and
\[ c_{ij} = c_{ij}(\Psi_z) \]
\[ = \int_{-\infty}^{0} h(s) \left( \int_{-\infty}^{0} h(t) I\{t < s(j/i)^{-1/2}\} d\Psi_z(t) \right) d\Psi_z(s(j/(j-i))^{-1/2}) \]
\[ + \int_{-\infty}^{0} h(s) h(s(j/i)^{-1/2}) \Psi_z(s(i/(j-i))^{-1/2}) d\Psi_z(s). \tag{2.17} \]

Following the proof of Lemma 2.1 one can easily verify that
\[ \text{var} \left( \sum_{R \leq i \leq n} \frac{1}{i} h(\xi_i) \right) - 2 \left( \sum_{R \leq i \leq n} \frac{1}{ij} (c_{ij} - d^2) \right) = O((\log \log n)^2) \]
and
\[ \sum_{R \leq i \leq j \leq n} \frac{1}{ij} (c_{ij} - d^2) \]
\[ = \int_{R}^{n} \int_{x}^{n} \frac{1}{xy} \left( \int_{-\infty}^{0} h(s) \int_{-\infty}^{0} h(t) I\{t < s(y/x)^{-1/2}\} d\Psi_z(t) \right) \]
\[ \times d\Psi_z(s(y/(y-x))^{-1/2}) - \left( \int_{-\infty}^{0} h(t) d\Psi_z(t) \right)^2 \]
\[ + \int_{-\infty}^{0} h(s) h(s(y/x)^{-1/2}) \Psi_z(s(x/(y-x))^{-1/2}) d\Psi_z(s) dy dx + o(\log n) \]
\[ = \int_{\log n}^{\log n} \int_{u}^{\log n} G^*(v,u) dv du + o(\log n), \]
where \( R = (\log n)^2 \) and
\[ G^*(v,u) = \int_{-\infty}^{0} \int_{-\infty}^{0} h(s) h(t) \Psi_z(t) I\{t < se^{-(v-u)/2}\} \]
\[ \times (1 - e^{-(v-u)/2})^{1/2} \Psi_z(s(1 - e^{-(v-u)/2})^{1/2} - \Psi_z(s)) dt ds \]
\[ + \int_{-\infty}^{0} h(s) h(se^{-(v-u)/2}) \Psi_z(s(e^{(v-u)} - 1)^{1/2} \Psi_z(s) ds \]
\[ = \overline{G}^*(v-u) \]
with \( \psi(t) = \Psi_z'(t) \). Repeating the proof of (2.16) we get that
\[ \overline{G}^*(y) \leq c_7 \exp(-c_8 y), \]
which implies that

\[
\int_{\log R}^{\log n} \int_{\log R}^{\log n} G^*(v, u) \, dv \, du = \int_{\log R}^{\log n - u} \int_{\log n - u}^{\log n} \widehat{G}^*(y) \, dy \, du \\
= \log n \int_{0}^{\infty} \widehat{G}^*(y) \, dy + o(\log n).
\]

Next, we consider a function which is in the domain of attraction of \( \mathcal{A} \).

**Lemma 2.3.** If the conditions of Theorem 1.1 are satisfied,

\[
F(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
1 - e^{-x} & \text{if } x > 0
\end{cases}
\]

and \( a(i) = \log i \) and \( b(i) = 1 \), then (2.1) holds where \( \sigma^2 \) is defined in (1.7).

**Proof.** It is easy to see that

\[
\sup_{-\infty < x < \infty} |H_i(x) - \mathcal{A}(x)| = O\left(\frac{1}{i}\right) \quad \text{as} \quad i \to \infty. \tag{2.18}
\]

We have the same rate of convergence in (2.6) as well as in (2.18), so following the proof of Lemma 2.1 we arrive at

\[
\text{var} \left( \sum_{R \leq i \leq \nu_n} \frac{1}{i} h(\xi_i) \right) = 2 \sum_{R \leq i \leq \nu_n} \frac{1}{ij}(c_{ij} - d^2) + o(\log n),
\]

where \( R = R(n) = (\log n)^2 \),

\[
d = d(A) = \int_{-\infty}^{\infty} h(t) \, dA(t)
\]

and

\[
c_{ij} = c_{ij}(A)
\]

\[
= \int_{-\infty}^{\infty} h(t) \int_{-\infty}^{\infty} h(t) \mathcal{I}(t < s + \log(j/i)) \, dA(t) \, dA(s + \log(j/i))
\]

\[
+ \int_{-\infty}^{\infty} h(s) h(s + \log(j/i)) A(s + \log(i/(j - i))) \, dA(s). \tag{2.19}
\]

Also,

\[
\sum_{R \leq i \leq \nu_n} \frac{1}{ij}(c_{ij} - d^2) = I_{n,2} + o(\log n),
\]

where

\[
I_{n,2} = \int_{R}^{n} \int_{x}^{n} \frac{1}{xy} \left( \int_{-\infty}^{\infty} h(s) \int_{-\infty}^{\infty} h(t) \mathcal{I}(t < s + \log(y/x)) \, dA(t) \right. \\
\times dA(s + \log(y/x)) - \left( \int_{-\infty}^{\infty} h(s) \, dA(t) \right)^2 + \int_{-\infty}^{\infty} h(s) h(s + \log(y/x)) \\
\times A(s + \log(y/(y - x)) \, dA(s)) \right) \, dy \, dx.
\]
Change of variables gives that
\[ \int_{\log R}^{\log n} \int_{u}^{\log n} L(v, u) \, dv \, du = \int_{\log R}^{\log n} \int_{0}^{\log n - u} \overline{L}(y) \, dy \, du, \]
where
\[ L(v, u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s) h(t) \lambda(t) \left( I \{ t \leq s + (v - u) \} \lambda(s + (v - u)) - \lambda(s) \right) \, dt \, ds 
+ \int_{-\infty}^{\infty} h(s) h(s + (v - u)) A(s - \log(e^{v - u} - 1)) \lambda(s) \, ds 
= \overline{L}(v - u), \]
where \( \lambda(t) = \lambda'(t) \). We can assume, without loss of generality, that the support of \( h \) is in \([-A, A]\) with some \( A > 0 \) and
\[ \int_{-\infty}^{\infty} h(t) \, dA(t) = 0. \]
If \( y > 2A \), then \( t \leq s + y \) for all \(-A \leq t, s \leq A\) and therefore \( \overline{L}(y) = 0 \). Hence
\[ I_{n,2} = \log n \int_{0}^{\infty} \overline{L}(y) \, dy + O(\log \log n), \]
which completes the proof of Lemma 2.3. □

Next, we collect some technical results on the norming and centering constants in (1.1).

**Lemma 2.4.** We assume that \( \delta > 0 \) and \( B > 1 \). If \( F \in \mathcal{D}(\Phi_2) \) and \( b(k) \) is defined in (1.2), then
\[ \max_{1 \leq m, k \leq B} \left| \frac{b(m)}{b(k)} - \left( \frac{m}{k} \right)^{1/2} \right| = o(1) \quad \text{as} \quad m \to \infty, \quad (2.20) \]
\[ \max_{1 + \delta \leq m, k \leq B} \left| \frac{b(m)}{b(m - k)} - \left( \frac{m}{m - k} \right)^{1/2} \right| = o(1) \quad \text{as} \quad m \to \infty. \quad (2.21) \]
If \( F \in \mathcal{D}(\Psi_2) \) and \( b(k) \) is defined in (1.3), then
\[ \max_{1 \leq m, k \leq B} \left| \frac{b(m)}{b(k)} - \left( \frac{m}{k} \right)^{-1/2} \right| = o(1) \quad \text{as} \quad m \to \infty, \quad (2.22) \]
\[ \max_{1 + \delta \leq m, k \leq B} \left| \frac{b(m)}{b(m - k)} - \left( \frac{m}{m - k} \right)^{-1/2} \right| = o(1) \quad \text{as} \quad m \to \infty. \quad (2.23) \]
If \( F \in \mathcal{D}(A) \), \( a(k) \) and \( b(k) \) are defined in (1.4), then
\[ \max_{1 \leq m, k \leq B} \left| \frac{b(m)}{b(k)} - 1 \right| = o(1) \quad \text{as} \quad m \to \infty, \quad (2.24) \]
\[ \max_{1 + \delta \leq m, k \leq B} \left| \frac{b(m)}{b(m - k)} - 1 \right| = o(1) \quad \text{as} \quad m \to \infty, \quad (2.25) \]
\[
\max_{1 \leq m, k \leq B} \left| \frac{a(m) - a(k)}{b(k)} - \log \frac{m}{k} \right| = o(1) \quad \text{as} \quad m \to \infty \quad (2.26)
\]

and
\[
\max_{1 + \delta \leq m, k \leq B} \left| \frac{a(m) - a(k)}{b(m - k)} - \log \frac{m}{k} \right| = o(1) \quad \text{as} \quad m \to \infty. \quad (2.27)
\]

**Proof.** If \( F \in \mathcal{D}(\Phi_\omega) \) or \( F \in \mathcal{D}(\Psi_\omega) \), then \( b(k) \) is a regularly varying sequence with exponent \( 1/\omega \) (resp. \(-1/\omega\)) and therefore \((2.20)--(2.23)\) follow from Theorem 1.2.1 of Bingham et al. (1987).

According to Theorem 8.13.4 in Bingham et al. (1987), there is a slowly varying function \( \ell \) such that
\[
\frac{U(x + u) - U(x)}{\ell(e^x)} \to 1 \quad \text{as} \quad x \to \infty
\]
for any \( u > 0 \). Observing that \( U \) is a monotone function, Polya’s theorem (cf. Bingham et al., 1987, p. 60) yields that
\[
\sup_{a \leq x \leq b} \left| \frac{U(x + u) - U(x)}{\ell(e^x)} - u \right| = o(1), \quad x \to \infty. \quad (2.28)
\]

The uniform convergence in \((2.28)\) implies immediately \((2.24)--(2.27)\).

We recall that \( c_{ij} \) are defined in \((2.9), (2.17) \) and \((2.19)\) and the definition depends on the extreme value distribution we have in the limit.

**Lemma 2.5.** We assume that the conditions of Theorem 1.1 are satisfied, \( \delta > 0 \) and \( B > 1 \). Then for any \( \varepsilon > 0 \) there is \( N \) such that
\[
\max_{1 \leq j \leq \delta} \| E h(\hat{\xi}) h(\hat{\xi}_j) - c_{ij} \| \leq \varepsilon
\]
if \( i > N \).

**Proof.** We use \((2.5)\). We assume that \( F \in \mathcal{D}(\Phi_\omega) \). The other two cases are very similar and their proofs will be omitted. Since \( \Phi_\omega \) is continuous, by \((1.1)\) we have that
\[
\sup_{-\infty < x < \infty} |H_k(x) - \Phi_\omega(x)| = o(1) \quad \text{as} \quad k \to \infty.
\]

So integration by parts and the bounded variation of \( h(t) \) give
\[
\sup_{-\infty < t < \infty} \left| \int_{-\infty}^{\infty} |h(t)| \, d|H_k(t) - \Phi_\omega(t)| \right| = o(1) \quad \text{as} \quad k \to \infty.
\]

Using again integration by parts, for any \( \eta > 0 \) we can find \( N_1 \) and \( N_2 \) such that
\[
\left| \int_{-\infty}^{\infty} h(s) \left( \int_{-\infty}^{\infty} h(t) I \{ t < sb(j)/b(i) \} \, dH_j(t) \right) \, dH_{j-i}(sb(j)/b(j-i)) \right| \leq \eta
\]
for any \( s \geq \underline{s} \).
if $i \geq N_1$ and $j - i \geq N_2$. By Lemma 2.4 and the fact that $h$ has a compact support we have that
\[
\sup_{-\infty < s < \infty} \max_{1 + \delta < j / i \leq B} \left| \int_{-\infty}^{\infty} h(t) I\{t < sb(j)/b(i)\} \, d\Phi_z(t) \right.
- \left. \int_{-\infty}^{\infty} h(t) I\{t < (j/i)^{1/\alpha}\} \, d\Phi_z(t) \right| = o(1) \quad \text{as} \quad i \to \infty
\]
and similarly,
\[
\sup_{-\infty < s < \infty} \max_{1 + \delta < j / i \leq B} \left| \Phi_z \left( s \frac{b(j)}{b(j - i)} \right) - \Phi_z \left( s \left( \frac{j}{j - i} \right)^{1/\alpha} \right) \right| = o(1) \quad \text{as} \quad i \to \infty.
\]
Hence, integration by parts yields
\[
\max_{1 + \delta < j / i \leq B} \left| \int_{-\infty}^{\infty} h(s) \left( \int_{-\infty}^{\infty} h(t) I\{t < sb(j)/b(i)\} \, d\Phi_z(t) \right) \, d\Phi_z(sb(j)/b(j - i)) 
+ \int_{0}^{\infty} h(s) (sb(j)/b(i)) \Phi_z(sb(i)/b(j - i)) \, d\Phi_z(s) - c_{ij} \right| = o(1) \quad \text{as} \quad i \to \infty,
\]
completing the proof. \[\Box\]

Now, we are ready to prove the main result of this section.

**Theorem 2.1.** If the conditions of Theorem 1.1 are satisfied, then (2.1) holds.

**Proof.** Since $h$ is bounded, for any $i < j$ we have
\[
|\text{cov}(h(\xi_i), h(\xi_j))| \\ = |\text{cov} \left( h(\xi_i), h(\xi_j) - h \left( \left( \max_{i < k \leq j} X_k - a(j) \right) / b(j) \right) \right)| \\ \leq c_7 E \left| h(\xi_j) - h \left( \left( \max_{i < k \leq j} X_k - a(j) \right) / b(j) \right) \right|.
\]
Observing that the random variable
\[
h(\xi_j) - h \left( \left( \max_{i < k \leq j} X_k - a(j) \right) / b(j) \right)
\]
is bounded and differs from 0 only if $\max_{1 \leq k \leq j} X_k$ is reached for some $1 \leq k \leq i$, i.e. with probability not greater than $i/j$, we get that
\[
E \left| h(\xi_j) - h \left( \left( \max_{i < k \leq j} X_k - a(j) \right) / b(j) \right) \right| \leq c_8 \frac{i}{j}
\]
and therefore
\[
|\text{cov}(h(\xi_i), h(\xi_j))| \leq c_9 \frac{i}{j} \quad \text{for all} \quad 1 \leq i < j < \infty.
\] (2.29)

We recall that $R = (\log n)^2$. Next for any $\delta > 0$ and $B > 1$ we write
\[
\text{var} \left( \sum_{R \leq i \leq n} \frac{1}{i} h(\xi_i) \right) = \sum_{R \leq i \leq n} \frac{1}{i^2} \text{var} \, h(\xi_i) + 2 \sum_{R \leq i < j \leq n;j/i > B} \frac{1}{ij} \text{cov} \, (h(\xi_i), h(\xi_j))
\]
\[ + \sum_{R \leq i < j \leq n, j/i \leq 1+\delta} \frac{1}{ij} \text{cov}(h(\xi_i), h(\xi_j)) \]

\[ + \sum_{R \leq i < j \leq n, 1+\delta \leq j/i \leq B} \frac{1}{ij} \text{cov}(h(\xi_i), h(\xi_j)) \]

\[ = \lambda_{n,1} + \cdots + \lambda_{n,4}. \quad (2.30) \]

Clearly,

\[ \lambda_{n,1} = O(1) \text{ as } n \to \infty, \quad (2.31) \]

and by (2.29) we have

\[ \lambda_{n,2} \leq c_{10} \sum_{R \leq i < j \leq n, j/i > B} \frac{1}{ij} \times \frac{i}{j} \leq \frac{c_{11}}{B} \log n \quad (2.32) \]

and

\[ \lambda_{n,3} \leq c_{12} \sum_{R \leq i < j \leq n, j/i \leq 1+\delta} \frac{1}{ij} \times \frac{i}{j} \lambda_{n,3} \leq \delta c_{13} \log n. \quad (2.33) \]

Let \( \varepsilon > 0 \) and choose \( B = 1/\varepsilon \) and \( \delta = \varepsilon \) in (2.30)–(2.33). Integration by parts and (1.1) yield that

\[ \lim_{i \to \infty} \text{E}h(\xi_i) = d \]

and therefore by Lemma 2.5 there is \( N = N(\varepsilon) \) such that

\[ \max_{(1+\delta) \leq j/i \leq B} |\text{cov}(h(\xi_i), h(\xi_j)) - (c_{ij} - d^2)| \leq \varepsilon \]

if \( i \geq N \). Hence, if \( n \geq n_0 \), then

\[ \left| \lambda_{n,4} - 2 \sum_{R \leq i < j \leq n, 1+\delta \leq j/i \leq B} \frac{1}{ij} (c_{ij} - d^2) \right| \]

\[ \leq 2\varepsilon \sum_{R \leq i < j \leq n, 1+\delta \leq j/i \leq B} \frac{1}{ij} \]

\[ \leq c_{14} \varepsilon \log(1/\varepsilon) \log n. \quad (2.34) \]

Putting together (2.2) and (2.30)–(2.34) we conclude that for any \( \varepsilon > 0 \)

\[ \limsup_{n \to \infty} \left| \frac{1}{\log n} \text{var} S(n) - \frac{1}{\log n} \sum_{R \leq i < j \leq n, 1+\varepsilon \leq j/i \leq 1/\varepsilon} \frac{1}{ij} (c_{ij} - d^2) \right| \]

\[ \leq c_{15} \varepsilon \log(1/\varepsilon). \quad (2.35) \]

Since (2.35) holds for any \( F \) in the domain of attraction of an extreme value distribution, Theorem 2.1 follows from Lemmas 2.1–2.3.
3. Proof of Theorem 1.1

We can and shall assume, without loss of generality, that $|h| \leq 1$. For any $i < k$ we define
\[ M_{k,i} = \max_{i < j \leq k} X_j \quad \text{and} \quad M_k = M_{k,0} = \max_{1 \leq j \leq k} X_j. \]

Let $r < p < q$ be positive integers and define
\[ X = \sum_{2^p < i \leq 2^q} \frac{1}{i} h \left( \frac{M_i - a(i)}{b(i)} \right) \]
and
\[ X' = \sum_{2^p < i \leq 2^q} \frac{1}{i} h \left( \frac{M_i - a(i)}{b(i)} \right). \]

**Lemma 3.1.** If the conditions of Theorem 1.1 are satisfied, then for any $d \geq 1$ we have
\[ E|X - X'|^d \leq (2(q - p))^d 2^{-(p-r)}. \]  
(3.1)

**Proof.** Let $2^p < i \leq 2^q$. Clearly, $M_{i,0} \neq M_{i,2^r}$ implies that the maximum of $X_1, \ldots, X_i$ is taken among the first $2^r$ terms and thus
\[ P\{M_i \neq M_{i,2^r}\} \leq 2^r / i \leq 2^{-(p-r)}. \]  
(3.2)

Now by $|h| \leq 1$ we have
\[ |X - X'| \leq \sum_{2^p < i \leq 2^q} \frac{2}{i} \left\{ M_i \neq M_{i,2^r} \right\} \]
and therefore (3.2) yields
\[ E|X - X'| \leq \sum_{2^p < i \leq 2^q} \frac{2}{i} P\{M_i \neq M_{i,2^r}\} \]
\[ \leq 2^{-(p-r)} \sum_{2^p < i \leq 2^q} \frac{2}{i} \]
\[ \leq 2(q - p)2^{-(p-r)} \]  
(3.3)

and
\[ |X - X'| \leq \sum_{2^p < i \leq 2^q} \frac{2}{i} \leq 2(q - p). \]  
(3.4)

Now (3.1) follows immediately from (3.3) and (3.4).

Let
\[ \delta_k = \sum_{2^k < i \leq 2^{k+1}} \frac{1}{i} \left( h \left( \frac{M_i - a(i)}{b(i)} \right) - Eh \left( \frac{M_i - a(i)}{b(i)} \right) \right). \]
Lemma 3.2. If the conditions of Theorem 1.1 are satisfied, then for any positive integers \(M\) and \(N\) we have

\[
E \left( \sum_{M < k \leq M+N} \delta_k \right)^2 \leq c_1 N \tag{3.5}
\]

and

\[
E \left( \sum_{M < k \leq M+N} \delta_k \right)^4 \leq c_2 N^3 \tag{3.6}
\]

with some constants \(c_1\) and \(c_2\).

Proof. We prove only (3.6) since the proof of (3.5) is similar; in fact simpler. It is easy to see that

\[
E \left( \sum_{M < k \leq M+N} \delta_k \right)^4 = \sum_{M < k \leq M+N} E \delta_k^4 + 6 \sum_{M < i < j \leq M+N} E \delta_i^2 \delta_j^2
\]

\[+ 4 \sum_{M < i < j < k \leq M+N} E \delta_i^2 \delta_j \delta_k + 12 \sum_{M < i < j < k \leq M+N} E \delta_i \delta_j \delta_k \delta_\ell
\]

\[+ 24 \sum_{M < i < j < k < \ell \leq M+N} E \delta_i \delta_j \delta_k \delta_\ell
\]

\[= S^{(1)} + \cdots + S^{(5)}.
\]

Since \(|h| \leq 1\) we get that \(|\delta_i| \leq 2\) and therefore

\[
S^{(1)} + S^{(2)} + S^{(3)} \leq c_3 N^2.
\]

Next, we prove that

\[
S^{(5)} \leq c_4 N^3.
\]

First, we show that if \(M < i < j < k < \ell \leq M + N\) and at least one of \(j - i\) and \(\ell - k\) is larger than \(N^{1/2}\), then

\[
|E\delta_i \delta_j \delta_k \delta_\ell| \leq c_5 N^{-4}.
\]

Assume that \(j - i \geq N^{1/2}\) and set

\[
\delta_i^* = \frac{1}{m} \left( h \left( \frac{M_{m,2^{i+1}} - a(m)}{b(m)} \right) - Eh \left( \frac{M_{m,2^{i+1}} - a(m)}{b(m)} \right) \right).
\]

Using Lemma 3.1 we obtain that

\[
E|\delta_i - \delta_i^*| \leq c_6 2^{-(j-i)} \leq c_7 2^{-N^{1/2}}.
\]

Similar estimates hold for \(E|\delta_k - \delta_k^*|, \ E|\delta_\ell - \delta_\ell^*|\) and thus

\[
|E\delta_i \delta_j \delta_k \delta_\ell - E\delta_i^* \delta_j^* \delta_k^* \delta_\ell^*| \leq c_8 2^{-N^{1/2}} \tag{3.10}
\]

for all \(M < i < j < k < \ell \leq M + N\), if \(j - i \geq N^{1/2}\). Observing that \(\delta_i\) and \(\{\delta_j^*, \delta_k^*, \delta_\ell^*\}\) are independent and \(E\delta_i = 0\), we conclude that the second expected value in (3.10)
equals 0 and therefore (3.9) is proven if \( j - i \geq N^{1/2} \). The case \( \ell - k \geq N^{1/2} \) can be treated similarly.

Relation (3.9) implies that the contribution of those terms in \( S(5) \) where at least one of \( j - i \) and \( \ell - k \) is greater than \( N^{1/2} \) is \( O(1) \). On the other hand, the contribution of the remaining terms is less than \( c_9 N^3 \), since the number of 4-tuples \( (i, j, k, \ell) \) satisfying \( M < i < j < k < \ell < M + N \), \( j - i \leq N^{1/2} \), \( \ell - k \leq N^{1/2} \) is clearly at most \( N^3 \). Hence (3.8) is proven.

Following the proof of (3.8) one can easily verify that

\[
ES(4) \leq c_{10} N^3
\]

and therefore (3.6) follows from (3.7) and (3.8).

Let us divide \([1, 1]\) into consecutive intervals \( \Delta_{1} = [p_1, q_1], \Delta_{1}' = [p_1', q_1'], \Delta_{2} = [p_2, q_2], \Delta_{2}' = [p_2', q_2'], \ldots \), where \( p_1 = 1 \), \( p_k = q_{k-1} \). We choose these intervals so that

\[
| \Delta_k | = [k^{1/2}] \quad \text{and} \quad | \Delta_k' | = [k^{1/4}],
\]

where \( | \Delta | \) denotes the length of the interval \( \Delta \). Set

\[
\beta_k = \sum_{2^{2k} < i < 2^{2k+1}} \frac{1}{i} h \left( \frac{M_i - a(i)}{b(i)} \right),
\]

\[
\beta_k' = \sum_{2^{2k} < i < 2^{2k+1}} \frac{1}{i} h \left( \frac{M_{i-2^{2k}} - a(i)}{b(i)} \right),
\]

\[
\eta_k = \sum_{2^{2k} < i < 2^{2k+1}} \frac{1}{i} h \left( \frac{M_i - a(i)}{b(i)} \right)
\]

and

\[
\eta_k' = \sum_{2^{2k} < i < 2^{2k+1}} \frac{1}{i} h \left( \frac{M_{i-2^{2k}} - a(i)}{b(i)} \right).
\]

By Lemma 3.1 for any integer \( d \geq 1 \) we have that

\[
E|\beta_k - \beta_k'|^d \leq c_{11} (q_k - p_k)^d 2^{(M_i - a(i))} \leq c_{12} k^{d/2} 2^{-k^{1/4}} \leq c_{13} k^{-4} \quad (3.11)
\]

and similarly,

\[
E|\eta_k - \eta_k'|^d \leq c_{14} k^{-4}. \quad (3.12)
\]

**Lemma 3.3.** If the conditions of Theorem 1.1 are satisfied, then

\[
\sum_{1 \leq i \leq k} (\eta_i - E\eta_i) = O(k^{5/8} \log k) \quad \text{a.s.}
\]

**Proof.** Applying (3.12) with \( d = 1 \) and using the monotone convergence theorem we get

\[
\sum_{1 \leq i \leq \infty} |(\eta_i - E\eta_i) - (\eta_i^* - E\eta_i^*)| < \infty \quad \text{a.s.} \quad (3.13)
\]
Also, (3.12) with \( d = 2 \) and Lemma 3.2 give
\[
\| \eta_k^* - E \eta_k^* \| = \| \eta_k - E \eta_k \| + O(1) = O(k^{1/8}),
\]
where \( \| \cdot \| \) denotes the \( L_2 \) norm. Thus,
\[
\sum_{1 \leq k < \infty} E|\eta_k^* - E \eta_k^*|^2 k^{-3/4} (\log k)^2 < \infty \quad \text{a.s. convergent.} \tag{3.14}
\]
Since \( \eta_1^*, \eta_2^*, \ldots \) are independent random variables with zero means, (3.14) implies that
\[
\sum_{1 \leq k < \infty} \eta_k^* - E \eta_k^* k^{5/8} \log k < \infty \quad \text{a.s.}
\]
and thus by the Kronecker lemma (cf. Chow and Teicher, 1988, p. 114) we have
\[
\sum_{1 \leq i \leq k} (\eta_i^* - E \eta_i^*) = O(k^{5/8} \log k) \quad \text{a.s.} \tag{3.15}
\]
Putting together (3.13) and (3.15) we obtain Lemma 3.3.

Let
\[
N_k = \sum_{1 \leq i < k} \left( [i^{1/2}] + [i^{1/4}] \right)
\]
and
\[
\lambda_n = \text{var} \left( \sum_{1 \leq k \leq n} \frac{1}{k} \eta_k \left( \frac{M_k - a(k)}{b(k)} \right) \right).
\]

**Lemma 3.4.** If the conditions of Theorem 1.1 are satisfied, then
\[
\sum_{1 \leq i \leq k} E(\beta_i^* - E \beta_i^*)^2 = \lambda_2^m (1 + O(k^{-1/8})).
\]

**Proof.** Lemma 3.2 implies that
\[
\text{var} \eta_i = O(i^{1/4})
\]
and (3.12) with \( d = 2 \) gives
\[
\text{var} \eta_i^* = O(i^{1/4}).
\]
Hence by the independence of \( \eta_1^*, \eta_2^*, \ldots \) it follows that
\[
\text{var} \left( \sum_{1 \leq i \leq k} \eta_i^* \right) = O(k^{5/4}). \tag{3.16}
\]
Using (3.16), the Minkowski inequality and (3.11), (3.12) with \( d = 2 \) we see that the first two of the quantities
\[
\text{var}^{1/2} \left( \sum_{1 \leq i \leq k} \beta_i^* \right), \quad \text{var}^{1/2} \left( \sum_{1 \leq i \leq k} (\beta_i^* + \eta_i^*) \right), \quad \text{var}^{1/2} \left( \sum_{1 \leq i \leq k} (\beta_i + \eta_i) \right)
\]
are
differ at most by $O(k^{5/8})$, while the second and third differ at most by $O(1)$. Since the third expression in (3.17) equals $\lambda_{2^n/2}^{1/2}$, we proved that
\[
\text{var}^{1/2} \left( \sum_{1 \leq i \leq k} \beta_i^* \right) = \lambda_{2^n/2}^{1/2} + O(k^{5/8})
\]
where the second equality follows from the observation that $\lambda_n$ is proportional to $\log n$ (cf. Theorem 2.1) and therefore
\[
\lambda_{2^n/2}^{1/2} \geq c_{14}(\log 2^{N_k})^{1/2} \geq c_{15} k^{3/4}
\]
with some $c_{14} > 0$ and $c_{15} > 0$. Since $\beta_1^*, \beta_2^*, \ldots$ are independent, Lemma 3.4 follows from (3.18).

**Lemma 3.5.** If the conditions of Theorem 1.1 are satisfied, then there is a Wiener process \( \{W(t), 0 \leq t < \infty\} \) such that
\[
\sum_{1 \leq i \leq k} (\beta_i - E\beta_i) = W(\lambda_{2^n/2}^{1/2}) + O(\lambda_{2^n/2}^{1/2}) \text{ a.s.}
\]
with any $\varepsilon > 0$.

**Proof.** Let
\[
Z_i = \beta_i^* - E\beta_i^* \text{ and } s_k^2 = \sum_{1 \leq i \leq k} E(\beta_i^* - E\beta_i^*)^2.
\]
By Lemma 3.2 we have
\[
E(\beta_k - E\beta_k)^4 = O((q_k - p_k)^4) = O(k^{3/2}). \tag{3.19}
\]
Using (3.11) with $d = 4$, (3.19) yields that
\[
EZ_k^4 = O(k^{3/2}).
\]
Also, by Lemma 3.4 we have
\[
s_k^2 = \lambda_{2^n/2}^{1/2}(1 + O(k^{-1/8})). \tag{3.20}
\]
Since $\lambda_N \sim c_{16} \log N$ (cf. Theorem 2.1) and $N_k \sim c_{17} k^{3/2}$ with some $0 < c_{16}, c_{17} < \infty$ we get for any $5/6 < \delta < 1$ that
\[
\sum_{1 \leq k < \infty} \frac{1}{s_k^2} \int_{x_k > x_k^{2\delta}} x^2 dP\{Z_k \leq x\} \leq \sum_{1 \leq k < \infty} \frac{1}{s_k^2} \int_{-\infty}^{x_k} x^4 dP\{Z_k \leq x\} \leq c_{18} \sum_{1 \leq k < \infty} k^{3/2} / \lambda_{2^n/2}^{1/2}
\]
\[
\leq c_{19} \sum_{1 \leq k < \infty} k^{3/2} / N_k^{2\delta} \leq c_{20} \sum_{1 \leq k < \infty} k^{3/2} / k^{3\delta} < \infty.
\]
Thus using the invariance principle in Theorem 4.4 of Strassen (1965) we can define a Wiener process $W$ such that

$$
\sum_{1 \leq i \leq k} \left( \beta_i - E\beta_i \right) - W(\lambda_k^2) = O(\lambda_k^{(1+\varepsilon)/2} \log s_k) \quad \text{a.s.}
$$

(3.21)

Since $\lambda_n \sim \log n$, (3.20) implies that

$$|\lambda_k^2 - \lambda_n| = O(\lambda_n^{11/12})
$$

and thus Theorem 1.2.1 of Csörgő and Révész (1981) on the increments of $W$ imply that

$$|W(\lambda_k^2) - W(\lambda_n)| = o(\lambda_n^{11/24+\varepsilon}) \quad \text{a.s.}
$$

(3.22)

Also, (3.11) with $d = 1$ and the monotone convergence theorem yield

$$\sum_{1 \leq i < \infty} |(\beta_i - E\beta_i) - (\beta_i - E\beta_i)| < \infty \quad \text{a.s.}
$$

(3.23)

Now Lemma 3.5 follows from (3.20)–(3.23).

After the preliminary results the proof of Theorem 1.1 will be very simple.

**Proof of Theorem 1.1.** By Lemmas 3.3 and 3.5 we have that

$$
\sum_{1 \leq i \leq k} \frac{1}{i} \left( h \left( \frac{M_i - a(i)}{b(i)} \right) - Eh \left( \frac{M_i - a(i)}{b(i)} \right) \right)
$$

$$= W(\lambda_n) + o(\lambda_n^{11/24+\varepsilon}) \quad \text{a.s.}
$$

(3.24)

with any $\varepsilon > 0$. Now if $2^{N_k} \leq n < 2^{N_k+1}$, then the random variable

$$\sum_{1 \leq i \leq n} \frac{1}{i} \left( h \left( \frac{M_i - a(i)}{b(i)} \right) - Eh \left( \frac{M_i - a(i)}{b(i)} \right) \right)
$$

differs from its value at $n = 2^{N_k}$ by at most

$$O \left( \sum_{2^{N_k} \leq i \leq 2^{N_k+1}} \frac{1}{i} \right) = O(N_{k+1} - N_k) = O(k^{1/2})
$$

$$= O(N_k^{1/3}) = O((\log n)^{1/3}) = O(\lambda_n^{1/3}).
$$

(3.25)

Also, Minkowski’s inequality and (3.25) imply for $2^{N_k} \leq n \leq 2^{N_k+1}$

$$|\lambda_n^{1/2} - \lambda_n^{1/2}| = O \left( \sum_{2^{N_k} \leq i \leq 2^{N_k+1}} \frac{1}{i} \right) = O(\lambda_n^{1/3})
$$

(3.26)

and thus $1/2 \leq \lambda_n / \lambda_n^{1/2} \leq 2$ for all $k$ large enough. By (3.26) we have that

$$|\lambda_n - \lambda_n^{1/2}| = O(\lambda_n^{5/6})
$$

and therefore using Theorem 1.2.1 of Csörgő and Révész (1981) we conclude

$$|W(\lambda_n) - W(\lambda_n^{1/2})| = o(\lambda_n^{5/12+\varepsilon}) \quad \text{a.s.}
$$
with any $\varepsilon > 0$. Thus (3.24) implies
\[
\sum_{1 \leq i \leq n} \frac{1}{i} \left( h \left( \frac{M_i - a(i)}{b(i)} \right) - Eh \left( \frac{M_i - a(i)}{b(i)} \right) \right) = W(n) + o(n^{1/24 + \varepsilon}) \quad \text{a.s.}
\]
with any $\varepsilon > 0$, completing the proof of Theorem 1.1. \qed

Acknowledgements

We are grateful to Renate Caspers and Ingo Fahrner for pointing out a missing term in the definition of $\sigma^2$ in the first version of our paper.

References