Limit distributions of some integral functionals for null-recurrent diffusions

R. Khasminskii

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

Received 13 October 1999; received in revised form 22 August 2000; accepted 23 August 2000

Abstract

The limit distributions as $T \to \infty$ of the functional $\int_0^T f(X(t))\,dt/T$ are found for one-dimensional null-recurrent Markov processes $X(t)$ under different assumptions concerning local characteristics of $X(t)$ and $f$. The conditions for the convergence of the distribution of this functional to each distribution in the class of generalized arc-sine law, studied by Lamperti and Watanabe, are obtained. © 2001 Elsevier Science B.V. All rights reserved.

MSC: primary; 60J60; secondary; 60J55

Keywords: Generalized arc-sine law; Null-recurrent diffusion

1. Introduction

Limit distributions of integral functionals for time-homogeneous Markov processes are well studied for ergodic positive recurrent processes. In particular, let $X^x(t)$ be the diffusion process ($X^x(0) = x$) of this type and let $\mu(\cdot)$ be its stationary distribution. Then for any $\mu$-integrable function $f(x)$ the law of large numbers in the following form is valid:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(X^x(t))\,dt = \int f(x)\mu(dx) \quad \text{a.s.}$$

Moreover under rather weak additional conditions, as $T \to \infty$, the central limit theorem also holds: for some constant $\sigma$

$$\frac{\int_0^T f(X^x(t))\,dt - T \int f(x)\mu(dx)}{\sigma \sqrt{T}} \xrightarrow{\text{distr.}} \mathcal{N}(0,1).$$

The solution of similar problem for nonrecurrent processes cannot be found in the united form. For instance, the limit distributions of integral functionals for such processes depend, as a rule, on initial point $x$. The interesting problem is an analysis of limit distributions of integral functionals in a null-recurrent diffusion process.

Partially supported by the NSF Grant DMS 9971608.

E-mail address: rafail@math.wayne.edu (R. Khasminskii).
We describe now some results at this area for the one-dimensional processes. At the beginning, we will consider the process \( X(t) \) in the canonical scale. It means that the generator of the process has the form

\[
L(x) = a(x) \frac{d^2}{dx^2} \quad x \in \mathbb{R}^1.
\]  

(1.1)

Assume that \( a(x) \) is a strictly positive and Lipschitz continuous function in any compact set \( K \subset \mathbb{R}^1 \). Denote \( p(x) = a(x)^{-1} \).

Below we assume the following:

**Condition A.** For some constants \( p_+, p_-, \beta \), satisfying the conditions \( \beta > -1 \), \( p_+ + p_- > 0 \), there exist limits

\[
\lim_{x \to \pm \infty} \frac{1}{X} \int_0^X |x|^{-\beta} p(x) \, dx = p_{\pm}
\]  

(1.2)

and, moreover, for any \( \varepsilon > 0 \)

\[
\sup_{|x| > \varepsilon} p(x)|x|^{-\beta} < \infty.
\]  

(1.3)

The following theorem is a corollary of Theorem 11.1, Chapter 4 in Khasminskii (1969) and Theorem 2.2 in Khasminskii and Yin (2000).

**Theorem 1.1.** Suppose that Condition A is valid and let \( \alpha = 1/(2 + \beta) \). Suppose also that the integral

\[
\tilde{f} = \int_{-\infty}^{+\infty} f(x) p(x) \, dx
\]  

(1.4)

converges absolutely and \( \tilde{f} \neq 0 \). Let \( G_\alpha(x) \) be a distribution function of the positive stable law with index \( \alpha \) so that

\[
\int_0^\infty \exp(-sx) \, dG_\alpha(x) = \exp(-s^\alpha), \quad s > 0.
\]

Then

\[
\lim_{T \to \infty} P \left\{ \frac{\int_0^T f(X^\alpha(t)) \, dt}{c_\alpha T^\alpha} < y \right\} = 1 - G_\alpha(y^{-1/\alpha}), \quad y > 0,
\]

where

\[
c_\alpha = \frac{\Gamma(1 + \alpha) \alpha^{-2\alpha}}{\Gamma(1 - \alpha)(p_+^{\alpha} + p_-^{\alpha})}.
\]

Theorem 1.1 does not describe the limit behavior of an integral functional if the integral in (1.4) diverges. For the special case, \( a(x) = \text{const} \), \( f(x) = 1_{\{x > 0\}}(x) \) the distribution of this functional was found by Lévy (1939). He proved that for any \( T > 0 \), \( 0 \leqslant x \leqslant 1 \)

\[
P \left\{ \frac{1}{T} \int_0^T 1_{\{x > 0\}}(W(t)) \, dt < x \right\} = \frac{2}{\pi} \arcsin \sqrt{x},
\]

where \( W(t) \) is a standard Brownian motion.
Later it was proved that this distribution (arc-sine law) is also the limit distribution of the number of positive values for the random walk with zero mean, and of the index the first maximum in the random walk on the time interval $[0, n]$ (see Feller (1966) and references there).

The limit distribution of integral functionals in the case $p(x)$ having limits in the Cesaro sense as $x \to \pm \infty$ was studied by Khasminskii (1999). The following result was proved there.

**Theorem 1.2.** Let the functions $p(x)$ and $f(x)$ satisfy the conditions: $p(x) < C < \infty$ for $x \in \mathbb{R}$ and for some constants $\tilde{p}_+, \tilde{p}_-, \tilde{f}_+, \tilde{f}_-$ with $\tilde{p}_+ + \tilde{p}_- > 0, \tilde{f}_+ \neq \tilde{f}_-$:

$$
\lim_{x \to \pm \infty} \frac{1}{x} \int_0^x p(x) \, dx = \tilde{p}_\pm,
$$

$$
\lim_{x \to \pm \infty} \frac{\int_0^x f(x)p(x) \, dx}{\int_0^x p(x) \, dx} = \tilde{f}_\pm.
$$

Let $X^x(t)$ be a Markov process with generator (1.1), $\gamma = (\tilde{p}_+/\tilde{p}_-)^{1/2}$. Then

$$
\lim_{T \to \infty} P \left\{ \frac{1}{T} \int_0^T f(X^x(t)) \, dt - \tilde{f}_-}{\tilde{f}_+ - \tilde{f}_-} < x \right\} = F_\gamma(x).
$$

Here $F_\gamma(x)$ is the proper probability distribution on $[0, 1]$ which is uniquely determined by the Stiltjes transform given by the equation

$$
\int_0^1 \frac{dF_\gamma(x)}{z + x} = \frac{z^{-1/2} + \gamma(1 + z)^{-1/2}}{z^{1/2} + \gamma(1 + z)^{1/2}}, \quad z > 0.
$$

In particular,

$$
F_1(x) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad 0 \leq x \leq 1.
$$

Lamperti (1958) introduced more general two-parameter class of random variables $Y_{x,A}$, $0 \leq x \leq 1, A > 0$ with values in $[0, 1]$ and the Stiltjes transform given by

$$
E \frac{1}{z + Y_{x,A}} = \frac{z^{x-1} + A(1 + z)^{x-1}}{z^x + A(1 + z)^x}, \quad z > 0.
$$

It was noticed by Lamperti (1958) that the probability distribution $F_{x,A}$ of $Y_{x,A}$ has the density

$$
f_{x,A}(x) = \frac{\sin \pi \alpha}{\pi} \frac{A \alpha^{x-1} (1-x)^{2-1}}{A^2 (1-x)^{2x} + 2Ax(1-x)^2 \cos \pi \alpha + x^{2\alpha}} I_{[0,1]}(x).
$$

It was proved by Lamperti (1958) that the distributions $F_{x,A}$ are all possible limit distributions for the occupation time of some sets for some stochastic processes with discrete time (even not necessarily Markovian). Necessary and sufficient conditions for convergence to the distribution $F_{x,A}$ with given $A, x$ are also given by Lamperti (1958).

Watanabe (1995) proved the necessary and sufficient conditions for convergence of the occupation time $\tau_T = \int_0^T I_{X_t > 0}(X^x(t)) \, dt$ of the set $\{x > 0\}$ for the one-dimensional
(generalized) diffusion process in the canonical scale. For nongeneralized process in the canonical scale (the process with generator (1.1)) Watanabe’s conditions for
\[
\frac{\tau_T}{T} \text{ distr. } Y_{\alpha,A}, \quad 0 < \alpha < 1
\]
can be written in the form
\[
\left| \int_0^{\pm x} p(y) \, dy \right| = |x|^\beta K_\pm(|x|) \tag{1.7}
\]
with slowly varying in the Karamata sense (see Feller (1966), Section 8.8) functions \(K_\pm(|x|)\), satisfying the condition
\[
\lim_{x \to \infty} \frac{K_+(x)}{K_-(x)} = A^{1/\alpha}. \tag{1.8}
\]
(Here, as above and below, \(\alpha(2 + \beta) = 1\).

The goal of this paper is to find the sufficient conditions guaranteeing the convergence in distribution of the integral functional to \(Y_{\alpha,A}\) for one-dimensional null-recurrent diffusion.

2. Main result

We suppose again that \(X^x(t)\) is a Markov process with the generator (1.1) where \(p(x) = a(x)^{-1}\) satisfies Condition A and \(f(x)\) satisfies the following condition.

**Condition B.** \(f(x)\) is a piece-wise continuous bounded function and
\[
\lim_{x \to \pm \infty} \frac{\int_0^x f(x)|x|^{-\beta} p(x) \, dx}{\int_0^x |x|^{-\beta} p(x) \, dx} = f_{\pm},
\]
where the constants \(f_+\) and \(f_-\) satisfy the condition \(f_+ - f_- \neq 0\).

**Theorem 2.1.** Let the functions \(a(x), f(x)\) satisfy conditions A and B. Denote \(A = (p_+/p_-)^\alpha\). Then
\[
\lim_{T \to \infty} \mathbb{P} \left\{ \int_0^T (f(X^x(t)) - f_-) \, dt \right\} < x = F_{\alpha,A}(x).
\]

**Remark 2.1.** Condition A of Theorem 2.1 follows from conditions (1.7) and (1.8), so for the special choice \(f(x) = 1_{\{x > 0\}}(x)\) this Theorem follows from Watanabe (1995). In particular, for the Wiener process \(\alpha = 1/2, A = 1\), and we have the Levi arc-sine law again.

**Proof.** Denote
\[
\eta_{t,T}(x) = \frac{\int_0^T (f(X^x(\tau)) - f_-) \, d\tau}{T(f_+ - f_-)}.
\]
Then the function
\[
u_{s,T}(t,x) = \mathbb{E} \exp\{-s\eta_{t,T}(x)\} \tag{2.1}
\]
is a solution of the problem (it is Feinman-Kac formula, see e.g. Stroock (1993))
\[
\frac{\partial u_{s,T}}{\partial t} = Ta(x) \frac{\partial^2 u_{s,T}}{\partial x^2} - \frac{s(f(x) - f_-)}{f_+ - f_-} u_{s,T}, \quad u_{s,T}(0,x) = 1. \tag{2.2}
\]

So the Laplace transform
\[
w_{s,\lambda}^{(T)}(x) = \int_0^\infty \exp\{-\lambda t\} u_{s,T}(t,x) \, dt \tag{2.3}
\]
satisfies the equation (recall that \( p(x) = a(x)^{-1} \))
\[
\frac{d^2 w_{s,\lambda}^{(T)}}{dx^2} = p(x) \left[ \left( \frac{s(f(x) - f_-)}{f_+ - f_-} + \lambda \right) w_{s,\lambda}^{(T)} - 1 \right]. \tag{2.4}
\]

Consider now a new function \( v_{s,\lambda}^{(T)}(x) = w_{s,\lambda}^{(T)}(xT^x) \) and denote \( \tilde{p}(y) = |y|^{-\beta} p(y) \). Then we have the equation for \( v_{s,\lambda}^{(T)}(x) \):
\[
\frac{d^2 v_{s,\lambda}^{(T)}}{dx^2} = |x|^{\beta} \tilde{p}(xT^x) \left[ \frac{s(f(xT^x) - f_-)}{f_+ - f_-} + \lambda v_{s,\lambda}^{(T)}(x) - 1 \right]. \tag{2.5}
\]

We deduce from (2.1) and (2.3) that the functions \( w_{s,\lambda}^{(T)}(x) \) and therefore \( v_{s,\lambda}^{(T)}(x) \) are bounded uniformly in \( x \in \mathbb{R}^1, T > 0 \) for \( \lambda > \lambda_0 \). So we can conclude from (2.5) that the function \( d^2 v_{s,\lambda}^{(T)}(x)/dx^2 \) is bounded uniformly in \( T \) as \( x > \delta \) for any \( \delta > 0 \), and the families \( v_{s,\lambda}^{(T)}(x) \) and \( (d/dx)v_{s,\lambda}^{(T)}(x) \) are compact in the space of continuous in \( x \) functions with uniform metrics with respect to \( T \) for \( \delta < x < K \) for any \( K > \delta > 0 \).

We need to check also that \( (d/dx)v_{s,\lambda}^{(T)}(x) \) is uniformly continuous in \( T \) over \([-\delta, \delta]\).

In fact we have from (2.5)
\[
\left| \frac{d}{dx} v_{s,\lambda}^{(T)}(h) - \frac{d}{dx} v_{s,\lambda}^{(T)}(0) \right| \leq C \int_0^h |x|^{\beta} \tilde{p}(xT^x) \, dx =: c|J|.
\]

Here and below we denote \( C, C_1 \) any uniformly in \( T,x \) bounded functions of \( s, \lambda \).

Let us choose the constant \( \gamma \in (0,1) \) satisfying the inequality \( \gamma < \beta + 1 \). Then for \( 0 < |h| < T^{-\gamma} \)
\[
|J| = T^{-z\beta} \left| \int_0^h p(xT^x) \, dx \right| \leq CT^{-z\beta} |h| < C|h|^\gamma
\]

(we used the boundness of \( p(x) \) for \( |x| < 1 \)). For \( h > T^{-\gamma} \) we have integrating by parts
\[
|J| = T^{-z\beta} \int_0^{T^{-\gamma}} p(xT^x) \, dx + \int_{T^{-\gamma}}^h x^\beta \tilde{p}(xT^x) \, dx
\]

\[
\leq CT^{x-1} + T^{-z} h^\beta \int_0^{hT^x} \tilde{p}(z) \, dz + |\beta| \int_{T^{-\gamma}}^h x^\beta \frac{\int_0^{xT^x} \tilde{p}(z) \, dz}{xT^x} \, dx
\]

\[
\leq Ch^{(1-x)/\beta} + C_1 h^{\beta+1}.
\]

(we used here the inequality
\[
x^{-1} \int_0^x \tilde{p}(z) \, dz < C
\]
following from (1.2) for \( |x| \geq 1 \).
Analogously, the same bound can be established for \( h < -T^{-2} \). It follows from this bound and (2.5) that \( v_{s,\lambda}^{(T)}(x) \) and \( (d/dx)v_{s,\lambda}^{(T)}(x) \) are the compact families with respect to \( T \) in any interval \([-K, K], K > 0\). Let \( T_n \to \infty \) is chosen so that, uniformly in \( x \),

\[
\lim_{n \to \infty} v_{s,\lambda}^{(T_n)}(x) = v_{s,\lambda}^{0}(x), \quad \lim_{n \to \infty} \frac{d}{{dx}}v_{s,\lambda}^{(T_n)}(x) = \frac{d}{{dx}}v_{s,\lambda}^{0}(x).
\]

Then we can conclude, by integrating (2.5) and using conditions A and B, that the function \( v_{s,\lambda}^{0}(x) \) satisfies, for \( x \neq 0 \), the equations

\[
\frac{d^2 v_{s,\lambda}^{0}(x)}{{dx}^2} = p_{-}|x|^{\beta}(\lambda x v_{s,\lambda}^{0}(x) - 1), \quad x < 0
\]

\[
\frac{d^2 v_{s,\lambda}^{0}(x)}{{dx}^2} = p_{+}|x|^{\beta}(\lambda s v_{s,\lambda}^{0}(x) - 1), \quad x > 0
\]

and the glueing conditions

\[
v_{s,\lambda}^{0}(-0) = v_{s,\lambda}^{0}(+0); \quad \frac{d}{{dx}}v_{s,\lambda}^{0}(-0) = \frac{d}{{dx}}v_{s,\lambda}^{0}(+0).
\]

It is easy to see that any bounded for \( x > 0 \) solution of (2.8) can be written in the form

\[
v_{+}(x) = \frac{1}{s + \lambda} + c_{1}g((s + \lambda)p_{+})x.
\]

Here \( c_{1} = c_{1}(s, \lambda) \) is a constant with respect to \( x \), and \( g(x) \) is a unique bounded solution, for \( x > 0 \), of the problem

\[
g'' - x^{\beta}g = 0; \quad g(0) = 1.
\]

Analogously for \( x < 0 \) any bounded solution of (2.7) has the form

\[
v_{-}(x) = \frac{1}{\lambda} + c_{2}g(-\lambda p_{-})x.
\]

The parameters \( c_{1}, c_{2} \) can be found from glueing conditions (2.9):

\[
c_{1} = \frac{z^{2-1}}{s(1+z)(z^2 + A(1+z)^2)}; \quad c_{2} = -\frac{A(1+z)^{2-1}}{sz(z^2 + A(1+z)^2)}; \quad z = \lambda/s.
\]

So we can conclude that

\[
v_{s,\lambda}^{0}(x) = \begin{cases} v_{+}(x), & x \geq 0, \\ v_{-}(x), & x < 0 \end{cases}
\]

with \( c_{i} = c_{i}(s, \lambda) \) given by (2.13). Thus, we proved (taking into account (2.3)) that for \( \lambda \geq \lambda_{0} \)

\[
\lim_{T \to \infty} v_{s,\lambda}^{T}(x) = v_{s,\lambda}^{0}(x) = \lim_{T \to \infty} \int_{0}^{\infty} \exp\{-\lambda t\} u_{s,T}(t, x T^{2}) \, dt.
\]

So for any \( t > 0 \) the integral

\[
\int_{0}^{t} \exp\{-\lambda_{0} \tau\} u_{s,T}(\tau, x T^{2}) \, d\tau
\]

has the limit for \( T \to \infty \). \( \square \)
Lemma 2.1. The family $w_s(t, xT^2)$ is continuous in $t \in [0, t_0]$ for any $t_0 > 0$ uniformly in $T$.

Proof. Denote $\zeta(t, x) = f((X^x(t)) - f_\pm)/(f_+ - f_-)$. It follows from condition B that $\zeta(t, x) < C$. So we have from (2.1)

$$|w_s(t + h, xT^2) - w_s(t, xT^2)| \leq E^{1/2} \left\{ \exp \left( -\frac{s}{T} \int_{tT}^{(t+h)T} \zeta(\tau, xT^2) \, d\tau \right) - 1 \right\}^2 \times E^{1/2} \exp \left( -\frac{2s}{T} \int_0^{tT} \zeta(\tau, xT^2) \, d\tau \right) \leq C(s, t_0)|h|.$$ 

It follows from Lemma 2.1 and (2.15) the existence of the limit

$$\lim_{T \to \infty} w_s(t, xT^2) = w_s(t, x) \quad (2.16)$$

and the equation

$$\int_0^\infty \exp\{-st\}w_s(t, x) \, dt = \nu_s(0)(x) \quad (2.17)$$

We obtain from (2.7), (2.8) and (2.17) that $w_s(t, x)$ is a unique solution of the problem

$$\frac{\partial w_s}{\partial t} = p_- x^{-\beta} \frac{\partial^2 w_s}{\partial x^2} - sw_s, \quad x > 0,$$

$$\frac{\partial w_s}{\partial t} = p_- |x|^{-\beta} \frac{\partial^2 w_s}{\partial x^2}, \quad x < 0,$$

$$w_s(0, x) = 1 \quad (2.18)$$

with glueing conditions at $x = 0$. Thus,

$$w_s(t, x) = E \exp \left\{ -s \int_0^t 1_{\{x > 0\}}(X_s^0(\tau)) \, d\tau \right\},$$

where $X_s^0(t)$ is a diffusion process with the generator

$$[p_-^{-1}1_{\{x > 0\}}(x) + p_-^{-1}1_{\{x < 0\}}(x)]|x|^{-\beta} \frac{d^2}{dx^2}$$

and $X_0^0(0) = 0$.

So $X_s^0(t)$ is the skew Bessel diffusion process, and the probability distribution of

$$\eta_T = \frac{1}{T} \int_0^T 1_{\{x > 0\}}(X_0^0(t)) \, dt$$

does not depend on $T$ and coincides with the distribution $Y_{s, A}, A = (p_+/p_-)^x$ (see Watanabe (1995), Theorem 2). So we have

$$w_s(0, 0) = E \exp \left\{ -s \int_0^t 1_{\{x > 0\}}(X_s^0(\tau)) \, d\tau \right\} = E \exp\{-st\zeta_1\}.$$
and therefore for any $\lambda > 0$, $|s| < \lambda$ and $z = \lambda/s$, 
\[ v_s^{(0)}(0) = \int_0^\infty e^{-\lambda t} u_s,t_0(t, 0) \, dt = E \int_0^\infty \exp\{-\lambda t - st\zeta_1\} \, dt = E(z + \zeta_1)^{-1}/s. \]

On the other hand, it follows from (2.10) and (2.13), the equality 
\[ v_s^{(0)}(0) = \frac{1}{s + \lambda} + c_1 = \frac{z^{x-1} + A(1 + z)^{x-1}}{s(z^x + A(1 + z)^x)}. \]

So, taking (1.5) into account the theorem is proved. □

**Remark 2.2.** Analogous to Remark 2 in Khasminskii (1999), it is easy to obtain from Theorem 2.1 that 
\[ \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T f(X^x(t)) \, dt = \frac{p^+_x f_+ + p^-_x f_-}{p^+_x + p^-_x}, \]

\[ \lim_{T \to \infty} \text{var} \left\{ \frac{1}{T} \int_0^T f(X^x(t)) \, dt \right\} = \frac{(1 - x)p^+_x p^-_x (f_+ - f_-)^2}{(p^+_x + p^-_x)^2} \]

if the functions $a(x)$, $f(x)$ satisfy conditions A and B. The first of these equations can be interpreted as the limit theorem concerning behavior of solution of the Cauchy problem for the parabolic equation $\partial u/\partial t = L(x)u$. So we obtain another proof of Theorem 3.1 in Khasminskii and Yin (2000).

It is well known (see, for instance, Khasminskii and Yin (2000) and references there) that a Markov diffusion process $X(t)$ with generator 
\[ L_1(x) = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} \]

can be reduced to the process with zero drift with help of the substitution 
\[ Y(t) = \int_0^{X(t)} w(z) \, dz, \quad w(x) = \exp\left(- \int_0^x b(z) p(z) \, dz \right). \]  

(2.19)

**Remark 2.3.** It is easy to check with help of (2.19) (see Khasminskii and Yin (2000) again) that condition A for the process with generator $L_1(x)$ can be transformed to the following condition (here $\mu(x) = p(x)w^{-1}(x)$).

**Condition A’.**

*For some constants $p_+, p_-$, satisfying the conditions $\beta > -1$, $p_+ + p_- > 0$, there exist limits* 
\[ \lim_{X \to \pm \infty} \frac{\int_0^X \int_0^x w(z) \, dz \cdot \beta \mu(x) \, dx}{\int_0^X \int_0^x w(z) \, dz} = p_\pm, \]  

(2.20)

*and, moreover, for any $\varepsilon > 0$*

\[ \sup_{|x| > \varepsilon} \frac{p(x)}{w^2(x)} \left| \int_0^x w(z) \, dz \right|^{-\beta} < \infty. \]

Analogous to condition B can be transformed to the following condition.
**Condition B’.**

\( f(x) \) is a piece-wise continuous bounded function and

\[
\lim_{x \to \pm \infty} \frac{\int_0^x |\int_0^x w(z) \, dz|^{-\beta} \mu(x) f(x) \, dx}{\int_0^x |\int_0^x w(z) \, dz|^{-\beta} \mu(x) \, dx} = f_\pm.
\]  

(2.21)

So making substitution (2.19) and using Theorem 2.1, we arrive at the following result.

**Theorem 2.2.** Let \( X(t) \) be a null-recurrent Markov process with generator \( L_1(x) \), and \( a(x), b(x), f(x) \) satisfy conditions \( A’ \) and \( B’ \). Then the assertion of Theorem 2.1 is valid for \( f_\pm, p_\pm \) from (2.20), (2.21).

**Example.** Consider the process \( X(t) \) with \( a(x) = 1 \) and \( b(x) \) having the compact support. Let also

\[
\lim_{x \to \pm \infty} \frac{1}{x} \int_0^x f(x) \, dx = f_\pm, \quad f_+ \neq f_-.
\]

Then it is easy to check that the conditions of Theorem 2.2 are fulfilled, \( \beta = 0, A = \exp(-\int_{-\infty}^{\infty} b(z) \, dz) \), so the limit distribution of integral functional coincides with the distribution \( F_\gamma \) from Theorem 1.2 with \( \gamma = A \).

**Acknowledgements**

I am grateful to the referee for the valuable suggestions that improved the paper.

**References**


