The dynamics of retirement saving — theory and reality

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Abstract

Pension problems and reforms are in the foreground of public interest and political action in many countries, yet economic theory offers inadequate support for finding viable solutions, because it is heavily loaded with simplifying concepts and unrealistic assumptions. These concepts and assumptions are briefly summarized in Chapter 1, while a generalized framework based on them is presented in Chapter 2. The basic stationary assumption is then relaxed in Chapter 3 what results in the conclusion that a profound, not just technical, but conceptual innovation is required. Chapter 4 outlines a few major issues and concepts for a more realistic pension economics. A summary is given in Chapter 5. © 2000 Elsevier Science B.V. All rights reserved.

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1. Concepts and assumptions in theory

Adults are supposed to work, to earn income and from that to pay for their consumption. Yet in the first and (for the majority) in the last years of human life people do not earn while they must consume. The question is, how are these two, non-earning stages of life financed? The simplified answer is: from the excess income of the stage in-between, in other words, from savings over the earning span. Thereby the rather narrow ‘conceptual framework’ is complete: the trinity of income, consumption and saving explain all aspects of the problem. Maybe a fourth concept, wealth (expectancy, assets, stocks) should be added, although it is nothing but accumulated saving.

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The problem, by definition, is cross-sectional as well as longitudinal. In any given moment, the young (children), the earners and the elderly (pensioners) live together in human society. On the other hand, all cohorts (generations, surviving individuals) proceed through these three stages of life with the passing of time.

It is extremely difficult, however, to handle cross-sectional and longitudinal aspects simultaneously. Theory needs a simplifying assumption: the ‘stationary population’ where all cohorts go through identical life-paths. Sometimes the assumption is quite crude: all individuals are supposed to work for $N$ years and live $M$ years thereafter. Sometimes the approach is more subtle, for example the probability of dying at age $k$ is a given number $m_k$. In some models life consists of two or three periods, in others many years of age are considered. Invariably, however, successive cohorts are identical. Consumption and/or earning depends on age but age-specific properties are constant over time.

The most important implication of the stationary assumption is the ‘Golden Rule congruence’. Longitudinal relations are always affected by the growth rate and the interest rate, which make longitudinal concepts numerically different from their qualitatively corresponding, cross-sectional counterparts. If and when, however, these two rates are equal (the Golden Rule case) then longitudinal proportions (in some models even quantities) are identical with the cross-sectional ones.

The stationary assumption makes room for the representative individual whose presence is required by orthodoxy, because his role is life-time ‘utility maximization’ under selected constraints. Utility is most often derived from consumption (or in some human capital models from work and/or leisure), while income (or consumption) is exogeneously given. Thus in most models optimization yields the ‘longitudinal allocation’ of consumption (or income) over the life-path. The optimal allocation is then considered viable for all individuals and cross-sectional, macroeconomic conclusions can be drawn.

Without exception, optimization is subject to the longitudinal (alternatively referred to as individual, competitive, actuarial) budget constraint, to be called here unambiguously the ‘zero-bequest constraint’, which calls for equality of life-time consumption with life-time income.

After this point, however, theory is branching off with respect to the cross-sectional environment. For the life-cycle theory of saving and in most human capital models non-zero aggregate saving is either explicitly required or implicitly presumed, as these theories are mainly concerned with wealth (positive or negative). To the contrary, overlapping generations (OLG for brevity) are built on the dual-constraint principle: besides longitudinal zero bequest, the cross-sectional (or social, conservative, pay-as-you-go, feasibility, etc.) ‘zero-saving constraint’ is also imposed and situations when both constraints are simultaneously satisfied, are called ‘equilibria’.

Technically, a certain clumsiness is characteristic of most studies. One reason is that the growth rate and the interest rate are separately denoted and treated, with the result that at least one of them must be constant over time (whether zero or not, does not really matter too much, except that zero is more convenient). The other reason — in OLG models — is the inconvenience inherent in dealing with two
concepts (longitudinal bequest and cross-sectional saving), substituting from here to there, watching two constraints simultaneously.

2. The generalized stationary framework

This section presents a more convenient, generalized descriptive framework, capable of accommodating most relevant theories and models, based on Augustinovics (1991). We introduce

1. A numéraire (e.g. aggregate cross-sectional labor income), to be denoted by $a_t$.

All other concepts are expressed quantitatively in terms of the numéraire, thereby they become relative, easily comparable over time and across countries. The lower right index refers to calendar year: the numéraire may be floating over time and no assumptions are required concerning its change.

2. The relative interest factor, to be denoted by $u$, which is the ratio of the interest factor over the growth factor. Both factors may be changing in time, only their relation has to be assumed time-invariant — but it does not have to be a presupposed number, it will be treated as a continuous variable which all longitudinal concepts are functions of. (Obviously $u = 1$ indicates the distinguished Golden Rule case.)

3. Age-specific cross-sectional shares in aggregate income and consumption as the basis of accounting, rather than the longitudinal allocation which is unrealistic because of the uncertainty concerning the future. (If desired, identical preferences and utility maximization could still be postulated to be acting in the background. The rationale for the given cross-sectional shares is, however, irrelevant as long as they are statistically observable.)

Let $T$ denote the age of the oldest cohort in the population, then formally for $k = 0, \ldots, T$

$$\hat{w}_{x,x+k} = a_{x+k}w_k \quad \hat{c}_{x,x+k} = a_{x+k}(1-s)c_k$$

$$\sum_k w_k = 1 \quad \sum_k c_k = 1$$

where the numéraire $a_{x+k}$ is aggregate labour income in year $x+k$. The aggregate saving rate is denoted by $s$, hence $a_{x+k}(1-s)$ describes aggregate consumption. The terms $\hat{w}_{x,x+k}$ and $\hat{c}_{x,x+k}$ indicate labour income and consumption in year $x+k$, respectively, at age $k$ of the cohort born in year $x$, while $w_k$ and $c_k$ denote the shares in aggregate totals of any cohort aged $k$.

Stationary assumptions concerning the economy are thus somewhat milder than usual, as the numéraire may freely change over time, fluctuations in growth and interest rates are accommodated. Only the relative discount factor $u$ (the relation between growth and interest) and the saving rate $s$ (the relation between aggregate income and consumption) are assumed to be constant over time. Hence cross-sectional as well as life-time income, consumption and saving of successive cohorts may differ in absolute numbers. Yet the basic stationary assumption is sustained, as
the cross-sectional, age-specific shares \( w_k, c_k \) are time-invariant, identical for all cohorts. (This assumption is stronger than, although hardly possible without, a demographically stable population.)

From the cross-sectional shares, longitudinal concepts characterizing the life-path of a cohort — hereafter denoted by symbols printed in bold and indexed by \( x \), referring to the time when the respective cohort was born — can be properly derived with the help of the numéraire and the relative discount factor. Most important of them is the

\[
B_x(u) = z_x \sum_k (w_k - (1 - s)c_k)u^{-k} \quad \text{bequest}
\]

i.e. life-time wealth, discounted to birth, of the cohort born in \( x \). The bequest equals zero if and only if

\[
s = S(u) = \frac{\sum_k (c_k - w_k)u^{-k}}{\sum_k c_k u^{-k}} \quad \text{zero-bequest saving function}
\]

i.e. if and only if the aggregate saving rate \( s \) equals \( S(u) \), the function that satisfies the longitudinal zero-bequest constraint at any value of the relative interest factor. By \( S(u) \) the condition for zero bequest can be assessed without postulating an almighty constraint. Observe that the numéraire has canceled, the shape of the \( S(u) \) function is time-invariant, as it depends exclusively on the age-specific shares, identical for all cohorts.

At any point of the life-path, the cohort’s life-time stock consists of (positive or negative) wealth, i.e. the difference between income and consumption already accrued in the past and expectancy, the similar difference to be expected in the future. Summing over together living cohorts yields human wealth and expectancy, i.e. stocks of the entire population. A somewhat tiresome and tricky calculation provides at \( u \neq 1 \)

\[
Q_x(u) = z_x \frac{\sum_k [w_k - (1 - s)c_k]u^{-k}}{1 - u} \quad \text{human wealth}
\]

\[
E_x(u) = z_x \frac{\sum_k [w_k - (1 - s)c_k]u^{-k} - s}{1 - u} \quad \text{human expectancy}
\]

At \( u \neq 1 \) both wealth and expectancy depend on the difference between the aggregate (cross-sectional) saving rate \( s \) and the relative factor of cohort bequest (the longitudinal ‘saving rate’). In other words, human stocks are functions of the difference between the longitudinal and the cross-sectional aspect, the difference that occurs even under the basic stationary assumption because the interest rate differs from the growth rate as reflected by \( u \neq 1 \). It is worth noting that if the bequest component is zero, i.e.
iff \( s = S(u) \) then \( Q(u) = S(u)/(1 - u) \) and \( E(u) = -S(u)/(1 - u) \)

the sign of wealth \( Q(u) \) is identical with the sign of zero bequest saving \( S(u) \) at \( u < 1 \) and, paradoxically, opposite at \( u > 1 \).

In the Golden Rule \( u = 1 \) case we omit the full form for \( s \neq 0 \) as of little interest and consider only

\[
Q_1(1) = \alpha \sum_k [c_k k - w_k k] \quad (u = 1 \text{ and } s = 0)
\]

\[
E_1(1) = \alpha \sum_k [w_k k - c_k k] \quad (u = 1 \text{ and } s = 0)
\]

when human stocks depend on the difference between the (weighted) average age of cross-sectional consumption and earning: wealth and expectancy are identical in absolute value and opposite in sign.

Thereby the major ingredients of the generalized stationary framework are complete. More details, further implications and interpretation are theoretically attractive, but would be superfluous in this paper, where we shall soon relax the basic stationary assumption. Nevertheless, a few general properties and the most frequent special cases deserve some attention. (Proofs will be omitted as they are easily derived through elementary analysis.)

It is easy to perceive that OLG models reside in the roots of \( S(u) \), where longitudinal bequest and cross-sectional saving are equally zero, while life-cycle and human capital theory are interested in the positive or negative intervals, with particular reference to the sign and magnitude of \( Q(u) \) and \( E(u) \). We have thus a unified and, therefore, comparative umbrella for all three major branches of theory.

The zero-bequest function \( S(u) \) has a trivial root at \( u = 1 \) because of the definition in (Eq. (2)). The reason is exactly the Golden Rule congruence: if proportions on the longitudinal life-path are identical with proportions in the cross-sectional population, then the sum of cross-sectional shares 'collected' over the entire life-path are identical with the corresponding cross-sectional aggregate, namely unity. Hence zero bequest requires (corresponds to) zero aggregate saving. This is a common and indeed trivial feature of all models built on the stationary assumption (although sometimes rather refined proofs are presented to reach the conclusion and Samuelson (1958) kept marvelling at his 'biological' interest rate).

In the 'non-trivial roots' of \( S(u) \) — if there are any — both human wealth \( Q(u) \) and expectancy \( E(u) \) also equal zero by (Eq. (5)) and (Eq. (6)). Such roots, therefore, while being distinguished 'equilibria' for OLG, are detrimental for life-cycle and human capital theory, usually evaded by careful selection of growth and interest rates.

The sign of the derivative \( S'(u) \) is determined by the difference between the average age of the longitudinal earning path and that of the consumption path:
The theoretical implications of the age-gap $Y(u)$ are not quite trivial but its analytical consequences are numerous, revealed to some extent by e.g. Arthur and McNicoll (1978) and Lee (1980). Let the population be called ‘young’ if $Y(u) > 0$, ‘old’ in the opposite case and ‘symmetrical’ if $Y(u) = 0$ holds. Then

1. In the Golden Rule trivial root zero aggregate saving usually does not result in zero stocks. By (Eq. (8)) wealth is negative (positive) if the population is young (old). Zero stocks result only in case of an incidentally symmetrical population. (In the latter case the trivial root is coincidentally also non-trivial).

2. In the trivial root the $S(u)$ function is increasing (decreasing) if the population is young (old); it has an extremum if the population happens to be symmetrical. Except for the latter case, the sign of $S(u)$ is thus changing in the trivial root.

It follows that there exists at least one interval of $u$ where $S(u)$ is positive and one where the function is negative.

Positive or negative saving for zero bequest? Yes, this is possible: interest rates smaller (larger) than the growth rate would shrink (inflate) longitudinal aggregates and if this effect works differently for the earning path than for the consumption path (because the population is young or old) then equating the longitudinal aggregates requires non-equal cross-sectional aggregates. Hard to swallow but plausible.

Non-zero stocks originating from zero saving at $u = l$, negative (positive) stocks originating from positive (negative) saving at $u > l$? This would be more difficult — although not impossible — to explain by a lengthy consideration. The questions of origin are, however, generally obscure in a stationary framework and the merits of pondering over them is questionable. (No wonder, therefore, that OLG concentrates on the non-trivial roots and usually neglects the problem of stocks, while life-cycle theory and human capital arguments are always careful in selecting growth and interest rates.)

What about the ‘number of non-trivial roots’ of $S(u)$, hence the number of positive and negative intervals? In (Eq. (4)) the denominator is positive, while the numerator is a polynomial in $u$, with $c_k - w_k$ as coefficients. By Descartes’s rule of signs, $S(u)$ has at most as many positive real roots as the number of sign-changes in the $c_k - w_k$ series. ‘At most’, however, does not mean that there are indeed as many roots — there may be less. Anyway, generality ceases at this point — the number of sign-changes depends on the conceptual and numerical specification of the particular stationary model.

In simple models, life consists of two or three periods, one of which is the earning period, preceded and/or followed by a consumption-dominated period (childhood and/or retirement). Consequently, the average age of earning is constant, while that of consumption is decreasing in $u$. The age-gap $Y(u)$ is monotonously increasing, it has at most one root, hence $S(u)$ has at most one extremum, thus at most two roots, one of them the trivial $u = l$. 

\[ S(u) = \frac{\sum_k w_k u^{-k}}{u \sum_k c_k} Y(u) \] where \[ Y(u) = \sum_k w_k u^{-k} k \] (10)
In the simplest, two-period cases, the earning period is either preceded or succeeded by a single consumption-dominated period. Obviously, childhood (education, training) before earning results in a young population and life-path, i.e. the age-gap $Y(u)$ is positive all over $u$, while retirement after earning implies an old population, i.e. $Y(u)$ is negative. The zero-bequest $S(u)$ function is, consequently monotonously increasing in the first and decreasing in the latter case, non-trivial root(s) are non-existent. For a two-period young (old) population, negative (positive) aggregate saving would balance the longitudinal life-path at $u < 1$ and the opposite holds at $u > 1$.

Systematizing and generalizing Samuelson’s consumption-loan idea, Gale (1973) recognized the mirror-image nature of the two two-period cases. He called the young population model ‘classical’ (referring to Irving Fisher) and the old population model ‘Samuelson’. He asserted that the only relevant ‘equilibrium’ (root) is the Golden Rule one. (Gale’s ‘no-trade’ equilibrium, in other words a flat zero saving all over $u$, is a mere mathematical construct, irrelevant from the substantial aspect, since it implies that both together living generations are self-sustaining, there is no need to finance non-earning periods of life.)

Gale’s seminal contribution is generally acknowledged with respect to the retirement saving problem, but his ‘classical’ (young) case has received little attention in the literature, except for some admiration towards the formal beauty of symmetry. This is a pity since the case is the most abstract skeleton of the human capital problem, in the same sense as the ‘Samuelson’ (old) case is generally accepted for the retirement issue.

The two-period case is a favourite in economic theory because of its simplicity and transparency. Nevertheless, the inherent unreality has been pointed out since long and by many (Tobin (1967) being one of the first and Simonovits (1995) one of the most recent critics), as in real life the earning span is both preceded and followed by a consumption-dominated stage (there are both children and old people in a society).

Analytically, however, the three-period case differs qualitatively from two periods: the age-gap $Y(u)$ is rising from negative to positive, hence it has a single root. $S(u)$ has a single extremum, hence there exists a single non-trivial root, at $u < 1$ if the population is young and at $u > 1$ if it is old. Either the $u < 1$ or the $u > 1$ interval is thus split into a positive and a negative portion with respect to $S(u)$. (With the exception of an eventually symmetrical population.)

Fig. 1 demonstrates the shape of the zero-bequest $S(u)$ (continuous line) and human wealth $Q(u)$ (dotted line) in all cases of two and three periods.

If the stages of life (whether two or three) are not represented by a single period each but consist of several periods (years), the above statements concerning the number of $S(u)$ roots, hence positive and/or negative intervals, still hold. When the number of periods exceeds three, however, the age-gap $Y(u)$ may have more than one roots, thus $S(u)$ may not be monotonously decreasing and/or increasing, there may be ups and downs within its positive and/or negative intervals.

The sign of $S(u)$ in the case of a two-stage old population, — positive at $u < 1$ and negative at $u > 1$, whether decreasing monotonously or not, — offers simple
explanations for several, seemingly complex theoretical issues. For example the famous social security paradox of Aaron (1966) states that unfunded (pay-as-you-go) pension schemes are more favourable at \( u < 1 \), while funded schemes are more advantageous at \( u > 1 \). Considering that unfunded schemes are supposed to be balanced cross-sectionally, i.e. to imply zero aggregate saving, it is clear that the zero saving requirement is ‘better’ than the need for positive saving, while zero is ‘worse’ if negative saving would suffice to achieve longitudinal balance. Another example is the controversy about the original zero interest rate in the life-cycle model (Modigliani and Brumberg (1954), Ando and Modigliani (1963)). Later Modigliani rightly claimed that the interest rate need not be zero, but he should have added that it needs to be smaller than the growth rate for the zero bequest condition to result in positive aggregate saving.

The shape of the \( S(u) \) function may be further spoiled if life consists of more than three stages. For example, if the earning span is interrupted by consumption-dominated stages because of unemployment or mid-life retraining, then the number of sign-changes in the \( w_k - c_k \) series is more than two, hence \( S(u) \) may have more than one non-trivial root. (Already Gale attempted to generalize to \( n + 1 \) periods, but when proving the uniqueness of ‘balanced equilibria’ — the non-trivial root of our \( S(u) \) function — he needed to inconspicuously compromise with the three-stage case.)

![Diagram](image)

Fig. 1. Typical shapes of \( S(u) \) and \( Q(u) \).
Factors shaping the zero-bequest $S(u)$ function are, however, intellectually challenging but realistically rather irrelevant niceties within the basically stationary framework.

3. Relaxing the stationary assumption

It is a fact of life that the life-paths of successive cohorts are different with respect to cohort size, longevity, booms or recessions at various periods of life, and several other factors. Consequently it is impossible for the age-specific, cross-sectional shares in income and consumption to remain constant over time. The shares $w_k$ and $c_k$ in the descriptive framework must be substituted by $w_{x,t}$ and $c_{x,t}$, respectively, indicating the shares in year $t$ of the cohort born in $x$. As opposed to (Eq. (1)), the starting point of accounting thus becomes

$$\hat{w}_{x,t} = \pi w_{x,t} \quad \hat{c}_{x,t} = \pi(1-s)c_{x,t}$$

(11)

with a crucial difference that has far-reaching implications:

$$\sum_x w_{x,t} = 1 \quad \text{and} \quad \sum_x c_{x,t} = 1 \quad \text{but} \quad \sum_t w_{x,t} \neq 1 \quad \text{and} \quad \sum_t c_{x,t} \neq 1$$

(12)

i.e. cross-sectionally the shares still add up to 1 (that’s why they are shares) but there is no reason why longitudinally, over the life-path the sum of shares ‘collected’ by a cohort should be exactly unity. The Golden Rule congruence has been impaired: even at $u = 1$ longitudinal proportions are not identical with cross-sectional ones.

The first concept to disappear with the stationary assumption is that of the representative individual. If the life-paths of successive cohorts are different they cannot be expected to have identical preferences. Even if preferences did not — miraculously — change with time, the actual parameters of earning and consumption would still be varying from cohort to cohort. Thereby the unified longitudinal utility maximization becomes conceptually infeasible.

It still remains possible, within the framework built on cross-sectional shares, to assert the zero-bequest function of the cohort born in $x$:

$$S_x(u) = \frac{\sum_k (c_{x,x+k} - w_{x,x+k})u^{-k}}{\sum_k c_{x,x+k}u^{-k}}$$

(13)

Because of (Eq. (12)), however, the convenient assurance of a trivial root at $u = 1$ is gone since the Golden Rule congruence is lost. The only case, when the existence of a unique root is guaranteed — not necessarily at $u = 1$ — is the simplest, solid two-period case, whether ‘young’ or ‘old’. In the three-period case $S_x(u)$ has at most two roots — very inconveniently, however, it may have no roots at all. In other words, it may be impossible at any relative interest factor $u$ to balance the life-path of the cohort born in $x$ by zero aggregate saving.
What is worse, $S_x(u)$ is cohort-specific. It would balance the life-cycle of cohort born in $x$ — whether by zero or non-zero aggregate saving — but it would certainly not, simultaneously, do the same for other cohorts, born before or after $x$. Yet there cannot exist more than a single aggregate saving rate in any one period of time. At this point, it is not only the formal analysis, it is the conceptual framework that collapses: the aggregate saving rate ceases to be an appropriate device for achieving zero bequest, for balancing the longitudinal life-path(s).

Conclusion: the concept of the balanced longitudinal path itself requires a fresh look. It may appear far-fetched, but the truth is that a seemingly simple, technical modification, the dismissal of the stationary assumption, results in the necessity of profound, conceptual rethinking. Bequest and aggregate saving must be substituted by appropriate concepts that reflect the case of the non-stationary population in reality.

4. Issues for pension economics

The financing of the two consumption-dominated stages of life is more or less institutionalized in modern societies. Child-raising and education are supported through tax-exemptions, children’s and maternity allowances, government-subsidized schools and universities, student loans — to an extent and in forms that vary from country to country. On the other hand, there exist various types of mandatory pension schemes in most countries and such schemes are largely or almost exclusively responsible for income in old age for the majority of the population.

It would be highly desirable for economic theory to consider the two problems and two groups of institutions in their mutual interdependence — in other words, to develop the realistic counterpart of the theoretical three-period case. Arguments could even be raised for an arrangement — maybe somewhat futuristic — where the actual financing of the two consumption-dominated stages would be linked (Augustinovics (1997)). In contemporary reality, however, several significant factors separate child-costs from retirement financing and this is well reflected in a split literature: one fraction about childhood and human capital, another about retirement. No wonder that the two-period cases of theory are still predominantly referred to, although the gap between a stationary theory and the non-stationary reality is usually ignored.

The rest of this paper is devoted to issues, mostly unresolved, waiting for thorough inquiries by the economics of mandatory pension systems. (Voluntary retirement saving, although prudent and desirable, is an entirely different matter that will not solve the problem of old-age income security for the majority of people.) The argument will be built around a tentative, illustrative rather than complete outline of an accounting framework that should be equally relevant for public or private, PAYG (pay-as-you-go) or funded pension schemes. This is not to say that the type of governance and the mode of finance are irrelevant, but these are aspects amply covered in the ongoing pension debates in most countries. Here we shall focus on more basic issues, common for all types of pension schemes: first of
all on demographic, economic and system-specific trends that determine cross-sectional and longitudinal balance or imbalance in a system.

We begin with two descriptive identities, to reveal the factors that shape the revenues and expenditures of a pension scheme. For the cohort born in \( x \), in calendar period (year) \( t \):

\[
V_{x,t} = \frac{H_{x,t}}{A_{x,t}} \frac{A_{x,t}}{V_{x,t}} = H_{x,t}A_{x,t}V_{x,t} \quad \text{contribution base} \quad (14)
\]

\[
P_{x,t} = \frac{L_{x,t}}{Z_{x,t}} \frac{Z_{x,t}}{P_{x,t}} = L_{x,t}z_{x,t}P_{x,t} \quad \text{pension expenditure} \quad (15)
\]

where \( H_{x,t} \), number of persons in active age; \( A_{x,t} \), number of contributors (employed); \( A_{x,t}/H_{x,t} \), employment intensity; \( V_{x,t} \), average wage; \( L_{x,t} \), number of persons in pensionable age; \( Z_{x,t} \), number of beneficiaries (pensioners); \( Z_{x,t}/L_{x,t} \), retirement intensity; \( P_{x,t} \), average pension.

Both identities begin with the crucial demographic fact, the number of persons in the respective age-span. It should be noted, however, that these numbers are not purely demographic: pensionable age is determined by the statutory retirement age, which may be — often is — different for men and women, for various occupations. This is the first indication of demographic trends being important, but not predominantly responsible for pension problems.

Secondly, not all active-age persons are contributors and not all pensionable-aged people are actually pensioners. Partly because early as well as delayed retirement is usually permitted (under certain conditions), and partly because the respective pension scheme may not cover the entire population, or some participants may not have acquired sufficient pension rights on their own. The resulting intensities \( a_{x,t} \) and \( z_{x,t} \) thus reflect the coverage by the pension scheme on one hand and the crucial economic effect on the other hand.

For historical reasons, most pension schemes are employment-based in the sense that actively employed (often including self-employed) persons are required to contribute, proportionally to their labour income (or a limited portion of it). Various pension systems relate in various ways to those who are not employed (unemployed, students, enlisted soldiers, housewives, disabled persons, etc.). The resulting complications are ignored in this stylized accounting framework — as they should not be in a veritable pension model.

Employment and retirement intensity, \( a_{x,t} \) and \( z_{x,t} \), respectively, of cohorts around the statutory retirement age tend to move in opposite directions, depending on economic cycles. Pension systems may stumble in times of severe and long-lasting recessions or crises, when output, employment and real wages shrink, when those who can, escape from unemployment into early retirement. For the more distant future of employment-based pensions, the spread of ‘atypical’ working arrangements on the labour market is likely to raise serious problems, as the traditional contribution base will probably be contracting to an extent far beyond the much-feared demographic effect.
Relations with the income tax system (e.g. wages may be gross and pensions net of tax, etc.), possible exemptions from contribution and non-compliancy are also ignored in our stylized framework. What remains, therefore, is average wage \( v_{x,t} \) and average pension \( p_{x,t} \) to be taken into account.

Assuming that contributions are due — and paid — according by a rate \( m_{x,t} \) out of the contribution base \( V_{x,t} \), the revenues and expenditures of the pension system, and/or individual cohorts, can be accounted for by aggregation. Elegant theorems can be hardly proved but crucial empirical analysis is possible and desirable. For transparency, we define cross-sectional and longitudinal average rates, applying a constant \( r \) technical discount factor in the latter cases (more refined methods of averaging are of course possible). Henceforth bold characters indicate longitudinal concepts:

\[
\begin{align*}
    a_t &= \sum_x A_{x,t} / \sum_x H_{x,t} \\
v_t &= \sum_x V_{x,t} / \sum_x A_{x,t} \\
z_t &= \sum_x Z_{x,t} / \sum_x L_{x,t} \\
p_t &= \sum_x P_{x,t} / \sum_x Z_{x,t} \\
m_t &= \sum_x m_{x,t} V_{x,t} / \sum_x V_{x,t} \\
f_t &= \sum_x m_{x,t} V_{x,t} / \sum_x V_{x,t} \\
\end{align*}
\]

Let us first consider the relation between expenditure and the contribution base cross-sectionally. If the \( m_t \) contribution rate corresponded exactly to their ratio, the annual balance in \( t \) would obviously be zero. Hence the ratio may be called the ‘internal contribution rate’ of the system:

\[
m_t = \frac{\sum_x P_{x,t}}{\sum_x V_{x,t}} = \frac{\sum_x L_{x,t} z}{\sum_x H_{x,t}} = d_t i_t = g_t r_t
\]

where \( d_t = \sum L_{x,t} / \sum H_{x,t} \), demographic dependency ratio; \( i_t = z_t / a_t \), intensity ratio; \( r_t = p_t / v_t \), ‘replacement’ rate; \( g_t = d_t i_t \), system dependency ratio.

System dependency \( g_t \) — the product of demographic dependency \( d_t \) and intensity ratio \( i_t \) — equals the ratio of beneficiaries over contributors. Prudent studies of pension issues discuss the concept, but it is often identified with the demographic ratio in simplified arguments for general public consumption. Empirical evidence, however, demonstrates that system dependency may seriously deteriorate for economic reasons, resulting from a sharp rise of the intensity ratio (declining employment and increasing retirement), even during demographically benign periods. Exactly this has happened, for example, during the 1990s in the Central-European transition countries (cf. Augusztinovics (1999)). Reversely, a
favourable decrease of the intensity ratio (shrinking unemployment and delayed retirement) may well counter-balance, at least to a large extent, the effect of population aging in the next century.

The ratio of average pension over average wage, \( r_t \), is generally called replacement rate, but quotation marks are justified since present pensions do not replace present wages; replacement is per se a longitudinal concept. (Cf. Kruse (1997).)

Anyway, as shown by the right-most segment of (Eq. (16)), the internal contribution rate finally boils down to the product of the system-dependency ratio and the replacement rate. Therefore, in principle the procedure is reversible, an ‘internal replacement’ rate can be defined as the ratio of the actual contribution rate over system-dependency:

\[
\hat{m}_t = g_t r_t, \quad \hat{r}_t = m_t / g_t
\]

The mutual interdependence of contribution and replacement rates is a crucial problem for pension system design. It is somewhat obscured by the distinction, fashionable in the pension literature, between ‘benefit defined’ and contribution defined pension systems. In pension politics, the interdependence is usually shielded from public debate. While in many countries there is a growing pressure to reduce contribution rates in the name of ‘competitiveness’, seldom is it made clear that in the long run the consequence may be mass poverty in old age.

The cross-sectional ‘balance’ (contribution revenue minus pension expenditure) of the pension scheme can be expressed in terms of the difference between actual and internal rates:

\[
D_t = m_t \sum V_{x,t} - \sum P_{x,t} = (m_t - \hat{m}_t) \sum V_{x,t} = (\hat{r}_t - r_t) g_t \sum V_{x,t}
\]

The balance, however, depends naturally also on the absolute magnitude of the aggregate contribution base, e.g. on the size of pension scheme (fund) or of the country. Therefore, actual and internal contribution and replacement rates seem to be more convenient tools for comparative analysis.

It is easy to perceive, even without formal description, that analogous concepts may be and should be defined longitudinally, combining the earning and retirement spans for each cohort. Either the longitudinal, internal contribution rate of the corresponding internal replacement rate would balance the cohort’s life-path with respect to the pension scheme. Obviously, internal (balancing) rates vary from cohort to cohort — as did the theoretical zero-bequest function \( S_x(u) \). Conclusion: either all relevant parameters of a pension scheme are cohort-specific and adjusted precisely from year to year — what seems to be practically impossible — or longitudinal balances could not be expected to be zero. Inter-cohort redistribution is inevitable in any pension system, quite apart and distinct from the usual, cross-sectional concept of ‘intergenerational transfers’ from the ‘young’ to the ‘old’.

Fig. 2 demonstrates the results of simplified calculations in the case of Hungary. All parameters, except the demographic ones, were kept constant in time and \( \rho = 1 \) was assumed. Hence pure demographic trends are revealed, based on official forecast until 2050 and simple projection after that. The sharply increasing line,
denoted by CR, represents the cross-sectional internal contribution rate $m_t$ over two centuries. Horizontal straight lines represent longitudinal internal rates $m_x$ over the earning span of cohorts born in the year indicated to the left of the line. It is evident that now, around 2000, applying the cross-sectional internal contribution rate would result in huge negative longitudinal balances for all cohorts presently in the labour force, but the situation would change around 2020. (The statutory contribution rate is much higher. The general picture is much less clear if economic parameters are also allowed to change.)

It should be added that desaggregation of birth cohorts by social properties (e.g. gender, educational attainment, occupation, etc., and/or participation in a given pension fund) would be absolutely necessary within a similar accounting framework. This has not been formally demonstrated here for brevity. The reader is requested to visualize a third index attached to each concept, beyond $x$ and $t$, say $n$ that would indicate social group $n$, born in $x$, in calendar year $t$. Then $V_{x,t,n}$ and $P_{x,t,n}$ would represent the group’s contribution base and pension, respectively, $m_{x,t,n}$ and $r_{x,t,n}$ their longitudinal internal rates, $D_{x,t,n}$ the life-time balance of their involvement with the pension system. Naturally, it would turn out that intra-cohort redistribution is inevitable as well.

The concept in pension parlance that corresponds to the theoretical zero-bequest constraint, i.e. to longitudinal balance, is ‘actuarial fairness’. This concept is often identified with the lack of redistribution, implying that at the time of death the present value of life-time benefits equals that of life-time contributions. It is usually assumed that funded schemes are actuarially fair while PAYG schemes are ‘redistributive’. That is a fallacy: all veritable pension schemes are redistributive by nature.

Firstly, because they are insurance pools against the mortality risk, redistributing among individuals according to longevity, from those who die early to those who live long. (Even in principle, full actuarial fairness at the individual level is possible only in so-called pension funds that permit cumulated personal contributions to be taken out in a lump sum on retiring, or transformed into a fixed-length annuity, or be inherited at any point of time in case of death of the insured person. Such funds
are, however, just pseudo-pension schemes, particular savings devices substantially not much different from banks or investment funds.)

Secondly, because they operate with parameters, at least a few of which are standardized, most often the statutory contribution rate. Hence they cannot really adjust to cohort- and group-specific parameters, — not to mention the uncertainty of future. Pension schemes are necessarily redistributing among birth cohorts and social groups. It is unfair to promote private pension funds by advertising the ‘your money remains yours’ slogan.

Thirdly, an intentional redistribution from the rich to the poor works in most public schemes, but it is a counter-productive addition to the pension institution. Social assistance and pension insurance should rather be separated in order to make the pension system transparent and incentives to contribute viable. For there is something that ordinary people, who know little about actuarial fairness, expect from the pension system. Maybe it could be called ‘pension fairness’, meaning that benefits, — as long as they last, i.e. until death — should be secure and proportional to previous contributions. (For example, two persons with very similar contribution histories should not receive substantially different pensions because they belong to different occupations, or retired at different points of time, or had been subjected to different rules of indexation.) Such fairness is not incompatible with redistribution resulting from various reasons of differences in longevity.

Denying the existence of redistribution, and/or fight endless controversies about privatization and the mode of finance does not seem to be a fruitful course for pension economics and pension reform proposals. Clear concepts and definitions of the various types of redistribution as well as generally accepted (acceptable) methods of measurement would be urgently needed. In lack of anything more refined, even the dispersion of cohort- and group-specific internal rates could provide quantitative, empirical evidence about the actual size and direction of redistribution in various pension schemes.

Such evidence could then provide guidance for public debate and political decisions on three crucial points: (1) the scope of the pension system, best measured by the average internal rates; (2) how much redistribution is acceptable, tolerable or unbearable for the society as a whole; and (3) how to avoid unbearable redistribution. These are the real issues for pension reform design. Financial balance should, of course, be kept in mind but it must not be the number one priority.

5. Summary

It maybe useful for the interested reader to provide a summary of the main points. Central is the observation that the two most popular and widely cited theoretical models of retirement saving — over-lapping generations and life-cycle theory — are based on a number of stationary assumptions.

However, some of these assumptions are relatively easy to relax:

1. It is not necessary to assume that all individuals have identical preferences if the model is built on empirically observable, cross-sectional distribution of income.
and consumption rather than on longitudinal allocation over the life-cycle which is difficult to derive without individual utility maximization.

2. The growth rate and the interest rate need not be constant over time if income and consumption are expressed as age-specific shares in terms of a freely changing aggregate numeraire rather than as absolute sums of money. In this case, only the relative interest factor (the ratio of the interest factor to the growth factor) has to be time-invariant.

3. The relative interest factor does not have to be exogeneously given or assessed as a ‘solution’ that satisfies two ‘budget constraints’; it may be treated as a continuous variable, the functions of which describe the entire range of possibilities.

4. The number of periods of human life (identical to the number of cohorts living together at a given point of time) may be realistically assumed to be large rather than limited to the special cases of two or three ‘illustrative’ stages of life (generations).

Under the resulting milder assumptions it is possible to develop a generalized stationary framework which accounts for the fundamental theoretical concepts affecting (or resulting from) the financing of retirement (and/or childhood as well). The cornerstone of that framework is the zero-bequest saving function which represents the aggregate cross-sectional saving rate that satisfies the zero-bequest (balanced longitudinal life-path) constraint, over the entire positive range of the relative interest factor. The analytical properties of this function are easily revealed. The most important of them is that the zero-bequest saving function has a trivial root where the relative interest factor equals one. In other words, in a ‘golden rule’ situation, when the interest rate equals the growth rate of the numeraire, zero aggregate saving results in zero bequest.

A number of well-known theoretical models and theorems can be obtained as special, restricted cases of this general framework. For example, Samuelson’s famous biological interest rate, Gale’s optimal steady state equilibrium and Aaron’s often-cited social security paradox can all be interpreted as manifestations of the trivial golden rule root of the zero-bequest saving function. In life-cycle models, the interest rate indeed does not have to be zero — as Modigliani later argued — but it must be smaller than the growth rate (the relative interest factor smaller than unity) for the zero-bequest criterion to result in positive retirement saving and wealth.

Within this generalized framework, however, the basic stationary assumption is still sustained: the age-specific parameters of survival, income and consumption are constant over time, i.e. identical for all successive cohorts. Briefly speaking, the economic population is still stationary.

It is technically possible to relax this basic assumption too and assess the zero-bequest saving function according to time-variant parameters. The result is, however, somewhat embarrassing theoretically. The trivial golden rule root disappears as proportions on the longitudinal life-path cease to be identical with proportions in the cross-sectional population. In certain cases the zero-bequest function has no roots at all, the implication being that for some cohorts there is no
such value of the relative interest factor at which zero saving would result in zero bequest. Furthermore, rather trivially, these functions are cohort-specific, they vary from cohort to cohort. Yet there cannot be more than a single aggregate saving rate at any point of time.

Consequently, the aggregate saving rate cannot be postulated as an appropriate device for balancing the longitudinal life-path of various cohorts living together in a non-stationary population. The entire conceptual framework of the retirement issue needs to be revisited if changing demographic and economic conditions are to be accounted for.

A relevant new pension economics might set off from concepts that are closer to the real world of pension systems. Demographic changes, employment and labour market arrangements, retirement behaviour should be considered; contribution rates and benefit levels should be compared. Cross-sectional aggregate revenues and expenditures of pension schemes, as well as life-time contributions and benefits of cohorts (and socially distinct sub-sets of them) should be assessed quantitatively, as outcomes of the various factors that affect them (e.g. the demographic and system dependency ratios, the replacement rate, etc.).

A simple, but powerful analytical and descriptive tool might be obtained by defining the internal contribution rate, the ratio of total pension benefits to the total contribution base (i.e. the wage fund in most pension systems). The internal contribution rate may be interpreted cross-sectionally for pension systems as well as longitudinally for successive cohorts. A statutory contribution rate equal to the internal one would obviously balance aggregate contributions and pensions; deviation from the internal rate results in surplus or deficit cross-sectionally, in positive or negative bequest longitudinally.

Even by the simplest empirical evidence, however, the longitudinal internal rate is naturally varying from cohort to cohort and the cross-sectional rate is fluctuating over time. It is impossible for these various internal rates to be identical. In principle, the statutory rate could be cohort-specific in order to equal the internal rate of individual cohorts, but in this case cross-sectional balance could not be reached. Or, reversely, the unique statutory rate could be adjusted annually to meet the internal cross-sectional rate, but then the life-path of various cohorts would not be balanced.

The conclusion is that there is no such thing as a pension system free of intergenerational and intertemporal redistribution. This crucial concept needs to be properly defined and quantified, rather than being assumed away by the theoretical zero saving and zero bequest ‘budget constraints’. In real life, endless debates about ‘public, pay-as-you-go’ versus ‘private, funded’ pension ‘tiers’ should not obscure the major issue, the types, directions and socially acceptable levels of redistribution.

References

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