Disequilibrium growth theory with insider–outsider effects

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Received 1 May 1998; received in revised form 8 September 1999; accepted 1 February 2000

Abstract

Ito has applied the non-Walrasian regime switching methodology to the Solovian neoclassical growth model and discussed the occurrence of regimes of full employment, overemployment and underemployment and the different dynamical systems (to be patched up) these regimes give rise to. We shall show in this paper that nothing of this sort really characterizes Solovian growth with non market-clearing real wages if over- or under-time work of the workforce within the firms (the insiders) is taken into account. This simple extension of the two-dimensional dynamical systems of Ito by one dimension implies that there is only one regime possible (the Goodwinian classical regime) with a 3D dynamics that are easily made globally stable and which moreover are often globally asymptotically stable, and this even more when substitution and endogenous growth are added to the model. © 2000 Elsevier Science B.V. All rights reserved.

JEL classification: E12; E13; E31

Keywords: Disequilibrium growth; Regime switching; Boundedness; Persistent fluctuations; Endogenous growth

1. Introduction

In an interesting paper, Ito (1980) has applied the non-Walrasian regime switching methodology to the Solovian neoclassical growth model. He there discussed in

* I have to thank two anonymous referees for suggestions that helped to improve this paper. Of course, the usual caveats apply.

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detail the occurrence of full employment, overemployment and underemployment and the different dynamical systems these possible regimes (to be 'patched up') give rise to. We shall show in this paper, however, that nothing of this sort really characterizes Solovian growth with sluggishly adjusting, non market-clearing real wages if one basic extension of this model is considered as relevant and assumed: the existence of over- or under-time work of the workforce within the firms (the insiders). This simple extension of the two-dimensional dynamical systems considered in Ito (1980) (by one dimension) gives the dynamics a completely new outlook, described by the interaction of two employment cycles as in Goodwin (1967), one for insiders and one for outsiders, where the actual employment of the insiders is governed by the marginal productivity rule (or more generally by the size of the capital stock), while the outside rate of employment follows the inside rate of employment with a time delay.

We show through this extension, augmented by smooth factor substitution and endogenous growth in later sections of the paper, that there is then only one regime possible — the classical regime of capital shortage — in the global dynamics of such a Solovian model with varying rates of inside and outside employment. Furthermore and more importantly, this 3D extension of the Solow model implies asymptotic stability for many parameter constellations of the model. If locally unstable, it allows — under simple additional nonlinearites — for global stability in an economically meaningful domain of this system with or without absolute full employment on the external labor market, but with no regime switches on the internal market. A different extension of neoclassical growth with fluctuating employment rates thus makes the non-Walrasian methodology basically redundant and allows for new assertions on local as well as global stability. We stress that at least the first of these assertions applies to non-Walrasian macrodynamics in general and thus provides a way out of the complicated phase diagram analysis generally suggested by this approach.¹

2. Fixed proportions in production

We start with the case of given output-capital and output-labor ratios, \( x = Y/K \), \( y = Y/L \), where we use \( Y \) for denoting the output level and \( K, L \) for the capital stock and the stock of labor. In such a situation, the Ito model is basically of the Goodwin (1967) growth cycle type,² if we neglect its regime switching aspect for the moment. We therefore start with a brief representation of this very basic model of cyclical growth which will be extended in various ways in this paper, including variable inside employment, substitution and endogenously generated technical change.

¹ See Chiarella et al. (2000) Ch.4, for detailed demonstrations of such an assertion.
² See Flaschel et al. (1997) for presentations (and for extensions) of this prototype model of cyclical growth.
The Goodwin (1967) growth cycle model is based on two laws of motions, one for real wages $\omega$ and one for the rate of employment $V = L^w/L$. We here denote by $L^w$ the employment level and by $L^s$ labor supply, where the latter is assumed to grow at the given natural rate $n = L^s/L^w$. We denote the rate of growth of a variable $z$ by $\dot{z}(=\dot{z}/z)$. Standard growth rate formulae then imply for the rate of growth of the employment rate $\dot{V}$, on the basis of the assumed fixed proportions technology, the general expression $\dot{V} = \dot{L} - n = \dot{Y} - n = \dot{K} - n$, i.e. this growth rate is determined in its endogenous evolution by the growth rate of the capital stock in this classical world of full capacity growth. The Goodwin model assumes that the growth rate of the capital stock, $K$, is determined by savings out of profits per unit of capital, which gives rise to $\dot{K} = s_c x (1 - \omega/y)$ if we assume a given rate of savings $s_c$ out of profits, since $\omega/y$ is the share of wages and therefore $x(1 - \omega/y)$ the rate of profit due to our above definitions of $y$, $x$. We thus get that the growth rate of the employment rate solely depends on real wages which gives one of the two laws of motion of the Goodwin (1967) model (still neglecting technical change here, see section 6 in this regard). Assuming in addition a real wage Phillips curve, as it can be derived from a conventional money wage Phillips curve by assuming myopic perfect foresight, here based on a ‘natural’ rate hypothesis in addition, provides the other law of motion of this growth cycle model, which in sum thus reads:

\[
\begin{align*}
\dot{\omega} &= \beta_\omega (V - \bar{V}), \quad \beta_w > 0, \\
\dot{V} &= s_c x (1 - \omega/y) - n
\end{align*}
\]

where $\bar{V} = \text{const} \in (0,1)$ denotes the ‘natural’ rate of employment of the Goodwin model. This ‘cross-dual’ dynamics, where the level of $\omega$ induces changes in $V$ and the level of $V$ changes of $\omega$, with its well-known prey–predator implications and interpretation, represents our point of departure for an alternative analysis of the situations of under- and over-employment in a growth context that completely bypasses the regime switching methodology of Ito (1980) and the non-Walrasian approach to macrodynamics.

In the place of Ito’s (1980) assumption that there is a switch to a new type of dynamics when the labor market reaches the full employment ceiling (which due to our assumption of a ‘natural’ rate of employment $\dot{V} < 1$ does not occur at the steady state as in Ito’s model), we assume as further flexibility in the production process of the economy the possibility of overemployment of the employed workforce in such cases. In the above model, we have the employment function $L = x K/y$ due to the assumption of fixed proportions in production. We assume for the time being that there is no limit with respect to available overtime work. We then have that firms can always produce what they want to produce by choosing an appropriate internal rate of employment $V^w = L/L^w$ of the labor force $L^w$ they employ.

---

1 The two natural rates $n, \bar{V}$ are made dynamically endogenous variables in Chiarella and Flaschel (1998).
Based on this internal rate of employment $V^w$, we assume next that firms adjust their labor force by recruiting new laborers from the external labor market as follows:

$$\dot{L}^w = \beta(L - L^w) + nL^w,$$

or equivalently:

$$\dot{L}^w = \beta(V^w - 1) + n.$$  (3)

This law of motion states that the workforce $L^w$ of firms is enlarged or reduced in a growing economy according to the difference between the actual employment $L$ of the employed, $L^w$, and the normal employment of the workforce of firms, also measured for simplicity by $L^w$, plus a term that reflects trend growth $n$. This employment policy of course comes to an end when the external labor market is exhausted (to be considered in Section 4). This however is generally not accompanied by a limit to further production due to the above distinction between actual employment $L$ and normal employment $L^w$.

Since $L \neq L^w$ is now possible, the Phillips curve (1) should be reformulated as follows:

$$\dot{\omega} = \beta_\omega(V - \overline{V}) + \beta_{\omega_2}(V^w - 1), \quad V = \frac{L^w}{L^s}, \quad V^w = \frac{L}{L^w}$$

(4)

to take account of the impact of the over- or underemployment of the employed on wage formation. This is our new formulation of the first differential equation of the Goodwin model (1,2). The second law of motion, for the internal rate of employment $V^w$, is according to Eq. (3) given by:

$$\dot{V}^w = \dot{L} - \dot{L}^w = \dot{K} - L^w = s_{x}(1 - \omega/y) - \beta(V^w - 1) - n$$  (5)

while the one for the external rate of employment $V$ is given by

$$\dot{V} = \dot{L}^w - \dot{L}^s = \beta(V^w - 1)$$  (6)

due to the role the rate $V^w$ plays in the employment policy of firms. The differential Eqs. (4–6) constitute our augmented Goodwin model, now with variable inside and outside employment rates. Note that this model is a linear model up to its use of growth rates in the place of simple time derivatives.

**Proposition 1.** The dynamical system (4)–(6) has a unique interior steady state given by:

$$\omega_0 = y\left(1 - \frac{n}{s_{x}}\right) > 0, \quad V^w_0 = 1, \quad V_0 = \overline{V}$$

if $s_{x} > n$ holds.

2. This steady state is locally asymptotically stable if and only if $\beta_\omega > \beta_{\omega_2}\overline{V}$ holds.

3. At the value $\beta^I_{\omega_2} = \beta_{\omega_2}\overline{V}$ of the parameter $\beta_{\omega_2}$, there occurs a Hopf-bifurcation of either subcritical, supercritical or degenerate type.

**Proof 1.** (See Appendix A). The concept of a Hopf-bifurcation is explained in detail in Wiggins (1990) and has been thoroughly discussed in the case of a Tobin type growth model in Benhabib and Miyao (1981). We here only briefly state that the obtained (generally non-degenerate) Hopf-bifurcations imply either the loss of a
stable corridor and the death of an unstable limit cycle as the Hopf-bifurcation value is approached or the birth of a stable periodic motion after the bifurcation point has been passed. The center-type stability of the Goodwin (1967) growth cycle is thus in particular made a locally implosive dynamics if inside workers dominate the real wage bargain and an explosive one in the opposite situation (at least locally). In the explosive case we have to add — as in Ito (1980) — the condition \( V \leq 1 \) to the model. We shall later on also assume — and motivate — that the growth rate of the labor supply will increase appropriately as the overemployment within firms becomes higher and higher which will take pressure from the labor market and thus help to avoid labor supply bottlenecks. This will lead us to bounded dynamics, but in the present situation still one where the share of profits \( 1 - \omega/y \) may fall below zero — which is not meaningful. We therefore add in a next step, as in Ito (1980), smooth factor substitution and the neoclassical theory of employment before we come to the discussion of global constraints from the side of labor supply and their implications.

Before closing this section let us stress that the above extension of the Goodwin (1967) growth cycle model is now based on two interacting cycles: the usual outsider cycle (which can be isolated by assuming \( \beta_j \approx \infty, \beta_{w_j} = 0 \) and which might be called the ‘US case’ of the model) and a new insider cycle (which can be isolated by \( \beta_i = 0, \beta_{w_i} = 0 \), the ‘Japanese case’ of the model). The astonishing thing is that the interaction of these two Goodwin cycles contributes to the stability of the general model, i.e. the presence of both variable inside employment with strong wage claim effects and the assumed lagged adjustment of outside employment produces convergence to the steady state in a Goodwin growth cycle setup. A first explanation of this may be seen in the fact that Eq. (6) inserted into Eq. (4) adds a derivative term to the postulated Phillips curve (4) which is known to be stabilizing. Note that this derivative term is already contained in Phillips’ (1958) original article and gave rise there to the explanation of so-called Phillips loops.

3. Smooth factor substitution

In the case of a neoclassical production function (with the usual properties) and the marginal productivity theory of the rate of employment

\[
Y = F(K, L) \quad \text{and} \quad \omega = F_1(K, L)
\]

we have to recalculate \( \dot{K} \) and on this basis the law of motion for the variable \( V^w \).

Denoting by \( k \) the actual capital intensity \( K/L \) we know from the Solow model of neoclassical growth that (7) gives rise to

\[
y = f(k), \quad x = f(k)/k, \quad \omega = f(k) - f'(k)k = g(k)
\]

for the now endogenous output — input ratios of the preceding section. For the function \( g \) there furthermore holds

\[
g'(k) = f''(k) - f'(k) - f''(k)k = -f'''(k)k > 0.
\]
i.e. the function $g$ is strictly increasing (due to decreasing marginal products of labor). We denote by $k = k(\omega)$ the inverse of $g$ and by $\varepsilon(\omega) = k(\omega)/k(\omega) > 0$ the elasticity of this function $k$. On the basis of these equations we then get:

$$\dot{L} = \dot{K} = s_c x(1 - \omega/y) - \varepsilon(\omega)\dot{\omega} = s_c f'(k(\omega)) - \varepsilon(\omega)\dot{\omega}.$$ 

This expression is now to be used in Eq. (5) in the place of only $s_c x(1 - \omega/y)$ in order to describe the dynamics of inside employment $V^w = L/L^w$ under neoclassical factor substitution and the neoclassical theory of employment. This, however, is the only change in the dynamical system (4)–(7) if such substitution is included.

**Proposition 2.** 1. The interior steady state of the dynamical system (4)–(6) with smooth factor substitution is of the same type as before, but now with an endogenous determination of $\gamma_0 = f(k_0)$, $x_0 = f(k_0)/k_0$ and $k_0 = k(\omega_0)$, where $\omega_0$ is given by the solution of the equation $n/s_c = f'(k(\omega_0))$.

2. This steady state is locally asymptotically stable if $\beta_{w_1} > \beta_{w_2} V - \delta$ holds for some suitably chosen $\delta > 0$. The size of $\delta$ can be chosen the larger, the larger the terms $\varepsilon(\omega)\beta_1\beta_{w_2}$ become.

**Proof 2.** (See Appendix A). As the expression for the Routh–Hurwitz condition $a_1a_2 - a_3 > 0$ in the proof shows, stability is indeed significantly increased by the inclusion of smooth factor substitution and the neoclassical theory of employment.

4. Effective supply constraints?

Having extended the Goodwin growth cycle approach by inside–outside labor market effects married with Solovian economic growth we now come to a specification of the bounds that can limit the evolution of such an economy. To do so, let us assume as an example that overtime work of the employed workforce $L^w$ is legally restricted to $L^w$, i.e. overtime work supplied by a person can be (and will be) at most as large as the normal working time of this person, i.e. we impose the inequality $L \leq 2L^w$ or $V^w \leq 2$. Furthermore: $V \leq 1$ must hold true as in the Ito (1980) model, due to the external labor market constraint. Finally, the rate of profit $x(1 - \omega/y)$ should be positive at all times.

Summarizing, our dynamical system with smooth factor substitution then reads, including the bounds $\omega < y$, $V^w \leq 2$, $V \leq 1$:

$$\dot{\omega} = \beta_{w_1}(V - \overline{V}) + \beta_{w_2}(V^w - 1)$$
$$\dot{V}^w = s_c f'(k(\omega)) - \varepsilon(\omega)[\beta_{w_1}(V - \overline{V}) + \beta_{w_2}(V^w - 1)] - \beta_1(V^w - 1) - n$$
$$\dot{V} = \beta_1(V^w - 1)$$

The above constraints represent all constraints that are needed from the perspective of non-Walrasian disequilibrium growth theory. Note here that $\omega, V^w, V > 0$ is automatically ensured (in finite time) due to the growth rate formulation of the
model, but that this does not yet exclude that limit points of trajectories which start in the positive orthant may lie on the boundary of it. This however is another topic that is not related to our questioning of the validity of the regime switching methodology of Ito (1980) and others, which is the topic of this paper.

Note here first that the viability condition $\omega < y$ (a positive profit share) is always fulfilled, since

$$\omega = f(k) - f'(k)k < f(k)$$

holds at all times in the Solow model.

Next we consider the condition $V^w \leq 2$. In order to guarantee that this condition holds true at each moment in time we make the following simple additional assumption on the growth rate $n$ of labor supply:

$$n = n(V^w), \quad n' \geq 0, \quad n(2) = \infty.$$  

This assumption states that the growth rate of labor supply increases when overtime work increases, up to infinity, when the legal barrier $V^w = 2$ is approached (in a continuous time framework!), due to rapid changes in the participation rate of households, due to the influx of labor from rural surroundings and also due to migration from other parts of the world. Note that all these changes may be due solely to workers’ decision making, but they may also be due to systematic efforts of the firms. Firms surely will attempt to recruit new labor force in case of significant labor shortage by influencing in particular the labor supply decisions of those households that so far were not part of the external labor market of the economy. In short, if labor tends to become the short side of the market, firms in market economies find ways to push up the growth rate of the labor supply sufficiently to avoid absolute supply bottlenecks. There may be many microeconomic aspects of the points just mentioned that deserve more detailed discussion. Our basic argument here however simply is that macroeconomic dynamics has to take account of these possibilities and flexibilities in the behavior of households and firms even on a preliminary level, instead of discussing the hypothetical consequences of hard constraints that have never existed in this form in developed market economies.4

With respect to the third restriction $V \leq 1$ it is possible, but not necessary, to assume that firms must reduce recruiting efforts on the ‘traditional’ labor market when the upper bound on the external employment rate is approached,5 increasing instead their activities for recruiting new workers as just discussed. In fact this constraint can become binding in the hard way, without any supply consequences as long as firm can use the internal labor market for more employment and production (which they always can if the above assumption on the rate $n$ holds).

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4 Note that this means that firms have to consider $V \leq 1$ and $n = L^*$ as two separate characteristics of the labor market.

5 Assuming $b = 0$ as $V \rightarrow 1$ says that it becomes more and more difficult for firms to recruit further workers from the existing labor market when the external rate of employment approaches 1. They then choose instead to operate more and more directly on labor supply decisions $L^*$. 
Proposition 3. 1. The dynamical system (8)–(10) with smooth factor substitution and supply constraints exhibits the domain \((0, \infty) \times (0,2) \times (0,1]\) as invariant subset which it thus cannot leave.

2. The classical regime of non-Walrasian disequilibrium analysis is the only regime that is possible in this domain, i.e. \(\omega = F(K,L)\) holds true at all times.

Proof 3. (See Appendix A). Adding three simple extrinsic nonlinearities to the linear growth model of Section 2 is thus sufficient to imply that there is no need for supply constraints as in non-Walrasian disequilibrium (growth) dynamics in order to keep such a dynamical system economically viable, in particular in the case where local asymptotic stability (see proposition 2) does not hold. The numerical properties of the dynamics considered in proposition 3 will be briefly discussed in the following section.

5. Numerical features of the dynamics

In order to investigate the dynamics (8)–(10) from the numerical point of view we choose a CES-representation of the assumed neoclassical production function which in intensive form is given by:

\[ y = f(k) = \gamma [ak^{-\eta} + (1 - a)]^{-1/\eta} \]  

with \(a \in (0,1), \eta \in (-\infty,0)\) (and with \(\rho = 1/(1 + \eta)\) the constant elasticity of substitution of this function). We assume \(\gamma = 1\) for simplicity. This CES production function implies the following equations as background for our system (8)–(10):

\[ \begin{align*}
    f'(k) &= a[a + (1 - a)k^{\eta}]^{-(1 + \eta)/\eta} \\
    \omega(k) &= f(k) - f'(k)k = (1 - a)[ak^{-\eta} + (1 - a)]^{-(1 + \eta)/\eta} \\
    k(\omega) &= \left(\frac{\omega}{1 - a}\right)^{-\eta/(1 + \eta)} - (1 - a) \left(\frac{\omega}{1 - a}\right)^{(1 + \eta)/\eta} \\
    \varepsilon(\omega) &= \frac{1}{1 + \eta} \frac{1}{1 - (1 - a)\left(\frac{\omega}{1 - a}\right)^{(1 + \eta)/\eta}}
\end{align*} \]

It gives rise to the following formulae for the steady state values of capital intensity and the real wage (on the basis of the expression obtained for \(\omega(k)\) and \(f'(k)\) \([= n/\kappa]\)):

\[ \begin{align*}
    k_\omega &= \left(\frac{n/\kappa}{a}\right)^{-\eta/(1 + \eta)} - a \left(\frac{\omega}{1 - a}\right)^{(1 + \eta)/\eta} \\
    \omega_\omega &= (1 - a)[ak^{-\eta} + (1 - a)]^{-(1 + \eta)/\eta}
\end{align*} \]

Note here that CES functions do not fulfill the Inada conditions for neoclassical production functions which means that there will not always exist a steady state solution with respect to the parameters \(n, \kappa, a\).
The numerical analysis of the dynamics with this type of production function gives rise to the following observations:

- The steady state of the dynamics appears to be globally asymptotically stable with respect to all initial values in the domain described in proposition 3 for much larger parameter sets than those characterized by assertion 2 in proposition 3.

- If locally unstable, the dynamics are however generally bounded and economically viable without hitting the full employment ceiling \( V = 1 \) (unless the NAIRU-rate \( \bar{V} \) is chosen very close to it). This result seems to be due to the fact that the profitability bound (based on marginal productivity theory and neoclassical use of smooth factor substitution) represents a very strong mechanism in the creation of the boundedness of the dynamics.

- If the full employment ceiling \( V = 1 \) is indeed hit by the trajectories of the dynamics this generally happens in conjunction with overtime work of approximately 20 percent which — depending on the exact formulation of the natural growth rate effect — is accompanied only by small increases in the growth rate of labor supply.

- If locally explosive, the considered 3D dynamics is generally fairly simple — giving rise to a unique stable limit cycle solely — but not to more complex types of attractors which are possible in such a setup due to the dimension of the dynamics.

In sum we can therefore state that the combination of Solovian growth with insider and outsider growth cycles is generally already sufficient to imply global stability in an economically meaningful domain of the resulting dynamics. The occurrence of absolute full employment on the external labor is however possible in such a dynamical system. It is however not accompanied — as in the original Goodwin (1967) growth cycle model — by excess capacity of firms (and a resulting revision in their investment behavior), but rather gives rise solely to medium sized cyclical variation in the employment rate of the employed workforce.

We close this brief summary of the numerical properties of our insider–outsider dynamics with a numerical example where the absolute full employment ceiling is hit on the external labor market and where the resulting consequences on the internal labor market, and on the growth rate of the labor supply, are as pronounced as we could find them to be. The simulations shown in Fig. 1 can therefore be characterized as already fairly exceptional with respect to possible parameter choices of the model.

The figure top left shows, in dimension 3, the stable limit cycle of the dynamics (8)–(10) with the additional nonlinearity in the growth rate of the labor supply we have assumed in Section 4. The two cycles that border this figure show its projection into the adjacent planes, now with the variable \( u \), the share of wages, in the place of \( v \), the real wage in order to show that profits remain positive along the shown cycle (and on the way to it). The figure bottom right finally shows the time

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6 The parameter values for this simulation of the model are: \( \beta_{w_1} = 0.5, \beta_{w_2} = 0.05, n = 0.05, \bar{V} = 0.99, x_c = 0.9, \beta_l = 0.1, \eta = 50, a = 0.5. \)
series for the outside and inside rate of employment (with the full employment ceiling for the first rate sometimes in operation and with a rate of employment of inside workers that stays below 130%). When inside employment approaches this level it is furthermore clearly visible that the rate of growth of labor supply responds to this as assumed in Section 4, but, as in the case of the inside rate of employment, only in a moderate way in order to satisfy the labor volume and its rate of growth demanded by firms.

6. Endogenous technical change

In this section of the paper we investigate to what extent our analysis of cyclical growth will change when modern discussions of the endogeneity of technical change
and growth are taken into account. There exist various possibilities in the literature to model endogenous technical change, see in particular Barro and Sala-i-Martin (1995) and Aghion and Howitt (1998). We here follow Barro and Sala-i-Martin (1995), Ch.4, see also Schneider and Ziesemer (1994), p.17, and use as representation of such technical change an approach based on Uzawa (1965) and Romer (1986), and synthesized by Lucas (1988). Other representations of endogenous change will not significantly alter the conclusions of this section which serves the purpose of illustrating the implications of an integration of the production of technological change into the extended growth cycle model of this paper.

The Uzawa-Romer-Lucas approach to endogenous technical change can be described by means of the following two equations, characterizing the productive activities of firms:

\[ \dot{A} = \eta L_w^w/L^w, \quad \eta > 0, \quad \text{the research unit,} \]
\[ \dot{Y} = K^\beta (A L)^{1-\beta} \frac{\zeta}{\zeta}, \quad \zeta > 0, \quad \text{the production unit.} \]

The employed workforce \( L^w = L_r^w + L_p^w \) is now split between the production of output (13), described by a Cobb–Douglas production function with productivity measure \( A \), augmented by the Romer externality \( \zeta \), and the production of productivity growth as described by Eq. (12). Note that we do not consider overtime work in the research sector, but as before in the production unit, where hours worked, \( L_1 \), can deviate as before from the normal employment \( L_w^w \) of the there employed workforce due to the pace of capital accumulation. We denote the ratio \( L_r^w/L_p^w \), the allocation of the workforce within the firm, by \( h \) which implies for the ratio \( L_r^w/L^w \) the expression \( h/(1 + h) \).

The production function (13) can be easily reformulated as follows:

\[ Y = K^\beta (A (1-\beta + \zeta(1-\beta)L_1))^{1-\beta} = K^\beta (BL_1)^{1-\beta} \]

It shows in this way that it is of the usual type (with Harrod neutral technical change), yet with a growth rate \( B = \dot{B} \) of the aggregate productivity index \( B \) that exceeds the growth rate of the individual productivity index \( \dot{A} \) of the firms, due to the Romer externality \( \zeta \):

\[ \dot{B} = (1 + \zeta/(1 - \beta)) \dot{A}. \]

We will apply this approach to technical change to the fixed proportions section of this paper, and leave the case of smooth factor substitution for future investigation.\(^7\) In the place of Eqs. (12,13) this gives rise to the following representation of produced productivity growth:

\[ \dot{A} = \eta \frac{h}{1 + h} \]
\[ \dot{Y} = \min \{ x_K A L_1 A^\zeta \} = \min \{ x_K A^{1+\zeta} L_1 \} = \min \{ x_K y L_1 \} \]

\(^7\)See Flaschel et al. (2000) for the consideration of smooth factor substitution in an IS-LM-PC model of endogenous technical change.
where, as in Section 2, the symbol \( y = Y/L_1 \) denotes labor productivity on the aggregate level, whose rate of growth is given by \( \ddot{y} = (1 + \xi)\dot{A} = (1 + \xi)\eta \frac{h}{1 + h} \). Let us consider \( h \) as a given parameter for the moment.

The equations constituting the model of Section 2, adjusted to the purposes of the present section, read:

\[
\dot{\omega} = \beta_{w_1}(V - \overline{V}) + \beta_{w_2}(V_w - 1) + \ddot{y}, \quad V = \frac{L_Y^w + L_s^w}{L_s} = (1 + h)V_1, \quad V_1 = \frac{L_1}{L_Y^w}
\]

(14)

\[
\dot{L}_1^w = \beta_l(V_1 - 1) + n, \quad V_1^w = \frac{L_1}{L_Y^w}
\]

(15)

\[
\dot{K} = s_c(1 - \omega/y)
\]

(16)

The employment policy of firms, Eq. (15), is based in this formulation on their production unit and there on the degree of over- or undertime work \( L_1 \) performed by production workers \( L_Y^w \) (while employment in the research unit is given by \( L_s^w = hL_Y^w \) in each moment of time). Wage claims are now made in view of labor productivity growth \( \ddot{y} \) which augments the real wage Phillips curve of section 2 in a straightforward way, see Eq. (14). There are thus only minor adjustments needed in these two equations of the dynamics in order to include produced productivity growth and the allocation of the workforce of firms into them. There is no change in the third equation, Eq. (16), which again gives the growth rate of the capital stock on the basis of the savings that are made out of profits.

In the presence of technical change we have to use the share of wages \( u = \omega/y \) in the place of real wages as state variable of the model. Furthermore, the dynamical equations must be reduced to expressions concerning the production unit solely in order to get an autonomous system of differential equations in the case of produced productivity growth \( (h > 0) \). This is easily done and gives rise to the following system of differential equations \( (V \leq 1, V^w \leq 2 \) again):

\[
\dot{u} = \beta_{w_1}(1 + h)V_1 - \overline{V} + \beta_{w_2}(V^w - 1),
\]

(17)

\[
\dot{V}_1^w = \dot{K} - \ddot{y} - \dot{L}_1^w = s_c(1 - u) - \beta_l(V_1^w - 1) - \left( n + (1 + \xi)\eta \frac{h}{1 + h} \right)
\]

(18)

\[
\dot{V}_1 = \beta_l(V_1^w - 1).
\]

(19)

This system is of exactly the same type as the dynamical system considered in Section 2. It thus gives rise to the same proposition as in this section. However, technical change, though produced by firms, is still basically of an exogenous nature here and thus demands in a next step the endogenization of the parameter \( h \) in order to arrive at a model which generates growth endogenously. In view of this necessity we propose as law of motion for the labor allocation ratio \( h \)

\[
\dot{h} = \beta_h((\dot{K} - \ddot{y}) - n) = \beta_h\left( s_c(1 - u) - \left( n + (1 + \xi)\eta \frac{h}{1 + h} \right) \right)
\]

(20)
where we interpret the expression $\dot{K} - \dot{y}$ as growth rate of labor demand, calculated by firms on the basis of their investment decisions concerning output and productivity increases. This rate is contrasted with the growth rate of labor supply available to fill this gap between capital stock and productivity growth. Firms therefore increase their efforts in raising labor productivity in the case of insufficient growth in the labor supply and vice versa. This is of course only a first step into the direction of endogenously generated technical change. Further extensions could for example concern the role of income distribution, but must be left for future reconsiderations in this paper.

For the extended dynamics (17)–(20) we get in the place of proposition 1:

**Proposition 4.** The dynamical system (17)–(20) exhibits a ray, parameterized by $h \in (0, \infty)$, of interior steady state solutions, given by:

$$u_0 = 1 - \frac{n + (1 + \xi)\eta}{s_\chi h} \frac{h}{1 + h}, \quad V_{10}^\infty = 1, \quad V_{10} = \frac{\bar{V}}{1 + h}$$

if the parameters of the model are chosen such that $\frac{n + (1 + \xi)\eta}{s_\chi h} < 1$ holds.

2. These steady states are locally asymptotically stable in all cases where the dynamics of Section 2 or the one for a given $h$ are locally asymptotically stable, i.e. endogenous technical change increases the parameter domain where local asymptotic stability prevails, combined with zero-root hysteresis now, since one of the eigenvalues of the linear part of the dynamics must be zero throughout.

3. If not locally asymptotically stable, which is the case if $\beta_h$ is sufficiently small and $\beta_{w\nu}$ sufficiently large, the system can be made locally asymptotically stable, again by way of a Hopf-bifurcation, if the parameter $\beta_h$ is made sufficiently large.

**Proof 4.** (See Appendix A). We thus in sum have that endogenous technical change and growth (with rate $(1 + \xi)\eta \frac{h}{1 + h}$) is easily integrated into the growth cycle model of this paper, adds to its stability, preserves its cyclical characteristics (at least to some extent) and most importantly, makes the generation of technical change and growth path-dependent, since shocks to its orbits will cause convergence to a different steady state position, due to the zero-root hysteresis now present in the dynamics. Furthermore, the steady state wage share now depends negatively on the parameters that characterize long-run productivity growth (including the size of $h$ that determines the allocation of workers to production and research activities). Of course, further investigations and extensions of the considered dynamics which add the effects of income distribution on the generation of technical change, the role of substitution in production, extensions of the model

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*Due to proposition 1.*
from the global point of view as considered in Section 4 should follow, but cannot be approached here due to space limitations.\footnote{See Chiarella et al. (2000), Flaschel et al. (2000) and Chiarella and Flaschel (1998) for further investigations on endogenous growth.}

7. Conclusions

We have shown in this paper that the non-Walrasian reconsideration of the Solow growth model by Ito (1980) which studies situations of over and under-employment, caused by a sluggish real wage dynamics, by means of differential inequalities or ‘patched-up’ dynamical systems, is misleading as there is indeed no regime-switching of this type close to or farther away from the Solovian steady state. Taking the possibility of over- (and under-) time work of the workforce of firms into account instead gives rise to an ordinary three-dimensional dynamics of Solow–Goodwin type with no regime switching at all, i.e. with the classical regime holding throughout.

Increasing the range of the adjustment possibilities that a market economy allows for therefore removes the non-smooth reaction patterns as assumed in non-Walrasian fix price theory which are thus only due to implausible rigidities in the assumed structure of the economy. Furthermore and perhaps more importantly, we have also shown that it is not difficult to design nonlinearities in the adjustment behavior of such an economy which not only globally prevent the occurrence of hard constraints on the output of firms, but which in fact produce a viable and basically smooth economic dynamics under all circumstances. Such a viability demonstration is generally not an easy task in the case of three dimensional dynamical systems.

These conclusions not only apply to the neoclassical disequilibrium growth theory of Ito (1980), but also to Keynesian growth models of Picard (1983), Solow and Stiglitz (1968), as well as many others, as shown in Flaschel (1999), Chiarella et al. (2000), Ch.4. They are also valid when endogenous growth is included in such model types as shown in Section 6 for the situation considered in Section 2. Consequently, though non-Walrasian economics has made us aware of the importance of supply–side constraints for Keynesian growth dynamics, it has vastly overstated the possibility of regime switches in such a setup due to its reliance on rigidities that do not really exist in market economies. Patching different dynamics together is therefore not a big issue in the macrodynamic analyses of market economies.
Appendix A. Mathematical proofs

A.1. Proof of proposition 1

1. Obvious (and as in the original Goodwin (1967) model).
2. The Jacobian of (4–6) at the steady state is given by

\[
J = \begin{pmatrix}
0 & \beta_w \omega_0 & \beta_w \omega_o \\
-\frac{s_c}{s} & -\beta_1 & 0 \\
0 & \beta_1 V_0 & 0
\end{pmatrix}
\]

Therefore, \(-a_1 = \text{trace } J = -\beta_1 < 0\) and \(-a_3 = \det J = -\beta_w \omega_0 \omega_1 x/y \beta_1 V_0 < 0\).

For \(a_2\) (the sum of the principal minors) we get \(a_2 = \beta_w \omega_0 \omega_1 x/y > 0\). According to the Routh–Hurwitz conditions, see Gantmacher (1971), we have to consider the positivity of \(a_1 a_2 - a_3\) in addition:

\[
a_1 a_2 - a_3 = \beta_w \omega_0 \omega_1 (x/y) (\beta_w - \beta_w V_0).
\]

Hence \(a_1 a_2 - a_3 > 0\) if and only if \(\beta_w > \beta_w V_0\). The assertion of a Hopf bifurcation at \(\beta_w = \beta_w V_0\) is then proved by means of the above expression for \(a_1 a_2 - a_3\) as in Benhabib and Miyao (1981).

3. As in Benhabib and Miyao (1981), due to the above expression for the coefficient \(a_1 a_2 - a_3\). □

A.2. Proof of proposition 2

The Jacobian of the dynamics with substitution is given (at the steady state) by the following matrix:

\[
J = \begin{pmatrix}
0 & \beta_w \omega_0 & \beta_w \omega_o \\
-\frac{1}{k(\omega)} & -\beta_1 - \varepsilon \beta_w & -\varepsilon \beta_w \\
0 & \beta_1 V_0 & 0
\end{pmatrix}
\]

since the expression

\[
r(\omega) = x(1 - \omega/y) = \frac{f(k(\omega))}{k(\omega)} \left(1 - \frac{\omega}{f(k(\omega))}\right)
\]

that now defines the rate of profit of the economy has the derivative \(r'(\omega) = -1/k(\omega)\):

\[
r'(\omega) = \frac{f'(k(\omega)) k(\omega) - f(k(\omega)) k'(\omega)}{k(\omega)^2} - \frac{k(\omega) - \omega k'(\omega)}{k(\omega)^2}
\]

\[
= \frac{k'(\omega) \left( f'(k(\omega)) k(\omega) - f(k(\omega)) + \omega \right)}{k(\omega)^2} \frac{1}{k(\omega)} = -\frac{1}{k(\omega)}
\]

because of \(f(\omega) - f'(k(\omega)) k(\omega) = \omega\). There follows \((V_0 = \bar{V})\)
\[ a_1 = \beta_1 + \epsilon \beta_{w_2} > 0 \\
\[ a_2 = \beta_w \omega_0 / k(\omega_0) + \beta_1 V_0 \epsilon \beta_{w_1} > 0 \\
\[ a_3 = \beta_1 V_0 \beta_{w_1} \omega_0 / k(\omega_0) > 0 \\
\[ a_4 a_2 - a_3 = \beta_1^2 V_0 \epsilon \beta_{w_1} + \epsilon \beta_{w_2}^2 \omega_0 / k(\omega_0) + \epsilon^2 \beta_{w_2} \beta_1 V_0 \beta_{w_1} + \beta_1 \epsilon \omega_0 / k(\omega_0) [\beta_{w_2} - V_0 \beta_{w_1}] \\
\]
\]

\[ A.3. \text{Proof of proposition 3} \]

The assumption made on the function \( n = n(V^w) \) implies:

\[ \hat{V}^w < 0 \quad \text{close to} \quad V^w = 2 \]

i.e. the state variable \( V^w \) cannot approach this border. Furthermore, \( V \leq 1 \) holds by assumption, while \( \omega < \gamma \) has already been shown to be the simple consequence of the marginal productivity theory of employment. \( \square \)

\[ A.4. \text{Proof of proposition 4} \]

1. Setting (17)–(20) equal to zero gives only three independent equations for the four state variables \( u, V^\gamma_1, V_1, h \), from which the shown steady state solutions are then easily obtained.

2. Inspection of the laws of motion (17)–(20) shows that the fourth law can be obtained form the second and the third in the following way:

\[ \dot{h} = \beta_h \hat{V}^w + \beta_h \hat{V}_1 \]

This implies that the determinant of the linear part of the system must be zero throughout, implying that one eigenvalue must always be zero, and it implies also the following algebraic relationship between the three state variables shown:

\[ h = \beta_h \ln V^\gamma_1 + \beta_h \ln V_1 + \text{const} \]

This relationship can be inserted into (17)–(19), thereby giving rise to an autonomous three dimensional dynamic system in the remaining state variables \( u, V^\gamma_1, V_1 \). These reduced dynamics must remain locally asymptotically stable if it was locally asymptotically stable for \( h = \text{const} \), as will be shown below, which due to the relationship \( h = \beta_h \ln V^\gamma_1 + \beta_h \ln V_1 + \text{const} \) shows that it will in particular converge to \( h = \beta_h \ln V_1 + \text{const} \). It will thus exhibit shock-dependence with respect to the value of \( h \) the system will converge to. The Jacobian of this system at the steady state is given by the sum of the Jacobian of the system without endogenous technical change

\[ J = \begin{bmatrix} 0 & \beta_{w_2} u_0 & \beta_{w_1} (1 + h) u_0 \\ -s x & -\beta_1 & 0 \\ 0 & \beta_1 V_1 & 0 \end{bmatrix} \]

and new terms as they derive from the above algebraic relationship for \( h \):
Recalculating the expressions for \( a_1, a_2, a_3 \) in the proof of proposition 1 for this extended Jacobian then shows that the addition of the matrix \( H \) to the original Jacobian \( J \) used to prove proposition 1 will increase all of the coefficients \( a_1, a_2, a_3 \) of the Routh–Hurwitz polynomial, and this by the summation of positive terms throughout. Furthermore, since \( a_3 \) is solely augmented by the expression \(-J_{21}J_{32}\beta_h\beta_0\), and \( a_1a_2 \) in particular by \(-J_{22}(J_{21}V_{10}\beta_0)\), and since \( J_{32} = \beta_0V_{10} = -J_{22}V_{10} \) holds, we get that also \( b \) is increased through the addition of the matrix \( H \). The stability conditions of the Routh–Hurwitz theorem are thus all improved when exogenous technical change (a given \( h \)) is extended by (20) in order to allow endogenous technical change.

3. On the basis of what has been shown in part 2. of the proof we know that \( a_1a_2 - a_3 \) is a quadratic function of \( \beta_h \) with only positive terms in front of the \( \beta_h^2 \). Since stability can only get lost via a negative term in \( a_1a_2 - a_3 \), we thus get that such instability can always be removed again by making the parameter \( \beta_h \) sufficiently large. Such switches from stability to instability and back to stability can only occur via Hopf-bifurcations as the reduction of the 4D dynamics to a 3D dynamics in part 2. of the proof immediately implies. □

References