The dynamics of capital accumulation in an overlapping generations model

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Abstract

This paper studies capital accumulation in a slightly altered, explicit OLG model. The fundamental difference with the standard model lies in the initial conditions. If a portion of the initial allocation of the capital stock is not assigned to retirees, the framework allowing for a genuine accumulation of capital is provided. Dynamic aspects of the resulting model are analyzed, including dynamics of the quotas of capital stock, shares of total output, partial influences of parameters, the connection between the depreciation rate and capital’s contribution to production, and the relation of the latter to saddle-node bifurcations and the existence of real-valued equilibria. It is demonstrated that the model is able to describe the initial phase of accumulation. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Standard OLG models are described by first order difference equations for which the capital intensity (capital per capita or the capital–labor ratio) in the next period depends exclusively on the savings habits of the generation working in the current period. Any increase in the capital stock from one generation to the next is accomplished through a reproduction rather than an accumulation mechanism (more capital begets more capital). Any genuine accumulation requires an equation
of motion expressing next period’s capital as a sum of current capital and the
difference between current production and consumption, the latter of which in-
cludes retirees’ consumption out of wages from the previous period. The accumula-
tion process in thus described by a second order relation. Of course, in the standard
model, retirees consume all of their previous savings and all of capital’s share of the
national product, which they receive as interest payments. The result is that capital
formation is due entirely to the savings of young workers, who must replace the
entire past capital stock and then some, if they are to see growth in capital
intensity.

There are behavioral assumptions that could lead to accumulation from one
period to the next, even if retirees hold all assets. One possibility is that retirees
refrain from spending all of savings and interest, as a consequence of either
altruistic bequests or uncertainty regarding lifespan, then at least some of capital
could remain in the economy as unconsumed capital (e.g. as savings of the young
out of bequests). OLG models with altruism have received much attention in the
literature. For an overview see Smetters (1999). Other recent models include Michel
the effects of uncertain lifespan are in Fuster (1999). All of these references assume,
however, that next period’s capital stock depends entirely on current savings (an
exception is Lines, 1999).

A second assumption, that retirees are not endowed with the entire capital stock,
could also lead to capital accumulation. The following is a study of the dynamics
of capital accumulation under the latter hypothesis. In Section 2 an OLG model is
developed. Existence and stability of steady states, and the dynamics in general, are
studied in Section 3. Concluding remarks are provided in Section 4.

2. Basic model

2.1. Worker-capitalists

The problem facing the representative agent is:

$$\max_{c_1, \theta} u = u(c_{1t}) + (1 + \theta)^{-1} u(c_{2t+1})$$

subject to

$$c_{1t} + s_t \leq w_t$$
$$c_{2t+1} \leq (1 + E_{\tau}r_{t+1})s_t$$
$$c(\cdot), w(\cdot), u(\cdot), r(\cdot) \in \mathbb{R}^+, \quad s(\cdot) \in \mathbb{R}_0^+$$
$$0 \leq \theta \leq \infty$$
$$E_{\tau}r_{t+1} = r_{t+1}$$

and variables are defined as follows: $c_{1t}$ is consumption in the working period at
time $t$, $c_{2t+1}$ is consumption in the retired period at time $(t + 1)$, $w_t$ is the wage in
period $t$, $s_t$ is savings in period $t$, $r_{t+1}$ is the interest rate on one-period loans at time $t+1$. The last line in Eq. (1) represents the assumption that individuals have perfect foresight. The single parameter is $\theta$ where $(1 + \theta)^{-1}$ is the subjective discount factor for future utility.

The instantaneous utility function assumed for consumption is the strictly concave, isoelastic function $u(\cdot) = \ln(\cdot)$, which meets the usual requirements of a utility function and guarantees that forward dynamics can be determined because substitution and income effects cancel each other exactly.

The first order condition determines the optimal amount of consumption and savings in the first period as $c^*_t, s^*_t = \frac{1 + \theta}{2 + \theta} w_t$

$$s^*_t(w_t) = w_t - c^*_t = \frac{w_t}{2 + \theta},$$

which is simply Keynes’ hypothesis that savings is a (constant) proportion of the individual’s income. Note that if the individual values utility in the retirement period as much as current utility ($\theta = 0$), he saves half of his wages for the second period. If the future has no weight at all, $\theta \to \infty$ and $s^*_t \to 0$.

Each member of the labor force $N$ is characterized by the same utility function and subjective discount factor so that total savings in any period is simply the product $S_t = N_s s_t$. It is assumed that the labor force grows according to

$$N_t = N_0(1 + n)^t - 1 < n < 1$$

where $n$ is given exogenously.

2.2. Technology

Total output $Y_t$ is produced by capital $K_t$ and labor $N_t$ with function $F$ that is homogeneous to the first degree, permitting output to be expressed in per capita terms (lower case variables)

$$Y_t = F(K_t, N_t) = K^a_t N^{1-a}_t \quad 0 < a < 1 \quad N_t \in \mathbb{R}^+, K_t \in \mathbb{R}_0^+$$

$$\frac{Y_t}{N_t} = y_t = k^*_t.$$  

The per capita production function is well-behaved and satisfies the Inada conditions.

2.3. Firms

While production technology is well-defined in the standard model, the organization of production is typically somewhat ambiguous. Diamond’s entrepreneurs in the competitive setup¹ become, in later analyses, profit-maximizing firms. The

¹'Capital demanders are entrepreneurs who wish to employ capital for production in period $t + 1.' (Diamond, 1965, p. 1130)
former approach seems to indicate a second type of agent, although no other characteristics are offered. In the latter case alternative interpretations can be considered. If firms are simply set up by retirees there are, at time \( t \), \( N_{t-1} \) of them, and their growth rate is the same as that of the work force. If, instead, firms are merely managed by retirees (see, e.g. Boldrin, 1992) their number and growth rate are irrelevant. In either case, the profit-maximizing behavior of retirees may need to be cast in the utility maximization problem. (Would these owners or managers seek to maximize profits, or maximize the return to their own capital?)

Essentially, in the standard OLG model, production is operated by a profit-maximizing algorithm which solves the problem:

\[
\max_{N_t, K_t} Y_t - w_t N_t - r_t K_t
\]

subject to

\[
Y_t = F(K_t, N_t).
\]

(5)

First order conditions for problem (5) give optimal levels of labor and capital as those for which respective prices are equivalent to respective marginal products. Given the production function in Eq. (4) these are:

\[
MP_N = w_t = (1 - \alpha)k_t^\alpha
\]

\[
MP_K = r_t = \alpha k_t^{\alpha - 1}.
\]

(6)

The algorithm employs the optimal factor levels and distributes wages and interest. Under the hypothesis that capital is entirely owned by retirees, the firm algorithm is the mechanism by which output is distributed between the young (worker) and the retired (capitalist) generations. Thus is the classical antagonistic framework of heterogeneous cohorts (workers and capitalists) transformed into that of homogeneous cohorts, heterogeneous generations. This formal description is particularly useful for the study of transactions between generations (for it focuses exactly on their opposing interests). It is less useful for studying the timepath of genuine capital accumulation (from one period to the next), and is certainly in great difficulty to explain the emergence of capitalism in a context of few retirees, high discount rates and/or subsistence wages.

Suppose, instead, that in the initial periods a portion of the capital stock is not assigned to retirees. Potential recipients of the unassigned part can be grouped as: (i) pure capitalists, entrepreneurs, speculators, or any other agents whose consumption patterns can be ignored in a first approximation (the extreme of the classical savings hypothesis); or (ii) the firm algorithm which, in addition to its other tasks, reinvests any of capital’s share remaining after distribution to retirees. The homogeneous cohorts vision is maintained by the latter assumption, but violated by the former. In either case, no further assumptions are necessary, as long as the other owners of capital refrain from consuming.
2.4. Equilibrium conditions

The equilibrium condition in the single good market, investment equals savings, is

\[ K_{t+1} - K_t = Y_t - C_t + C_t 
= r_tK_t + w_tN_t - N_t \delta_t - (1 + r_t)N_{t-1} s_{t-1} 
= N_t \delta_t + r_tK_t - (1 + r_t)N_{t-1} s_{t-1} \tag{7} \]

where the second line makes use of Euler’s theorem for a first-degree homogeneous production function. Notice that the dissavings of the retired generation is given by the last term on the RHS. If all of capital’s share of production goes to retirees \( N_{t-1} s_{t-1} \equiv K_t \equiv S_{t-1} \) and the model reduces to the standard version. The initial conditions permitting the relaxation of this hypothesis are studied in Section 3.4.

The labor market follows the standard model. Labor supply is inelastic, and the market is in equilibrium when \( w_t \) in Eq. (6) induces firms to hire \( N_t \).

The demand for capital is given by the marginal product function

\[ KD_t = N_t \left( \frac{2}{r_t} \right)^{1/1-x} \]

Under the hypotheses the supply of capital available at \( t \), rearranging Eq. (7) is:

\[ KS_t = S_{t-1} + (1 + r_{t-1})(K_{t-1} - S_{t-2}) \]

The equilibrium rate of rental for capital services is that which satisfies Eq. (6) and induces firms to hire the capital services available (and fixed) at time \( t \).

3. The dynamics of accumulation

3.1. Existence

So far the variable capital has been treated as net capital. Assume now that depreciated or obsolete capital diminishes the general level of capital. This provides an opportunity to specify any effects of varying the rate of depreciation. The dynamics of the stock of capital variable is given by including a radioactive decay depreciation \( \delta K \) in Eq. (7):

\[ K_{t+1} = (1 + r_t - \delta)K_t + N_t \delta_t - (1 + r_t)N_{t-1} s_{t-1} \]

or dividing through by \( N_t \), in per capita terms:

\[ k_{t+1} = \frac{1 + r_t - \delta}{1 + n} k_t + \frac{s_t}{1 + n} - \frac{1 + r_t}{1 + n} s_{t-1}. \]

After substituting optimal marginal products and savings functions, the basic equation of the genuine accumulation model can be written as:
\[ k_{t+1} = \beta (1 - \delta) k_t + (x\beta + \nu)k_t^x - \beta \nu k_{t-1}^x - x\beta \nu k_t^{x-1} k_{t-1}^{x-1} \]  
(8)

with \( \beta = (1 + n)^{-1} \), \( \nu = \beta (1 - x)/(2 + \theta) \) and \( \beta > 0, \nu > 0 \) under the hypotheses.

Notwithstanding the highly nonlinear form of the per–capita capital accumulation equation it is possible to determine the functional form for the fixed points. Let \( \bar{k} \) denote equilibrium. Then substituting in Eq. (8) and rearranging the terms it follows that

\[ \bar{k}(x\beta \nu \bar{k}^{2x-2} - (x\beta + \nu - \beta \nu)\bar{k}^{x-1} + [1 - \beta (1 - \delta)]) = 0. \]  
(9)

By setting \( z = \bar{k}^{x-1} \), the full parentheses in Eq. (9) can be written as a linear quadratic equation and the equilibria are easily found:

\[ \bar{k}_1 = 0 \]
\[ \bar{k}_{2,3} = z_{2,3}^{1/2-x-1} \]
\[ z_{2,3} = \frac{1}{2\nu}[(x + \nu) \pm \sqrt{(x + \nu)^2 - 4(n + \delta)x\nu}]. \]  
(10)

The solutions for \( z_{2,3} \) are either 2 or 0 real roots. Since roots of complex numbers are complex, a condition for the existence of a non-trivial fixed point is that \( z_{2,3} \) be real. If \( -1 < n < -\delta, \) \( z_{2,3} \) are always real. However, applying Descartes’ theorem gives \( z_2 > 0 \) but \( z_3 < 0. \) In that case there is a fixed point at the origin, a positive real fixed point, and a third fixed point which can be complex or real, depending on the value of \( x. \) Then the branch of fixed points \( \bar{k}_3 \) will not be continuous over the parameter \( x. \) In any case, for \( n \) over that range, the existence of at least one non-trivial equilibrium is guaranteed.

It is unlikely that the labor force declines at a rate faster than capital depreciates. But if \( -\delta < n < 1, \bar{k}_{2,3} \) are potentially complex. In fact, the condition for a positive discriminant can be written as:

\[ \delta < \frac{(x - \nu^2)^2}{4\nu}. \]  
(11)

If condition (11) is not satisfied \( \bar{k}_{2,3} \) are complex, leaving a single fixed point at the origin. This suggests a direct link between existence of non-trivial equilibria and the size of the depreciation rate (more on this point in Section 3.5). The relation between the rate of depreciation and existence of positive equilibria in the standard OLG model is well-known, see, e.g. Boldrin (1992), Jones and Manuelli (1990). An example of an existence condition is \( \lim_{k \to -\infty} R(p) = \rho > \delta. \) In the present second-order model, if Eq. (11) is satisfied, Descartes’ theorem ensures that there are, in addition to the trivial fixed point, two positive real fixed points.

There are four parameters involved in the determination of the equilibrium points: \( x, n, \theta, \) and \( \delta. \) The manifolds for \( \bar{k}_2 \) and \( \bar{k}_3, \) setting \( \theta = 1, \delta = 0.1, \) are shown in Fig. 1 left and right, respectively, over a wide range for the technology coefficient and labor growth rate. Note that \( \bar{k}_2 \) is the smaller value, \( \bar{k}_3 \) the larger and that some combinations of parameters imply complex values. For \( \delta = 0 \) however, the manifolds are real over the ranges of parameter values considered.
Fig. 1. Fixed point manifolds: left, \( k_1 \); right, \( k_3 \).
3.2. Stability

Setting $x_t = k_{t-1}$ the second order equation in Eq. (8) becomes the first order system:

$$k_{t+1} = \beta (1 - \delta) k_{t} + (x_0 + v)\bar{k}_{t}^{-1} - \beta \nu v x_{t}^{-1} - \alpha \beta \nu k_{t}^{-1} x_{t}^{-1}$$

$$x_{t+1} = k_{t}.$$  \hfill (12)

Local stability properties are studied by means of the linear approximation given by the Jacobian matrix $J$, which for system (12) is:

$$J = \begin{pmatrix} J_{11} & J_{12} \\ 1 & 0 \end{pmatrix}$$

where

$$J_{11} = \beta (1 - \delta) + \alpha (x_0 + v)\bar{k}_{t}^{-1} + \alpha \beta \nu (1 - \alpha) k_{t}^{-1} x_{t}^{-1}$$

$$J_{12} = - \alpha \beta \nu x_{t}^{-1} - \alpha^2 \beta \nu k_{t}^{-1} x_{t}^{-1}.$$  

The Jacobian, as well as the dynamic equation of the model, calculated at the first equilibria \((\bar{k}, \bar{x}) = (0, 0)\) are undefined. However, inserting near-zero values for $k$ \((k = 10^{-5} \text{ and } k = 10^{-7})\) into the Jacobian gives one large eigenvalue \((7.2 \times 10^5 \text{ and } 4.5 \times 10^8, \text{ respectively})\), and a second eigenvalue less than unity \((0.429 \text{ for both})\). The origin is locally unstable.

Numerical simulations confirm that, in the case for which there are no equilibria in the first quadrant, trajectories from positive initial values tend rapidly towards negative infinity. Galor and Ryder (1989) refer to a unique attracting equilibrium at the origin as ‘contraction phenomenon’. Although the local stability characteristic of the fixed point at the origin is different (a saddle rather than a stable node), since the qualitative dynamical implications are the same (the economy collapses) the term ‘contraction’ will be applied.

If the non-contraction condition given by Eq. (11) is satisfied, two other equilibria exist and the Jacobian and its eigenvalues can be calculated. The

![Fig. 2. Fixed point branches over 0.001 < \alpha < 0.3.](image)
branches of steady states are given by Eq. (10). In Fig. 2 an example of the branches of \( \bar{k}_2 \) and \( \bar{k}_3 \) is provided under the assumptions that \( \theta = 1 \), \( \delta = 0 \). The technology coefficient varies over \((0.001, 0.3)\) for two values of the labor force growth rate, \( n = 0.01 \) (the steeper curve) and \( n = 0.1 \). Eigenvalues were calculated at various values and the local stability of the branches is represented by solid curves (stable) and dashed curves (unstable). The upper branch represents values of \( \bar{k}_1 \), which are, locally, stable nodes (one eigenvalue near 1, the other close to 0). The lower curve represents values of \( \bar{k}_2 \), which are saddle points locally.

If the initial values are less than the lower branch in Fig. 2 the economy contracts, that is, for any given parametric configuration, the take-off condition for initial capital intensity is

\[
k_0, k_1 > \bar{k}_2.
\]  

(13)

If the initial values are greater than \( \bar{k}_2 \) but less than \( \bar{k}_3 \), the economy monotonically grows to the steady state value of \( \bar{k}_3 \). For initial values greater than the equilibrium value (in simulations up to \( k_0 = 10^6 \)) the economy declines monotonically to \( \bar{k}_3 \). The basin of attraction for \( \bar{k}_3 \) is then, at least \((\bar{k}_2,10^6)\). The take-off condition will be discussed in more detail in Section 3.5, one now turn to the dynamics of the allocation of capital’s share of the output.

### 3.3. Capitalists

Essentially, what differentiates the genuine accumulation from the standard model can be summed up as a relaxing of the initial conditions. For clarity’s sake, let capital be owned by retirees \((r)\) and firms \((f)\). Then, under the assumptions, one can write

\[
\begin{align*}
K_t &= K_t^r + K_t^f \\
K_t^r &= N_{t-1} s_{t-1} = S_{t-1} \\
K_t^f &= K_t - S_{t-1}.
\end{align*}
\]  

(14)

Two initial conditions are required to solve (or to simulate) the system in Eq. (12), \( k_0, k_1 \), and setting \( t = 1, t - 1 = 0 \) in Eq. (14) one has

\[
\begin{align*}
K_t^r &= S_0 = N_0 s_0 = N_0 \frac{(1 - \alpha)k_0^r}{(2 + \theta)} \\
K_t^f &= K_1 - K_t^r = K_1 - N_0 \frac{(1 - \alpha)k_0^r}{(2 + \theta)}.
\end{align*}
\]

Assume that at \( t = 0 \), \( K_0 \) is very small and that all of capital’s share is given to retirees. If the value assigned to \( K_1 > K_t^r \), that is, if

\[
k_1 > \nu k_0^r
\]  

(15)

firms receive a positive endowment. If \( k_1 \leq \nu k_0^r \), the framework reduces to the standard model for which retirees own the whole of the capital stock. Condition
Fig. 3. Trajectories of capital quotas and capital density.

(15) is extremely sensitive to the discount parameter, so that for high discounting (as one might expect in an early phase of capitalism), firms will almost surely be endowed with some of the capital stock.

At time $t$, the quotas held by retirees and firms of the total capital are $\gamma_r'$ and $1 - \gamma_r'$, respectively. Their respective shares of total output due to owning capital are $\Gamma_r'$ and $\alpha - \Gamma_r'$, respectively, where

$$
\gamma_r' = \frac{K_r'}{K_t} = \frac{v k_{t-1}^x}{k_t} \quad \Gamma_r' = \alpha \gamma_r' \\
1 - \gamma_r' = \frac{K_f'}{K_t} = 1 - \frac{v k_{t-1}^x}{k_t} \quad \alpha - \Gamma_r' = \alpha(1 - \gamma_r').
$$

Fig. 3 presents a typical trajectory with initial conditions satisfying Eq. (15) under the parametric configuration

$$
\alpha = 0.25 \quad n = 0.1 \quad \theta = 1 \quad \delta = 0.1 \quad k_0 = 0.17 \quad k_1 = 0.2.
$$

The capital intensity increases to equilibrium value $\bar{k}_3 = 1.15$. The retirees’ equilibrium quota of capital can be calculated, substituting the unique, positive equilibrium $\bar{k}_3$ (where it exists) in Eq. (16). Then $\bar{\gamma}_r' = v/\bar{k}_3^{1-x}$, giving 0.21 for the standard parametric configuration employed in Figs. 3 and 4.

Fig. 4 represents the timepaths, for the same configuration, of the various shares of total output: workers’ and capital’s constant shares $1 - \alpha$ and $\alpha$, respectively; the firms’ and retirees’ shares at $t$ acquired through rights on capital’s share as in Eq. (16); the lifetime share for agents retired at $t$, $(1 - \alpha) + \alpha \gamma_r'$.

3.4. Infinite discounting

The stability properties of the branches of fixed points $\bar{k}_2$ and $\bar{k}_3$ are unchanged by varying $n$ and $\alpha$ over ranges typically assumed in economic models. That is not the case for $\theta$ and $\delta$. The consequences for stability of the discount parameter are studied in this section, those of the depreciation rate are studied in Section 3.5.
Recall that \((1 + \theta)^{-1}\) is the subjective discount factor for utility in the retired phase and varies from \(\theta = 0\) for which half of the wage is saved for the future, to \(\theta = \infty\) for which none of the wage is saved. Even large increases in \(\theta\) after \(\theta = 1\) have little effect on the value of the fixed point \(\bar{k}_3\). However in the limit

\[
\lim_{\theta \to \infty} y = \frac{1 - \alpha}{(1 + n)(2 + \theta)} = 0
\]

and the genuine accumulation model of Eq. (9) reduces to:

\[
k_{t+1} = \frac{1 - \delta}{1 + n} k_t + \frac{\alpha}{1 + n} k_t^x,
\]

This is a first order difference equation with fixed points at

\[
k_1 = 0 \quad k_2 = \left(\frac{\delta + n}{\alpha}\right)^{1/\alpha - 1},
\]

the latter always real if \(n > -\delta\). The eigenvalue

\[
\frac{\partial k_{t+1}}{\partial k_t} = \frac{1 - \delta}{1 + n} + \frac{\alpha^2}{1 + n} k_t^{x-1}
\]

is undefined at \(\bar{k}_1\) but

\[
\lim_{k_1 \to 0} \frac{\partial k_{t+1}}{\partial k_t} = \infty
\]

and the origin is a repeller. As long as \(\bar{k}_2\) is real \((n > -\delta)\) its eigenvalue is less than unity and it is a local attractor. As for its basin of attraction, simulations suggest that any initial value less than \(\bar{k}_2\) and not exactly 0 is attracted to it, as well as all higher initial values used in simulation (up to \(10^6\)).

That is to say, even if workers save nothing for their retirement phase, there is a real, positive, stable capital intensity to which the economy moves from any positive initial value of the ratio. In such a case, workers do not supply capital and all of capital’s share goes to firms who drive the accumulation process.

Fig. 4. National income shares.
3.5. Bifurcations

The fixed points of a first order discrete system are called hyperbolic if and only if no eigenvalues of the linearized system lie on the unit circle. Roughly speaking at non-hyperbolic points topological features of the system (such as the number of stationary points, or their stability) are not structurally stable. Bifurcation theory describes the way these features vary as parameter values are varied. For a system in two dimensions the stability conditions ensuring that both eigenvalues remain within the unit circle are:

1. $1 + \text{tr}J + \det J > 0$
2. $1 - \text{tr}J + \det J > 0$
3. $1 - \det J > 0$

where $\text{tr}J$ is the trace of the Jacobian matrix, $\det J$ is the determinant. The violation of any single inequality while the other two simultaneously hold leads to, respectively, a flip (real eigenvalue passes through $-1$), a fold (real eigenvalue passes through $1$), a Neimark (modulus of a complex eigenvalue pair passes through $1$) bifurcation. For the equilibria $k_{2,3}$, condition (i) is always satisfied, eliminating the possibility of a flip bifurcation. Condition (ii) and (iii) are not easily proven analytically. However, the determinant of the Jacobian is less than $1$ over the combinations of parameter ranges studied and would seem to be always satisfied over common ranges assumed in the literature for the parameters, eliminating Neimark bifurcations.

That leaves only the possibility of a generic fold (also known as a saddle-node) bifurcation if condition (ii) is violated. It is typical in a fold bifurcation that, as a real eigenvalue gets larger than $1$, real fixed points become complex or vice versa. Simulations suggest that such is the case for the accumulation model. In Fig. 5 an example of the branches of $k_2$ and $k_3$ is provided for $n = 0.1$, $\theta = 1$, over $\alpha \in (0.001, 0.3)$ for four values of the depreciation rate $\delta = 0, 0.05, 0.1, 0.15$. Again, upper values are the $k_3$ stable nodes, lower values are the $k_2$ saddles and at the point at which the two branches converge and disappear, the fixed points $k_{2,3}$ become complex. At $\delta = 0$ the branches observed in Fig. 2 are seen as an extreme
case of the fold bifurcation. As the rate of depreciation increases, the range of \( \alpha \) admitting of real, non-trivial equilibria diminishes until by a value of \( \delta = 0.25 \) no real fixed point branches appear for the indicated range of \( \alpha \) (e.g. fixed points calculated at \( \delta = \alpha = 0.3 \) are complex-valued).

For non-zero depreciation, values of \( \alpha \) to the left of the fold bifurcation point the unique equilibrium is the origin, a saddle point which sends trajectories to \( -\infty \). For those combinations of parameter values the only dynamical behavior possible for the model economy is collapse. In order to avoid the contraction trap high depreciation rates must be accompanied by high values of \( \alpha \), the elasticity of output with respect to the capital input.

To the right of the bifurcation point a positive fixed point attracts over at least \((\bar{k}_2, 10^6)\). If the initial capital intensity values satisfy the take-off condition (13), growth to a steady state is assured. As an idea of the relative order of magnitude necessary for initial values with respect to the stable equilibrium value \( \bar{k}_2 \), consider that the standard configuration used in Figs. 3 and 4 gives \( \bar{k}_2 = 0.16, \bar{k}_3 = 1.15 \). That is, initial values must be larger than one-seventh of the equilibrium value for take-off. An early capitalism scenario of low-\( \alpha \) technology \((\alpha = 0.05)\) and a high discount rate \((\theta = 20)\) gives a similar ratio. On the other hand, a technology transfer scenario of high-\( \alpha \) technology \((\alpha = 0.25)\) combined with a medium–high discount rate \((\theta = 10, \text{or } 8\% \text{ of earnings are saved})\) gives \( \bar{k}_2 = 0.02 \) very low, and 2 orders of magnitude smaller than \( \bar{k}_3 = 1.30 \). Fig. 5 and other numerical simulations suggest that over typical ranges, the parameters have the following partial effects on the take-off value

\[
\frac{\partial \bar{k}_2}{\partial \alpha} < 0 \quad \frac{\partial \bar{k}_2}{\partial n} < 0 \quad \frac{\partial \bar{k}_2}{\partial \theta} < 0 \quad \frac{\partial \bar{k}_2}{\partial \delta} = 0
\]

and on the equilibrium value

\[
\frac{\partial \bar{k}_3}{\partial \alpha} > 0 \quad \frac{\partial \bar{k}_3}{\partial n} < 0 \quad \frac{\partial \bar{k}_3}{\partial \theta} > 0 \quad \frac{\partial \bar{k}_3}{\partial \delta} < 0
\]

where, in particular, the take-off value is greatly influenced by \( \theta \).

4. Concluding remarks

The genuine accumulation model was shown to differ from the standard model through the initial conditions which, if they satisfy a weak assumption on magnitudes, ensure that a portion of the initial capital stock is not assigned to retirees. It is assumed that the consumption of non-retirees can be ignored, so that their return on ownership of the capital stock is entirely reinvested. The model admits, under

2 It should be noted that, although the bifurcation might appear to be transcritical (a special type of fold bifurcation) in the zero depreciation case, there is no exchange of stability as the upper curve always represents the stable fixed point. In fact, we are observing back-to-back fold bifurcations (and pieces of left hand wings of other branches are visible near the origin).
certain parametric configurations a unique, stable, positive equilibrium to which the system converges monotonically.

The model serves as a framework for studying the sensitivity of the dynamics involved in capital accumulation to the various parameters in relations regarding savings, investment, capital depreciation and the growth rate of the labor force. As regards the latter, a larger workforce earns more and saves more and can more easily cover the consumption of retirees for a net savings to be invested. On the other hand, since accumulation is in a per capita sense, the faster the labor force grows the more difficult it is to increase capital density. Holding all other parameters fixed, the lower the growth rate, the higher the stable steady state value and the take-off value.

The faster capital depreciates, the lower the equilibrium value of capital density. More importantly, it was shown that the size of the depreciation parameter is crucial for the existence of non-trivial steady states, that is, for avoiding contraction. Values greater than around 0.25 determine complex-valued fixed points under normal ranges of the other parameters leaving a single unstable fixed point at the origin, which sends all trajectories off to negative values and the collapse of the economy. It is also interesting to note that for $\delta < 0.25$ the higher $\delta$ the higher must be $\alpha$ in order to get real valued equilibria. In other words, capital’s contribution to production must be high if capital depreciates or becomes obsolete quickly. That relation leads to fold bifurcations which are directly linked to the problem of existence. If $\delta = 0$ and the model is interpreted in net capital terms, a real equilibrium value is always available over the same parametric configurations, suggesting that such an approach may be misleading about long run dynamics.

The discount rate has a slight, positive effect on the stable equilibrium value, a large, negative effect on the take-off value $k_2$, and, naturally, a negative effect on the equilibrium quota of capital owned by retirees. In the extreme case that individuals totally discount the future ($\theta = \infty$) a positive, real, stable equilibrium exists nevertheless. This non–trivial equilibrium is an important difference with the Diamond-type OLG models, for which no savings from wages means no investment, no capital, no economy. The accumulation process can be initiated in the present model because the return to capital is allocated to others who use it as a source for investment.

The higher the technology coefficient, ceteris paribus, the higher is equilibrium capital intensity, and the lower is the quota of retirees in the total capital stock. While the effect of $\alpha$ on lifetime earnings cannot be analytically determined, simulations (and logic) suggest that the reduction in workers’ share due to a higher $\alpha$ is not compensated by any possible gains due to ownership of a (smaller quota) of an increased capital share over standard parametric ranges.

References