A mathematical theory of traffic hysteresis

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Abstract

This paper presents a mathematical theory for modeling the hysteresis phenomenon observed in traffic flow. It proposes that acceleration, deceleration and equilibrium flow should be distinguished in obtaining speed-concentration and/or occupancy relationships, such that the phase transitions from one phase to another can be correctly identified. The analysis shows that the speed–concentration curves obtained following this approach are hysteresis loops, as predicted by the theory. The paper also gives a discussion of the general properties of the proposed modeling equations and examines the relationship between traffic hysteresis and stop-start waves observed in traffic flow. © 1998 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Fundamental to traffic flow modeling and control are the basic relationships between three traffic states: flow rate (q), speed (v) and concentration (ρ). The continuum theory developed independently by Lighthill and Whitham (1955) and Richards (1956) (the LWR theory), for example, assumes that there exists an unique equilibrium flow-concentration curve or, equivalently, a speed–concentration curve, but it does not prescribe a form for this relationship, if it exists. Although a great deal of experimental evidence suggests that some kind of relationship does exist between these three basic quantities, it is an open question as to which functional forms should be adopted to describe these relationships. In practice, a specific form is either postulated or obtained through data-fitting.

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Haight (1963) presented three ways to deduce these fundamental relationships of traffic flow: (1) from statistical modeling; (2) from car-following, and (3) from fluid analogies. A number of well-known functional forms for speed and concentration are derived from steady-state car-following theories. They include the Greenshields’ (1934) linear model, Greenberg’s (1959) logarithmic model, Underwood’s (1961) exponential model, and the one-parameter family polynomial models
\[
v = v_f \left(1 - \left(\frac{\rho}{\rho_j}\right)^n\right)
\]
where \(n\) is the parameter, \(v_f\) is free flow speed and \(\rho_j\) is jam concentration.

All the aforementioned functions are smooth and describe stationary flow behavior. Two observed traffic flow phenomena appear to challenge the notion of a smooth speed-concentration curve: one is the sudden speed drop often observed in empirical speed-concentration phase plots and the other is traffic hysteresis, a phenomenon characterized by that the acceleration and deceleration flow have different speed-concentration curves. The first phenomenon has led some researchers to believe that speed-concentration curves are inherently discontinuous, and to propose two-regime or catastrophe models to fit observed data (e.g. Edie, 1961; Acha-Daza and Hall, 1994). Those discontinuous models appear to provide a better fit to traffic data than the smooth models (e.g. Drake, Schofer and May, 1967), nonetheless they do not provide an explanation as to what causes the abrupt phase transitions in traffic flow, nor do they offer a plausible explanation to traffic hysteresis. The question that what is the correct form of the fundamental relationship between speed and concentration is yet to be answered.

In the new monograph on traffic flow theory (TRB, 1997), Hall describes rather well the difficulties that the transportation research community faces in answering the aforementioned question. According to Hall, “the problem for traffic flow theory is that these curves are empirically derived. There is not really any theory that would explain these particular shapes, except perhaps for Edie et al. (1980), who propose qualitative flow regimes that relate well to these curves.” As Hall further points out, “the task that lies ahead for traffic flow theorists is to develop a consistent set of equations that can replicate this reality”.

In this paper we propose a system of equations that explains qualitatively the phase transitions depicted by observed speed-concentration plots. In the sections to follow, we first classify traffic flow into three kinds—acceleration, deceleration and (strong) equilibrium, then analyze the phase plots for speed and occupancy (a surrogate of concentration) of different types of flows. Next we deduce traffic hysteresis from a system of equations, highlight the dangers of mixing different types of flows in traffic stream modeling, and discuss the connection between stop-start waves and traffic hysteresis.

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1More interestingly, he postulated a fifth boundary condition for zero concentration and showed that all flow-concentration relationships formerly derived from various assumptions violated this fifth boundary condition. The fifth boundary condition was surprisingly forgotten in the study of fundamental relationships of traffic flow until recently when Del Castillo and Benitez (1995) prescribed a fifth boundary condition for the curve at jam concentration.

2This speed drop typically occurs near a critical concentration where flow is nearly maximal before the speed drop.
2. Some basic concepts

In dealing with a dynamic system, it is important to characterize, if one can, the basic types of system behavior. Two commonly used notions concerning the behavior of a dynamic system are *equilibrium* and *nonequilibrium*. Any state of a dynamic system is either an equilibrium state or a nonequilibrium state, but not both at the same time. For example, the equilibrium states of a finite dimensional autonomous dynamic system governed by the following equation

\[ \frac{dx}{dt} = \Phi(x) \]

where \( x \in \mathbb{R}^n \) and \( \Phi \) is a linear or nonlinear mapping, are members of the set

\[ E(x) = \{ x : \Phi(x) = 0 \} \]

Any state that is not in \( E(x) \) is a nonequilibrium state.\(^3\)

The same notions can also be used to characterize traffic flow dynamics (whose states are elements of certain infinite dimensional function spaces). Below are our definitions of equilibrium and nonequilibrium in traffic flow.

Let \( \rho(x, t) \) and \( v(x, t) \) be the traffic concentration and travel speed at location \( x \) and time \( t \), we say that traffic is in *equilibrium* when the following temporal stationarity condition holds for every \( t \in \mathbb{R}_+ \) at any location \( x \in \mathbb{R} \):

\[ \frac{\partial v}{\partial t} = 0; \frac{\partial \rho}{\partial t} = 0 \quad (1) \]

and that traffic is in *nonequilibrium* if the following temporal non-stationarity condition holds for some time interval \( [t_1, t_2] \subset \mathbb{R}_+ \) at some location \( x \):

\[ \frac{\partial v}{\partial t} \neq 0; \frac{\partial \rho}{\partial t} \neq 0 \quad (2) \]

where \( \mathbb{R}_+ \) is the set of nonnegative real numbers, \( \mathbb{R} \) is the set of real numbers and \( t_2 \geq t_1 \).

Two remarks need to be made about this definition: (1) the definition of nonequilibrium allows for a singleton of an interval \( [t_1, t_2] \), \( t_1 = t_2 \). This accounts for cases arising from shock waves \{\( \rho(x - st), v(x - st) \)\} separated by two constant states \( (\rho \pm, v \pm) \), where \( s \) is the speed of the shock and (2) the definition of equilibrium does not exclude \( \frac{\partial v}{\partial x} \neq 0; \frac{\partial \rho}{\partial x} \neq 0 \), therefore vehicle acceleration \( \frac{dv}{dt} \) may not be nil even under equilibrium conditions because of the non-zero convection terms.

\(^3\)An equilibrium state can further be classified as stable or unstable, a notion which is not important in the context of our analysis.
It should be noted, however, $\frac{dv}{dt} = 0$ implies $\frac{d\rho}{dt} = 0$ in the LWR theory.\(^4\)

If, in addition to (1), one also has $\frac{\partial v}{\partial x} = 0$; $\frac{\partial \rho}{\partial x} = 0$ then we say that traffic is in strong equilibrium.

Further we say that a traffic stream during some time period $t_1, t_2$ is in acceleration phase if

$$\frac{dv}{dt} > 0$$

and deceleration phase if

$$\frac{dv}{dt} < 0$$

Strong equilibrium, acceleration and deceleration phases completely characterize the dynamic modes of a traffic system. In the sections to follow, this characterization scheme will be used to identify speed-concentration relationships in both strong equilibrium and acceleration/deceleration flows. Because most experimental observations are made at specific locations during finite time intervals, our subsequent analysis will use a local version of the above characterization, that is, we focus on part of the $x - t$ plane $(X, T), X \subset \mathbb{R}, T \subset \mathbb{R}_+$. \(^3\)

3. Empirical speed-concentration relationships

This section examines experimental data to find clues as to what kind of models are likely to be needed to describe the speed-concentration relationship. In this process one should separate different phases in traffic flow (e.g. equilibrium,\(^5\) acceleration and deceleration) and examine this relationship for each phase, for it is wise to start with the assumption that different type of flows may have different speed-concentration relationships. No harm is done if it is proven otherwise.

Based on our definitions in Section 2, strong equilibrium flow corresponds to those traffic conditions that travel speed is constant, which is equivalent to that flow is increasing linearly with

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\(^4\)Proof: the LWR theory reads:

$$\rho_t + (\rho v)_x = 0, v = v_c(\rho(x, t))$$

If $v_t = 0$, one has $v_t = v'_c(\rho)(\rho_t$. Note that $v'_c(\rho) < 0$, therefore $\rho_t = 0$. Consequently $(\rho v)_x = 0$, which leads to $\rho_x = 0$.

Now $\frac{d\rho}{dt} = v'_c(\rho)\frac{d\rho}{dt} = v'_c(\rho)(\rho_t + v\rho_x) = 0$, therefore $v_t = 0 \Rightarrow \frac{d\rho}{dt} = 0$.

If $\frac{d\rho}{dt} = 0$, one has $v'_c(\rho)(\rho_t + v\rho_x) = 0$. But $\rho_t + (\rho v)_x = 0$, therefore $v\rho_x = (\rho v)_x$, which results $\rho v_x = 0$ or $v_x = 0$.

From $\frac{d\rho}{dt} = v_t + v v_x$ one obtains $v_t = 0$. That is, $\frac{d\rho}{dt} = 0 \Rightarrow v_t = 0$.

\(^5\)In the subsequent texts equilibrium always means strong equilibrium unless noted otherwise.
respect to concentration for light traffic conditions\(^6\) but is constant for other traffic conditions. This criterion is consistent with the cumulative curve approach used by Cassidy to identify equilibrium flow-density curves, which has yielded strong evidence that a smooth equilibrium flow-concentration curve exists in traffic flow (Cassidy, 1996). An earlier study conducted by Del Castillo and Benitez (1995), who identified stationary data from measurements on European highways, also yielded smooth equilibrium speed-concentration curves.\(^7\) What distinguishes these results from the results of other studies that claim the speed-concentration curve is not smooth is the usage of data—in both Cassidy’s work and Castillo and Benitez’s study, only stationary data are used, while other studies used data that contain transients (which is not consistent with the equilibrium assumption). Should the other authors use stationary data, they might have also obtained smooth equilibrium speed-concentration curves as Cassidy, and Castillo and Benitez did.

In contrast to the limited number of empirical studies on equilibrium speed-concentration relationships, a large amount of literature exists for nonequilibrium speed-concentration relationships. Examples include Edie (1961), Drake, Schofer and May (1967), Cedar (1975), (1976), Cedar and May (1976), Easa and May (1980), Gunter and Hall (1986), Hall, Allen and Gunter (1986), Persaud and Hurdle (1988a), Banks (1989), Acha-Daza and Hall (1994).\(^8\) The majority of these studies support the notion that nonequilibrium speed-concentration curves are discontinuous, for it is often observed that travel speed experiences a sharp drop at a particular concentration level. Although these studies do not provide a theory to explain what causes this speed drop, the notion of discontinuous phase trajectories in traffic dynamics is not exotic: systems theory tells us that sudden phase transitions often occur in complex nonlinear systems, and traffic flow dynamics is highly nonlinear.

It appears, however, that a simple\(^9\) discontinuous nonequilibrium curve suggested by these studies and a smooth equilibrium curve together does not provide a consistent theory of traffic flow. To see this, we examine the equilibration process of traffic flow whose equilibrium curve is smooth and nonequilibrium curve is discontinuous. Without loss of generality, we assume that traffic is initially at an arbitrary equilibrium state B, as shown in Fig. 1, and that it accelerates gradually to another equilibrium state B'. There are two paths that can realize this phase transition: (1) path BB' along the equilibrium curve and (2) path BA'A''B'. Along the first path the phase transitions are infinitely small, which is exactly how phase transitions occur in the LWR theory. Along the second path, however, traffic has to make two jumps BA' and A''B' to reach state B'. Because B and B' are arbitrary points on the equilibrium curve, the latter phase transition path is rather odd, to say the least. One cannot avoid such jumps, however, in dealing with phase transitions from a point on the transient curve to a point on the equilibrium curve (e.g. from A to B').

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\(^{\text{6}}\)This is because speed is independent of concentration in light traffic.

\(^{\text{7}}\)It should be pointed out that occupancy instead of concentration was used in the aforementioned studies. It can be shown that occupancy is equivalent to concentration under equilibrium conditions but only somewhat related to concentration under nonequilibrium conditions.

\(^{\text{8}}\)Some of these studies concern flow-concentration relationships, equivalents of speed-concentration relationships because \(\text{flow} = \text{speed} \times \text{concentration}\).

\(^{\text{9}}\)In the context of this paper, a simple curve refers to a curve that is not closed, has no folds and contains at most finite number of discontinuous points.
The rather peculiar nature of the traffic equilibration process, as described above, results from insisting on a *unique* transient curve. If one allows transient curves to vary with flow conditions, one would not be forced to include the jumps in phase transitions. One still can, however, have phase curves with infinite slopes, as AA' in Fig. 2. Phase curve AA' in Fig. 2 differs from jumps BA' and A'' B' in Fig. 1 in that it is physically realizable and that all the states along AA' in Fig. 2 are admissible, while all states on BA' and A''B' except the end points in Fig. 1 are not.

The assertion that there is no unique transient speed-concentration curve can be checked with experimental data. Fig. 3 shows a portion of occupancy and speed data measured by detectors on a freeway in San Francisco (Skabardonis et al., 1996). Although the original data were collected for each freeway lane at a sampling interval of 30 s, the data in Fig. 3 are averages over all lanes. First, all data points were used to plot the speed–occupancy graph, and the results are shown in Fig. 4. One can clearly infer from this figure that speed generally decreases when occupancy increases, but cannot make any definitive assertions regarding phase transitions. It appears, however, that a simple nonequilibrium phase curve, whether it is smooth or discontinuous, may not be enough to describe all phase transitions.

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**Fig. 1.** Phase trajectories of traffic equilibration, unique transient curve.

**Fig. 2.** Phase trajectories of traffic equilibration, flow-dependent transient curves.
Hoping that it would reveal more structure of nonequilibrium phase transitions in traffic flow, speed–occupancy curves for acceleration and deceleration flows were drawn separately. Fig. 5 shows the phase trajectories for traffic flow in the time period [200,325], during which a major traffic disturbance passed through the detector location. A lot more can be said about phase transitions depicted in this figure than those in Fig. 4:

1. There is more than one nonequilibrium curve.
2. Acceleration and deceleration are asymmetric.
3. Phase trajectories form hysteresis loops.
4. Both acceleration and deceleration curves are nearly smooth.
5. Mixing acceleration and deceleration flow generates discontinuity.

All these assertions, if also true to general flow, will have significant implications in traffic flow theory. Now the question is how general these assertions are: do they apply to traffic flow in other time periods or other locations? To answer this question, we plotted phase trajectories using traffic data from different time periods at the same site and at different sites. Figs. 6 and 8 show the phase trajectories for time period [50,97] at Site 6, and time period [200,300] at Site 4 (upstream of Site 6, refer to Fig. 7 for a time history of occupancy at this site), respectively. These plots appear to confirm all the assertions. Moreover, they show that mixing acceleration and deceleration flow also generates folds. One cannot, however, take complete comfort in generalizing these results, because they may be unique to that particular freeway where data were collected. It would be more convincing if similar assertions can be made for other locations. In this respect we note two early studies on traffic stream models, one by Treiterer and Myers (1974) and the other by Maes (1979). In the first study, Treiterer and Myers used aerial photography to measure traffic concentration and speed, and tracked a vehicle stream as it entered and left a

Fig. 3. Time history of occupancy and speed at Site 6.
disturbance. The speed–concentration curve obtained by Treiterer and Myers is shown in Fig. 9. In the second study, Maes examined the relationships between flow, occupancy and speed under transient traffic conditions caused by incidents. The data was measured by inductive loop detectors and sampled at 1-min intervals. Fig. 10 shows a speed-occupancy curve similar to those obtained by Maes. It is obvious that all but the last assertions can be made from the results of both studies.\textsuperscript{10}

Clearly, there is strong empirical evidence that the speed–concentration curves for transient traffic are not unique, and it appears that there are two branches in such curves—one for acceleration traffic and the other for deceleration traffic. A careful reader would notice, however, that there is a difference between the curves obtained by the author and Treiterer and Myers and by

\textsuperscript{10}The two studies do not provide any evidence either supporting or contradicting the last assertion.
Maes: the acceleration and deceleration curves in the former switched positions in the middle to form two hysteresis loops while those curves in the latter form a single hysteresis loop that spans the whole range of traffic flow conditions. This discrepancy will be explained in Section 6. Before that, we shall propose a system of equations to model traffic hysteresis.

Fig. 5. Phase plots obtained by distinguishing flow types, Site 6, time [200,325].

Fig. 6. Phase plots obtained by distinguishing flow types, Site 6, time [50,97].
4. The mathematical theory

This section proposes a system of equations that is used to deduce traffic hysteresis in Section 5. The derivation of the equations are based on the following conjectures:

1. Drivers only respond to frontal stimuli.
2. There is a speed \( v_c(\rho) \) to which drivers attempt to adopt whenever it is possible.
3. When going through a strong traffic disturbance, drivers’ responses go through three stages: (1) anticipation dominant phase (phase A); (2) balanced anticipation and relaxation phase (phase B) and (3) relaxation dominant phase (phase C), although the order of stage (1) and (3) can be reversed depending on the nature of the disturbance.

Both conjecture 1 and conjecture 2 can be justified based on empirical evidence. Conjecture 3, however, needs elaboration. Note that drivers usually are aware of traffic conditions further ahead of them when traffic concentration is large (a strong disturbance), such that they adjust their speeds even before the traffic disturbance reach them if the spacing ahead allows for such an adjustment. Therefore the anticipation effect is dominant in traffic dynamics. On the other hand, when traffic is heavy, a driver usually could not see that a traffic disturbance is coming until it reaches him, therefore his response to the disturbance is retarded. Stage 2 is the connection between the two extremes and it could be omitted when the transition from stage 1 (or 3) to stage 3 (or 1) is abrupt.

The aforementioned conjectures lead to the following micro-macro models for vehicle \((n+1)\) at a position \(x_{n+1}\):

\[
\frac{dx_{n+1}(t)}{dt} = v_c(\rho(x_{n+1} + \Delta, t)), \text{ phase A} \tag{4}
\]
where $\tau$ is a relaxation time constant and $\Delta > 0$ is a small distance. It is assumed that $v_e'(\rho) < 0$ and $(\rho v_e(\rho))'' < 0$.

With a proper choice of $\tau$ and $\Delta$, Eqs. (4),(5) and (6) leads to the following respective equations:

\[
\frac{dx_{n+1}(t + \tau)}{dt} = v_e(\rho(x_{n+1} + \Delta, t)), \text{ phase B) (5)}
\]

\[
\frac{dx_{n+1}(t + \tau)}{dt} = v_e(\rho(x_{n+1}, t)), \text{ phase C (6)}
\]

The conservation of mass $\rho t + (\rho v)_x = 0$, when coupled with (7), results in a single equation:

\[
\rho = v_e - \tau\rho v_e'(\rho)^2 \rho_x
\]

\[
\rho = v_e - \tau \frac{dv}{dt} + \rho v_e'(\rho)^2 \rho_x
\]

\[
\rho = v_e - \tau \frac{dv}{dt}
\]

Fig. 8. Phase plots obtained by distinguishing flow types, Site 4, time [200, 300].

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11The derivation of (8) from (5) is detailed in Zhang (1998). Eqs. (7) and (9) can be obtained in the same manner as (8).
\[ \rho t + f_x(\rho) = \tau(\beta(\rho)\rho)_x \]  

(10)

where \( f_x(\rho) = \rho v_e(\rho) \) and \( \beta(\rho) = (\rho v'_e(\rho))^2 \geq 0 \); when coupled with Eq. (8), forms a 2\( \times \)2 system:

\[ \rho t + (\rho v)_x = 0 \]

(11)

Fig. 9. Trajectory for \( \rho(x(t), t) \) vs. \( v(x(t), t) \) obtained by Treiterer and Meyers.

Fig. 10. Trajectory for \( 0(x, t) \) vs. \( v(x, t) \) obtained by Maes.
where \( g' (\rho) = \rho (v'_e (\rho))^2 \); and when coupled with (9), results in another 2×2 system:

\[
\rho t + (\rho v)_x = 0
\]

(13)

\[
v_t + \left( \frac{1}{2} v^2 \right)_x = \frac{v_e (\rho) - v}{\tau}
\]

(14)

Eq. (10) is a ‘viscous’ conservation law. It admits smooth travelling waves of the form \( \phi (x - st) \), where \( s \) is the speed of the travelling wave and \( \phi \) is a smooth and monotone function of its argument. When the relaxation time \( \tau \) approaches zero, the travelling waves of (10) approach the shock waves of the inviscous conservation law

\[
\rho_t + f_s (\rho)_x = 0
\]

(15)

whose speed is given by \( s [\rho] = [f'_s] \), where \( [\rho] \) denotes the difference in \( \rho \) on both sides of the shock. Note that this inviscous conservation law is none other than the LWR theory, and its characteristics are

\[
\hat{\lambda}_s (\rho) = f'_s (\rho) = \rho v'_e (\rho) + v_e (\rho)
\]

On the other hand, systems (11) and (12), and system (13) and (14) are fundamentally different than the viscous conservation law: both systems are hyperbolic. In fact, system (11) and (12) is strictly hyperbolic because its characteristics are

\[
\hat{\lambda}_1 (\rho, v) = \rho v'_e (\rho) + v < -\rho v'_e (\rho) + v = \hat{\lambda}_2 (\rho, v).
\]

and system (13) and (14) is highly degenerate because its characteristics coalesce

\[
\hat{\lambda}_1 (\rho, v) = v = \hat{\lambda}_2 (\rho, v)
\]

In general, discontinuities can develop spontaneously in hyperbolic systems of conservation laws. Weak solutions are therefore introduced for such systems. It is beyond the scope of this paper to discuss the nature of solutions of systems (11) and (12), and (13) and (14). We do point out that system (11) and (12) is linearly stable (to small perturbations) but (13) and (14) is linearly unstable. The stability result for (11) and (12) can be found in Zhang (1997), and that for (13) and (14) can be found in the appendix of this paper.

5. Mathematical deduction of traffic hysteresis

We shall study mainly speed–concentration curves of traffic flow that contain strong travelling/shock or rarefaction waves, because deceleration (associated with a travelling/shock wave) or

\[\text{A travelling/shock wave is said to be strong if the magnitudes of } [\rho] \text{ and } [v] \text{ are large across the shock or travelling wave.}\]
acceleration (associated with a rarefaction wave) of a vehicle stream is most prominent under these two scenarios, which should in turn highlight the fundamental structure of nonequilibrium speed–concentration relationships. Because strong travelling/shock or rarefaction waves are often associated with large concentration gradients, a limit argument will be used to perform our mathematical derivation. That is, we use in our analysis the right hand side of

\[
\frac{\partial \rho}{\partial x} = \lim_{\epsilon \to 0} \frac{\rho(x + \xi, t) - \rho(x - \xi, t)}{\epsilon}
\]

(16)

to approach the left hand side of (16) when traffic concentration gradient becomes very large. Furthermore, we impose a consistency condition between relaxation time, concentration gradient and acceleration. Specifically we require that

\[
\frac{\epsilon}{\tau} = 0(v_e); \quad \tau \frac{dv}{dt} = 0(v_e)
\]

such that \(v \geq 0\) always holds.

Now we can deduce traffic hysteresis from the proposed system of equations. First, we examine the phase A transition of a vehicle stream going through the strong waves. Here (7) applies. Suppose that traffic stream travels into a denser traffic region, that is, \(\frac{\partial \rho}{\partial x} \to +\infty\). The second term on the RHS of (7) is always nonnegative. Together with the consistency requirement, we conclude that the speed during deceleration, \(v_d\), satisfies

\[
0 \leq v_d \leq v_e
\]

(17)

Following a similar argument, we can obtain the following for the speed \((va)\) of a vehicle stream travelling into a less dense region with a drastic decrease in concentration:

\[
0 \leq v_e \leq v_a
\]

(18)

Giving a unique equilibrium speed–concentration relationship, we therefore have

\[
v_d \leq v_e \leq v_a
\]

(19)

for rapidly changing traffic conditions associated with either a strong travelling/shock or rarefaction wave, i.e., the acceleration curve is above the deceleration curve for such conditions. It should be noted that such drastic changes are most likely to occur at moderately high traffic concentrations.

Next, we deduce from the theory the relative positions of the acceleration and deceleration branches of the speed–concentration curve in the high concentration region (phase C). Here (9) applies.
If a vehicle stream is accelerating, \( \frac{dv}{dt} \geq 0 \), according to (9) we have
\[
v_a \leq v_e
\]
and if a vehicle stream is decelerating, \( \frac{dv}{dt} \leq 0 \), we obtain
\[
v_d \geq v_e
\]
Combining the above two expressions, we have
\[
v_a \leq v_e \leq v_d
\]
for the same concentration level, that is, the acceleration branch is below the deceleration branch in the dense concentration region.

Because the acceleration and deceleration branches switched positions at extremes, there must be at least one point in the region \((0, \rho_j)\) where the two branches meet with the equilibrium curve (\(\rho_j\) is the jam concentration):
\[
v_a = v_e = v_d
\]
For such an event to occur, we must require:
\[
\frac{dv}{dt} + \rho v_e^2 \frac{\partial \rho}{\partial x} = 0
\]
\[\text{(22)}\]
\[\text{i.e.,}\]
\[
\frac{dv}{dt} = -\rho v_e^2 \frac{\partial \rho}{\partial x}
\]
\[\text{(23)}\]
in the intermediate stage (phase B) governed by (8).

It should be noted that (23) is the acceleration given by the LWR theory. The LWR theory can therefore be interestingly viewed as a nonequilibrium model with acceleration/deceleration speed–concentration curves collapsed to the equilibrium curve. When neither anticipation nor relaxation effects are dominant, (8), together with the conservation equation (of mass), forms a theory in its own right.

Synthesizing the aforementioned results, we obtained a sketch of the speed–concentration curve for transient flow associated with strong travelling/shock or rarefaction waves; it has the form shown in Fig. 11, which is independent of location, time, and means of measurement, and accords with the nonequilibrium speed–occupancy curves obtained through distinguishing acceleration and deceleration and the experimental observations of Treiterer and Myers (1974). Although the sketch has only two hysteresis loops, our analysis does not exclude the possibility of triple or more hysteresis loops, as shown in Fig. 12.
Fig. 11. Speed–concentration curves $(\rho(x(t), t), v(x(t), t)$ predicted by theory.

Fig. 12. Speed–concentration curves with multiple rendezvous points (segments).
6. A comparison with other hypotheses

It is clear that with the proper treatment of traffic data, the experimentally obtained speed-occupancy curves accord well with theoretical predictions. Why, one may wonder, have many previous studies obtained results that are strikingly different than the results presented here? The root cause of this difference, we believe, is the treatment of data. Without a clear definition of traffic equilibrium and a traffic theory to guide the proper processing of raw data, many previous studies lumped together every bit of data collected, which has led to two serious problems: (1) the data contain inherently different traffic phenomena, and (2) aggregation of data over longer time intervals than the acceleration-deceleration ‘wave length’ adds considerable distortion to obtained flow relationships. For example, Maes (1979) used one-minute interval data from stop-and-go traffic to obtain a speed-occupancy curve consisting of a single hysteresis loop (Fig. 10), which accords to neither our theoretical predictions nor other empirically obtained speed–occupancy or speed–concentration curves. We shall argue that it was data aggregation (that mixes different phases of flow) that caused this discrepancy. Because Maes’ data are not available to us, we cannot directly verify our conviction. We therefore use an analogy to support our assertion.

Fig. 13 shows the speed at a location on a San Francisco freeway, sampled in 10-s intervals. It clearly depicts a stop-and-go traffic condition. We plotted the occupancy–speed for acceleration and deceleration flow for the range between the dotted lines on Fig. 13, and the result is shown in Fig. 14. Clearly, there is also a small hysteresis loop for congested traffic (the fold near occupancy 22% and the jump near occupancy 30% on 14 are probably attributable to the aggregation of data across lanes), which accords with theoretical predictions. Note that the average frequency of the stop-start wave shown in Fig. 13 is about 0.017, or 1 min per cycle. Aggregation of data longer than 30 s will mix the acceleration and deceleration effects, and thus distort the true relationship between occupancy and speed. The longer the sampling interval, the more severe this distortion becomes. To see this, we aggregated the same data over 1 min intervals, and drew the phase portrait for the aggregated speed and occupancy. Fig. 15 shows the data after aggregation, and Fig. 16 shows the speed-occupancy curves for both acceleration and deceleration flows. Not surprisingly, Fig. 16 has the same structure as Fig. 10.

In light of stop-start waves occurring at high occupancies, the hysteresis loop shown in Fig. 10 is not physically feasible because such loops predict that traffic has to return from heavily congested flow to free flow to complete a hysteresis loop (it should be noted that a single hysteresis loop is possible if traffic is ‘moderately’ congested, with the positions of acceleration and deceleration branches opposite to those in Fig. 10). This difficulty can be removed if jumps between acceleration and deceleration branches are allowed. There is, however, no clear empirical evidence to support the contention that such jumps did occur in traffic flow. In fact, such jumps are physically impossible because they require vehicles to switch from acceleration to deceleration (or vice versa).

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13The data are from the same study (Skabardonis et al., 1996) as those used in Section 3.
14As a side note, we should point out that the width of a ‘strong’ traffic shock is typically two to three effective vehicle lengths (about 100 ft) and the shock passes a specific location in about 2–3 s. The data sampling interval if a detector, however, is usually 30 s or longer, which smoothes considerably the speed and concentration profiles along a shock path. To obtain an undistorted speed–concentration curve, we shall argue, the sampling interval of a detector should be at least as short as the time it takes for a shock to pass that detector, or as short as half of the acceleration–declaration wavelength in congested traffic.
without a finite transition. It is possible, however, that one of the branches (acceleration or deceleration branch) is nearly discontinuous. Physically this corresponds to a situation that a platoon of vehicles accelerate or decelerate simultaneously without diffusing, which leads to a sudden increase or drop of speed with nearly no change in concentration.

The speed–concentration curves predicted by our theory, on the other hand, can provide a plausible explanation for stop-start waves in congested traffic flow. Because of the multiple

Fig. 13. Stop-start waves in congested regime.

Fig. 14. Hysteresis in congested traffic.
hysteresis loops in these curves, traffic can get ‘trapped’ in a loop located in the high occupancy area. The time required to escape these congested hysteresis loops could, as one often observes, last much longer than the time needed to get into these hysteresis loops. Still, one does not know why it takes such a long time for traffic to break away from these hysteresis loops. This, we hypothesize, is more a matter of psychology than physics, and is worth further investigation.

Fig. 15. Aggregated speed and occupancy.

Fig. 16. Phase plot, aggregated speed and occupancy.
7. Concluding remarks

The hysteresis phenomenon in traffic flow have been known to transportation researchers for decades, yet no theory to-date has been able to model such a phenomenon. We have in this paper made an attempt to model traffic hysteresis using a traffic theory. We have shown that the predictions of the theory have accorded well with certain empirical observations, and provided an explanation for those for which theoretical predictions do not fit empirical results. In the process, we provided guidelines for proper data treatment in traffic flow analysis and clarified certain misconceptions in the study of traffic flow relationships.

We would like to point out that the emphasis of this paper is not to obtain mathematical formulas for those transient curves. Rather, it is directed towards gaining a better understanding of the general structural properties of those transient phenomena in traffic flow reflected in the \((\rho, v)\) phase plane. A limitation of the analysis is that one could not identify from the theory the exact locations where the transitions between the three phases (Phases A, B, and C) occur. To do this, one needs a unified theory that should include (7), (8) and (9) as special cases. The development of such a theory is a lofty goal of traffic flow theorists, and the author believes that careful gathering, analysis and synthesis of experimental data are critical to the success of this effort.

It is interesting to note that nearly three decades ago Newell (1965) had proposed two speed-spacing curves for transient flow and postulated a set of driving rules to explain instability in traffic flow. The predictions from Newell’s theory accord well with the later observations made by Treiterer and Myers (1974). Moreover, his theory also provided a behavioral explanation for the formation of hysteresis loops. Unfortunately Newell’s theory has gone unnoticed for several decades. A probable reason for this negligence is perhaps the difficulty in obtaining the mathematical formulae for the acceleration and deceleration curves. As a description of transient phenomena, one would expect these curves to be quite sensitive to traffic flow and roadway conditions, and the difficulty in gathering precise data adds to the complexity of the problem.

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Appendix. Stability analysis of (13) and (14)

Assume that \(\rho_0\) and \(v_0 = v_c(\rho_0)\) are the steady-state solutions of (13) and (14) and

\[\rho = \rho_0 + \xi(x, t), \quad v = v_0 + w(x, t)\]
are the perturbed solutions of (13) and (14) with \( \xi(x, t) \) and \( w(x, t) \) small perturbations to the steady-state solution. Substituting the perturbed solutions into (13) and (14), neglecting higher order terms, expanding and combining remaining terms, one has:

\[
\dot{\xi}_t + v_0 \dot{\xi}_x + \rho_0 w_x = 0 \tag{A1}
\]

\[
w_t + v_0 w_x = \frac{\dot{\xi}_x - w}{\tau} \tag{A2}
\]

Using (A1) to eliminate \( w \) from (A2), one obtains the following second order equation:

\[
\dot{\xi}_t + c_0 \dot{\xi}_x = -\tau(\dot{\xi}_{xx} + 2v_0 \dot{\xi}_{xt} + v_0^2 \ddot{\xi}_{tt}) \tag{A3}
\]

or

\[
\dot{\xi}_t + c_0 \dot{\xi}_x = -\tau(\partial_t + v_0 \partial_x)^2 \xi \tag{A4}
\]

where \( (\partial_t + v_0 \partial_x) \xi \equiv \dot{\xi}_t + v_0 \dot{\xi}_x \) is the wave operator and \( c_0 = (\rho_0 v_e(\rho_0))' \)

If we take the Fourier transform of both sides of (A4)

\[
\mathcal{F}(\xi(x, t)) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \xi(x, t)e^{i(\lambda x - kt)}dxdt \equiv \tilde{\xi}(\lambda, k)
\]

one gets

\[
\tau(k - v_0 \lambda)^2 \dot{\tilde{\xi}} + i(k - c_0 \lambda) \tilde{\xi} = 0 \tag{A5}
\]

The stability of (13) and (14) is determined by the characteristic roots of

\[
\tau(k - v_0 \lambda)^2 + i(k - c_0 \lambda) = 0 \tag{A6}
\]

If the imaginary parts of its characteristic roots are positive, this system is unstable, otherwise it is stable.

Let \( k = v_0 + iy \) be the roots of (A6), then one has

\[
y^2 + \beta y - i\beta(v_0 - c_0)\lambda = 0, \quad \beta = \frac{1}{\tau}
\]

whose solutions are

\[
y_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 + 4i(v_0 - c_0)\lambda \beta}}{2}
\]

After some algebraic manipulations, one finally arrives at
where \( \gamma = 4(u_0 - c_0)\lambda \). It is clear that unless \( \gamma \to 0 \), or the wave length \( \frac{1}{2} \to \infty \), there will be always one characteristic root \( k \) whose imaginary part is positive. Note that (13) and (14) is considered to be a model for highly density traffic, the wave lengths arising from such traffic are likely in the order of miles. One therefore can practically conclude that (13) and (14) always have an unstable mode. The cause of this instability is that drivers’ responses to stimuli are retarded.

References


