Continuum modeling of multiclass traffic flow

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Received 7 August 1998; received in revised form 31 March 1999; accepted 8 April 1999

Abstract

In contemporary macroscopic traffic flow modeling, a distinction between user-classes is rarely made. However, it is envisaged that both the accuracy and the explanatory ability of macroscopic traffic flow models can be improved significantly by distinguishing classes and their specific driving characteristics. In this article, we derive such a multiple user-class traffic flow model. Starting point for the derivation of the macroscopic flow model is the user-class specific phase–space density, which can be considered as a generalization of the traditional density.

The gas-kinetic equations describing the dynamics of the multiclass Phase–Space Density (MUC-PSD) are governed by various, interacting processes, such as acceleration towards a class-specific desired velocity, deceleration caused by vehicle interactions and the influence of lane changing. The gas-kinetic equations serve as the foundation of the proposed macroscopic traffic flow models, describing the dynamics of the class-dependent spatial density, velocity and velocity variance. The modeling approach yields explicit relations for both the velocity and the velocity variance. These equilibrium relations show competing processes: on the one hand, drivers accelerate towards their class-dependent desired velocity, while on the other hand, they need to decelerate due to interactions with vehicles from their own class and asymmetric interactions with vehicles from other classes. Using the operationalized model, macroscopic simulations provide insight into the model behavior for different scenarios. © 2000 Published by Elsevier Science Ltd. All rights reserved.

1. Introduction

Contemporary traffic control measures aim to pursue a more efficient use of existing infrastructure by recognizing the importance of distinguishing user-classes, such as trucks, buses, passenger-cars, and high-occupancy vehicles, and managing these classes differently. Examples of
such control instruments are uninterrupted passage for buses at ramp-metered on-ramps, dynamic truck overtaking prohibitions, and dynamic lane allocation control.

Due to the complexity of controlling heterogeneous traffic flow operations, caused by among others the interaction between user-classes and the interplay between the available traffic, a model-based approach is needed. That is, future research should focus on theory development regarding generic traffic flow theory for the synthesis and analysis of heterogeneous traffic flow. Also, emphasis should be on analyzing the problem of optimizing traffic control of heterogeneous flows. To this end, macroscopic multiple user-class traffic flow models are invaluable. This article presents a macroscopic traffic flow model that distinguishes user-classes having distinct traffic characteristics.

The macroscopic model described here is derived from mesoscopic principles. Mesoscopic models describe the behavior of small groups of vehicles of a specific user-class, classified by their position, velocity and desired velocity at an instant in time. These groups are aggregated with respect to the velocity and the desired velocity of the vehicles of a user-class:

\[
\int_{\text{velocity}} \int_{\text{desired velocity}} \text{mesoscopic} \rightarrow \text{macroscopic}
\]

resulting a macroscopic model. The proposed model generalizes the models proposed by Helbing (1996) by macroscopically describing traffic flow operations for multiple user-classes.

The article is organized as follows. We first present a very short overview of the state-of-the-art in multiclass continuum traffic flow modeling. Subsequently, we derived the multiclass gas-kinetic flow equations that are the mesoscopic foundation of the macroscopic model derived in the ensuing section. In this section we also discuss the expressions for the equilibrium velocity and velocity variance following from the modeling approach. We conclude the article with results from macroscopic simulation to show the plausibility of the multiclass model.

2. Traffic flow modeling approaches

In this article, we aim to develop a macroscopic traffic flow model that enables the synthesis and analysis of multiclass traffic flow. Examples of these user-classes are person-cars, recreational traffic, vehicles at various levels of driver-support, and articulate and non-articulate trucks. Such a model is necessary to improve modeling the effects of (class-specific) traffic control on the heterogeneous traffic stream.

Research on the subject of traffic flow modeling started some 50 years ago when Lighthill and Whitham (1955) presented an article concerning a macroscopic modeling approach based on an analogy between the dynamics of particles in a fluid and vehicles in a traffic stream. During the last five decades a variety of flow models has been developed aiming to improve on the original effort of Lighthill and Whitham (1955). These modeling approaches can be categorized according to a number of criteria, such as operationalization (simulation, analytical model), representation of the processes (deterministic, stochastic), scale (discrete, continuous), and level-of-detail (microscopic, mesoscopic, macroscopic). In this article, we will focus on continuum models. These are models characterized by continuous independent variables. The considered models are either
mesoscopic or macroscopic, respectively describing groups of vehicles or the collective vehicular stream. Furthermore, we only consider deterministic analytical models.

**Macroscopic modeling approaches and modeling issues:** Several macroscopic continuum models have been proposed in the literature. Basically, each of these can be generalized towards a multiclass traffic flow model. A well-known model is the so-called kinematic model or simple continuum model first presented by Lighthill and Whitham (1955). This model consists of the so-called conservation-of-vehicle equation, describing the dynamics of the density $r$ in relation to the flow $q$:

$$\frac{\partial r}{\partial t} + \frac{\partial q}{\partial x} = 0.$$  \hspace{1cm} (2)

The model is completed by a speed–density relation $V = V^c(r)$, and the relation $q = rV$.

This simple model has been successfully applied and captures the basic properties of the vehicular flow. However, several problems with this oversimplified description of traffic have been reported (cf. Lyrintzis et al., 1994; Kerner et al., 1996):

1. The kinematic model does not admit deviations from the speed–density $V(x, t) = V^c(r(x, t))$ relation. In other words, drivers react instantaneously to changing traffic conditions.

2. Discontinuous solutions result irrespective of the smoothness of the initial solution. This contradicts the observed real-life flow behavior.

3. The LWR-model is unable to capture the formation of localized structures, phantom-jams, hysteresis, and stop–start waves under specific conditions. This is unfortunate, since realistic modeling these phenomena is imperative for describing real-life traffic flow.

The **Payne-type models** \(^1\) remedy these flaws by adding to the conservation-of-vehicles equation a dynamic equation describing the dynamics of the velocity $V$. In its most general form, this velocity equation equals:

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = \frac{V^c(r) - V}{\tau} - \frac{1}{r} \frac{\partial P}{\partial x} + \frac{\eta}{r} \frac{\partial^2 V}{\partial x^2},$$  \hspace{1cm} (3)

where $P$ is the so-called traffic pressure and $\eta$ the traffic viscosity coefficient. In Lyrintzis et al. (1994) and Kerner et al. (1996) different model specifications (i.e. choices for $P$, $\eta$, and $V^c(r)$) are successfully used to simulate traffic flow operations. Indeed, introducing a dynamic velocity equation remedies the issues (1)–(3). However, in Daganzo (1995), the following points of interest not considered by Payne-type models are stated:

1. Vehicles are anisotropic particles that mainly react to downstream traffic conditions.

2. The interaction between the vehicle in the flow is asymmetric in that the slow vehicles remain virtually unaffected by faster vehicles.

3. Unlike particles in a fluid, vehicles have personality that remains unaffected by prevailing traffic conditions.

Finally, Helbing (1996) proposes a third dynamic equation for the velocity variance $\Theta$:

$$\frac{\partial \Theta}{\partial t} + V \frac{\partial \Theta}{\partial x} = 2 \frac{\Theta^c(r) - \Theta}{\tau} - 2 \frac{P}{r} \frac{\partial V}{\partial x} + \frac{\kappa}{r} \frac{\partial^2 \Theta}{\partial x^2}.$$  \hspace{1cm} (4)

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\(^1\) The frequently used term higher-order models is purposely omitted, since the Payne model only contains first-order derivatives.
where \( P = r\Theta, \Theta^r(r) \) the equilibrium velocity variance, and \( r \) denotes the kinematic coefficients. In addition to resolving the aforementioned issues, the model of Helbing (1996) also considers:

1. Finite spacing requirements of the vehicles, important due to the relative scale of processes of interest.

2. The quantitative effects of finite reaction and braking times.

The model presented in the article can be considered as a multiclass generalization of the model of Helbing (1996). Consequently, it inherits the desirable properties of the latter model, which will be illustrated in the sequel of this article.

**Multiclass approaches to mesoscopic and macroscopic flow modeling:** The number of mesoscopic or macroscopic models developed is small. In Daganzo (1994b) a generalized theory to model freeway traffic in the presence of two vehicle types and a set of lanes reserved for one of the vehicle classes is presented. It refers to the case of a long homogeneous freeway, described with the Cell-Transmission model (cf. Daganzo, 1994a). However, all vehicles have identical driving characteristics (acceleration times, free-flow velocities, etc.) and are only different in their lane-use behavior. In other words, the differences in among other acceleration performance and behavior, maneuverability, and maximum velocities are not described. The model can be considered as a generalization of the original LWR-model, thereby inheriting the shortcomings of the latter.

In Hoogendoorn and Bovy (1996) a *semi-discrete* (that is, continuous in time, and discrete in space) multiclass generalization of the Payne model (cf. Payne, 1971) is proposed. This model describes the asymmetric interaction between vehicles of the respective user-classes heuristically, and lacks theoretical foundation. Finally, let us note that Helbing (1997) presents a tentative multiclass model.

**Outline of the derivation of macroscopic MUC model:** In this article we present a macroscopic multiclass traffic flow model based on gas-kinetic principles. The derivation approach can be considered as a multiclass generalization of the approach followed in, among others, Prigogine and Herman (1971) and Helbing (1996). Let us briefly outline the approach used to establish the macroscopic multiple user-class traffic flow model.

First, we derive the *multiclass gas-kinetic equations* describing the dynamics of the multiclass Phase–Space Density. These equations enable modeling the process of drivers accelerating towards the class specific desired velocity and the process of drivers decelerating due to impeding vehicles of the same or other classes. Subsequently, the *method of moments* is applied to the MUC gas-kinetic equations, in order to establish partial differential equations describing the dynamics of the class specific density, velocity and velocity variance. From this modeling approach, expressions for the *equilibrium velocity* and *equilibrium velocity variance* result. These convey both *acceleration processes* towards the class dependent desired velocity as well as *deceleration processes* due the within and between user-class interactions.

**3. Gas-kinetic modeling of multiclass traffic**

This section introduces the so-called gas-kinetic equations describing multiple user-class traffic operations. These equations are a multiclass generalization of the single-class gas-kinetic equation
presented in Paveri-Fontana (1975). We will show that these multiclass gas-kinetic equations are given by the following set of partial differential equations:

\[
\frac{\partial \rho_u}{\partial t} + v \frac{\partial \rho_u}{\partial x} + \frac{\partial}{\partial v} \left( \rho_u \frac{\omega_u(v,v^0) - v}{\tau_u} \right) = \frac{\partial \rho_u}{\partial t} \bigg|_{\text{int}} + \frac{\partial \rho_u}{\partial t} \bigg|_{\text{int}}
\]

for all \( u \in \mathcal{U} \), where \( \mathcal{U} \) denotes the set of user-classes \( u \). Eq. (5) expresses the dynamic changes in the so-called multiclass Phase–Space Density (MUC-PSD) \( \rho_u \) due to:

1. **Convection** because of the inflow and outflow of vehicles (term \( C \) of (5)).
2. Acceleration towards the acceleration velocity \( \omega_u(v,v^0) \) (term \( A \) of (5)). The acceleration velocity reflects the expected velocity to which a vehicle of class \( u \) with velocity \( v \) and desired velocity \( v^0 \) accelerates.
3. **Interactions without immediate overtaking opportunity**, expressed by the multiclass collision equations (term \( I \) of (5)).

In the remainder of this section we will elaborate upon the MUC-PSD, and present the derivation approach of the gas-kinetic equations (5).

**Multiclass Phase–Space Density:** The gas-kinetic equations derived in this section describe dynamic changes in the MUC-PSD. This notion is defined as follows. The joint probability density function \( \phi_u(v,v^0|x,t) \) describes the joint probabilities of the velocity \( L_u \) and desired velocity \( L_u^0 \). Using \( \phi_u \), we define the MUC-PSD \( \rho_u(x,v,v^0,t) \) by the expected number of vehicles on \( (x,t) \) having a velocity \( v \) and a desired velocity \( v^0 \) per unit roadway length:

\[
\rho_u(x,v,v^0,t) \triangleq \phi_u(v,v^0|x,t)r_u(x,t),
\]

where \( r_u(x,t) \) equals the expected number of vehicles of class \( u \) per unit roadway length. Note that for the density \( r(x,t) \), being the expected number of vehicles per unit roadway length, it holds that \( r(x,t) = \sum_u r_u(x,t) \).

**Derivation of the multiclass gas-kinetic equations:** Several very complex and interacting processes cause the dynamic behavior of the traffic stream: drivers accelerate towards their desired velocity, vehicles interact, followed by either an immediate lane change or by a deceleration, or drivers adapt their desired velocity to prevailing road, ambient or weather conditions. In this section, we present dynamic equations describing temporal changes in the traffic conditions caused by these processes, resulting in the multiclass gas-kinetic equations.

These mesoscopic traffic dynamics are governed by what will be referred to as **continuum** and **non-continuum** processes. Continuum processes yield **smooth changes** in the MUC-PSD due to inflow and outflow in the phase–space \( \mathcal{Y} \) consisting of vectors \( y = (x,v,v^0) \). Non-continuum processes reflect **non-smooth** changes in the MUC-PSD, caused by among others vehicular interactions followed by either an **immediate lane change** or **deceleration** of the impeded vehicle.

**Qualitative discussion of changes in the MUC-PSD:** Let us first qualitatively discuss the different processes causing changes in the MUC-PSD. To this end, Fig. 1 shows the traffic processes relevant to multiclass traffic operations in cell \( x \), defined by the infinitesimal region
In the multiclass gas-kinetic derivation, we will distinguish convection, acceleration, and deceleration processes. Let us elaborate upon these processes.

1. **Convection:** The MUC-PSD \( \rho_u(x, v, v^0, t) \) changes due to the inflow of vehicles from cell \( x - dx \), and the outflow of vehicles to cell \( x + dx \) (A and B).

2. **Acceleration:** Let \( \omega_u(x, v, v^0, t) \) denote the acceleration velocity of vehicles \( \rho_u(x, v, v^0, t) \). Vehicles from class \( u \) in cell \( x \) driving at velocity \( v \) accelerate in order to attain their acceleration velocity \( \omega_u \) resulting in an increase \( \rho_u(x, w, v^0, t) \) at the expense of \( \rho_u(x, v, v^0, t) \) (see (C)). Vehicles accelerating from \( w < v \) to \( v \) cause \( \rho_u(x, v, v^0, t) \) to increase (D).

3. **Interaction:** When a vehicle driving with velocity \( v \) catches up with a slower vehicle, it either needs to reduce its velocity, or perform an immediate lane change. In the first case, the number of vehicles in \( \rho_u(x, v, v^0, t) \) decreases (E); \( \rho_u(x, v, v^0, t) \) increases when fast vehicles are impeded by vehicles driving with velocity \( v \) (F).

**Derivation of the gas-kinetic equations:** Before presenting the multiclass gas-kinetic equations, let us briefly discuss the assumptions underlying the gas-kinetic flow equations:

1. The desired velocity is an observable concept that is meaningful to drivers.
2. The lengths of the vehicles are neglected.
3. No user-class transitions are allowed.
4. Free-flowing vehicles accelerate towards their desired velocity; constrained vehicles accelerate to the expected velocity of the platoon-leaders.
5. Impeded vehicles not able to immediately overtake *instantaneously* adapt to the velocity of the impeding vehicle; immediately overtaking vehicles remain at the same velocity.
6. Impeding vehicles are unaffected by the impeded vehicle.

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\(^2\) The infinitesimal vehicles assumption is remedied in the macroscopic MUC model.
Continuum processes (Convection, acceleration and desired velocity adaptation): By balancing the flows through the boundaries of a differential volume \( \partial x, \partial v, \partial v_0, \partial t \), we can establish the following multiclass generalization of gas-kinetic model of Paveri-Fontana (1975) (for details, we refer to Hoogendoorn and Bovy, 1998a).

\[
\frac{\partial \rho_u}{\partial t} + \nabla \left( \rho_u \frac{\partial y}{\partial t} \right) = \left( \frac{\partial \rho_u}{\partial t} \right)_{NC},
\]

where \( y \in Y \), the \( \nabla \)-operator is defined by:

\[
\nabla \triangleq \left( \frac{\partial}{\partial x} \frac{\partial}{\partial v} \frac{\partial}{\partial v_0} \right),
\]

where \((\partial \rho_u/\partial t)_{NC}\) reflects the changes caused by non-convective processes. Let us first specify the respective terms in \((\partial y/\partial t)\).

Clearly, the total derivative \( \partial x/\partial t \) equals the velocity \( v \):

\[
\frac{\partial x}{\partial t} = v.
\]

The total derivative \( \partial v/\partial t \) reflects the acceleration process. Let us assume that the acceleration processes can be described realistically by an exponential acceleration law:

\[
\frac{\partial v}{\partial t} = \frac{\omega_u(x, v, v^0, t) - v}{\tau_u(x, v, v^0, t)},
\]

where \( \omega_u(x, v, v^0, t) \) and \( \tau_u(x, v, v^0, t) \) respectively denote the class-specific acceleration velocity and the acceleration time. In contrast to the original mesoscopic single-class traffic flow model proposed by Prigogine and Herman (1971) this term describes individual relaxation towards the desired velocity, instead of the collective relaxation towards a traffic composition dependent equilibrium velocity. This is a very important model feature. We argue that it is more realistic to assume that drivers aim to traverse the roadway at their individual desired velocity and are possibly restricted as a result of interactions with slower vehicles, than to assume that drivers relax towards a velocity which they chose based on the prevailing average traffic conditions.

Different specifications of \( \omega_u \) and \( \tau_u \) have been proposed to model the acceleration process in the single-class case. These describe either of the two extreme cases:

1. All vehicles in the traffic stream are able to accelerate towards their individual desired velocity \( v^0 \), given their class-specific acceleration time \( \tau^0_u \) (equivalent to single-class model of Paveri-Fontana, 1975).

2. Only free-flowing vehicles accelerate towards their desired velocity \( v^0 \) given their acceleration time \( \tau^u_0 \). Constrained (platooning) vehicles do not accelerate at all (see Helbing, 1997).

Case 1 yields the following specification of the acceleration law:

\[
\omega_u = v^0 \quad \text{and} \quad \tau_u = \tau^0_u \quad \Rightarrow \quad \frac{\partial v}{\partial t} = \frac{v^0 - v}{\tau^0_u},
\]

while case 2 yields:

\[
\omega_u = \theta_u v + (1 - \theta_u) v^0 \quad \text{and} \quad \tau_u = \tau^0_u \quad \Rightarrow \quad \frac{\partial v}{\partial t} = \frac{v^0 - v}{\tau^0_u/(1 - \theta_u)},
\]

where \( \theta_u = \theta_u(x, v, v^0, t) \) denotes the expected fraction of platooning vehicles of class \( u \). Neither of these extreme cases describes traffic acceleration realistically (Hoogendoorn, 1998). Rather, we
assume that free-flowing drivers accelerate towards their desired velocity \( v_0 \), given their acceleration time \( \tau_0 \), while platooning drivers accelerate towards the average acceleration velocity of the platoon-leaders, given the acceleration times of the latter vehicles. For free-flowing vehicles (indicated by the index \( f \)) we then have:

\[
\omega^f_u = v_0 \quad \text{and} \quad \tau^f_u = \tau_0.
\]

Hoogendoorn (1998) shows that the aggregate-class expected acceleration time \( \tau^f(v) \) and the aggregate-class desired velocity \( \omega^f(v) \) of free-flowing vehicles driving with velocity \( v \) satisfies:

\[
1/\tau^f(v) = \sum_{s \in \mathbb{U}} \frac{\int \rho^f_s(v, w_0^0)/\tau_0^0 \, dw_0^0}{\int \rho^f_s(v, w_0^0) \, dw_0^0},
\]

and

\[
\omega^f(v) = \sum_{s \in \mathbb{U}} \frac{\int \rho^f_s(v, w_0^0)(w_0^0/\tau_0^0) \, dw_0^0}{\int \rho^f_s(v, w_0^0)(1/\tau_0^0) \, dw_0^0},
\]

respectively, where the free-flowing MUC-PSD \( \rho^f_s(1 - \theta_u)\rho_u \). We assume that all vehicles in a platoon have the same velocity \( v \), and the acceleration velocity \( \omega^f \) and time \( \tau^f \) of vehicles in a platoon is equal to the acceleration velocity and time of the free-flowing platoon-leader. Furthermore, we assume that the leader of a platooning vehicle driving with velocity \( v \) is drawn uniformly from the population of free-flowing vehicles driving with velocity \( v \). Then, using \( \omega^f \) and \( \tau^f \) we find for the constrained vehicles \( c \) of class \( u \):

\[
\omega^c_u = \omega^f \quad \text{and} \quad \tau^c_u = \tau^f.
\]

The resulting acceleration law becomes:

\[
\frac{dv}{dt} = \frac{\omega_u - v}{\tau_u} \triangleq (1 - \theta_u) \frac{v_0 - v}{\tau_0_u} + \theta_u \frac{\omega^f - v}{\tau^f}.
\]

Finally, we assume that drivers do not change their desired velocity during their trip. That is, the derivative of desired velocity with respect to time satisfies:

\[
\frac{dv_0}{dt} = 0.
\]

By substitution of (9), (17), and (18) into (7), we find:

\[
\frac{\partial \rho_u}{\partial t} + v \frac{\partial \rho_u}{\partial x} + \frac{\partial}{\partial v} \left( \rho_u \frac{\omega_u(v, v_0^0) - v}{\tau_u} \right) = \left( \frac{\partial \rho_u}{\partial t} \right)_\text{NC} \quad \text{for all} \quad u \in \mathbb{U}.
\]

Non-continuum processes (vehicular interaction): Let us now consider the right-hand side of (19). We can distinguish two types of non-continuum processes, namely adaptation of the desired velocity distribution to a reasonable desired velocity distribution, and the deceleration caused by vehicle interactions. With respect to the former, the assumption that the desired velocity distribution is adapted quickly to the reasonable desired speed justifies neglecting the adaptation process. Let us thus elaborate only upon the deceleration process.

An interaction event is defined by the event that a fast vehicle catches up with a slower vehicle. When an interaction occurs, the fast vehicle needs to perform a remedial manoeuvre to prevent a collision. Consequently, we will refer to the impeded vehicle as the active party, while the impeding
vehicle is referred to as the passive party. The used terminology stems from the concepts of active and passive overtaking (cf. Leutzbach, 1988).

Let us now consider interacting vehicles of \( \rho_u(x, v, v^0, t) \). Interactions can either yield an increase or a decrease in \( \rho_u(x, v, v^0, t) \), respectively denoted by \( (\hat{\partial}\rho_u/\hat{\partial}t)_{\text{int}}^- \) and \( (\hat{\partial}\rho_u/\hat{\partial}t)_{\text{int}}^+ \). That is:

\[
\left( \frac{\hat{\partial}\rho_u}{\hat{\partial}t} \right)_{\text{NC}} = \left( \frac{\hat{\partial}\rho_u}{\hat{\partial}t} \right)_{\text{int}}^- + \left( \frac{\hat{\partial}\rho_u}{\hat{\partial}t} \right)_{\text{int}}^+.
\]

Let \( \rho_{u,s}(x, v, w, t) \) denote the joint phase–space density of classes \( u \) and \( s \), with \( v = (v, v^0) \) and \( w = (w, w^0) \) respectively denoting the velocity and desired velocity of vehicles 1 and 2. In other words, \( \rho_{u,s} \) denotes the expected number of vehicle pairs of a vehicle of class \( u \) having a velocity \( v \) and a desired velocity \( v^0 \) and a vehicle of class \( s \) having a velocity \( w \) and a desired velocity \( w^0 \) per unit roadway length. Let \( p_u = p_u(x, v, w, t) \) denote the probability that a vehicle of class \( u \) can immediately overtake the slower vehicle. Then, using the results established in Leutzbach (1988), we can show that the decrease-rate \( (\hat{\partial}\rho_u/\hat{\partial}t)_{\text{int}}^- \) due to vehicles in \( \rho_u(x, v, v^0, t) \) interacting with slower vehicles equals:

\[
(\hat{\partial}\rho_u/\hat{\partial}t)_{\text{int}}^- = \sum_{s \neq u} \int_{v' = 0}^v \int \left( 1 - p_u(v, v') \right) \rho_{u,s}(v, v^0, v', w^0) \, dw^0 \, dv'.
\]

Increases in \( \rho_u(x, v, v^0, t) \) due to vehicles in \( \rho_s(x, v, v^0, t) \) with \( w > v \) interaction with slower vehicles in \( \rho_u(x, v, v^0, t) \) are equal to:

\[
(\hat{\partial}\rho_u/\hat{\partial}t)_{\text{int}}^+ = \sum_{s \neq u} \int_{v' = v}^\infty \int \left( 1 - p_u(v', v) \right) \rho_{u,s}(v', v^0, v, w^0) \, dw^0 \, dv',
\]

where we have assumed that the slower vehicles are unaffected by the interaction.

The class–specific immediate overtaking probability \( p_u = p_u(x, v, w, t) \) depends on a number of factors, such as the traffic conditions on the destination lanes, the maneuverability and length of the vehicle, the difference between the current velocity \( v \) and the velocity \( w \) of the impeding vehicle, the composition of traffic (e.g. percentage of heavy-vehicles), etc.

The number of interactions between vehicles having speed \( v \) and \( w \) respectively are equal to the ‘interaction rate’ equal to the relative speed \( w - v \) and the mutual phase density. Paveri-Fontana (1975) established that assuming vehicular chaos justified the factorization approximation \( \rho_{u,s} = \rho_u \rho_s \), reflecting probabilistic independence of the vehicles. Expressions (21) and (22) constitute a multiclass generalization of the so-called collision equations (see Leutzbach, 1988).

Hoogendoorn (1998) shows that the assumption of vehicular chaos is flawed, except for free-flow conditions. That is, for increasingly dense traffic, the assertion that velocities are independent is not realistic. To remedy this, the author proposes a traffic flow theory based on description of the traffic stream as a collection of vehicle platoons represented by their platoon leaders (the free-flowing vehicles) to describe the increase in velocity-correlation for increasingly dense traffic (see Hoogendoorn, 1999). Assuming that only the unconstrained platoon leaders (rather than all

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3 For the sake of notational simplicity, only the dependency of the immediate overtaking probability \( p_u \) on \( v \) and \( w \) is shown.
vehicles) move independently infers substituting $q^f_s$ for $q_s$ into expressions (21) and (22) (see Hoogendoorn, 1998), where $q^f_s$ equals the free-flowing vehicles in $q_s$. Substitution of the resulting equations (21) and (22) into (19) establishes the multiclass gas-kinetic equations. Finally, let us remark that in the remainder of this article, we assume that the resulting expressions for (21) and (22) can be approximated by:

$$\left( \frac{\partial q_u}{\partial t} \right)_{\text{int}}^- = \sum_{s \in \mathbb{V}} (1 - \pi_{u,s}) \int_{v'} \int_{w' = 0} (w' - v) \rho_u(v, v^0) \rho^f_s(w', w^0) \, dv^0 \, dw,$$

(23)

$$\left( \frac{\partial q_u}{\partial t} \right)_{\text{int}}^+ = \sum_{s \in \mathbb{V}} (1 - \pi_{u,s}) \int_{v' = v}^{\infty} \int_{v'' = v} (v' - v) \rho_u(v', v^0) \rho^f_s(v, w^0) \, dv^0 \, dv',$$

(24)

where we have used $q_{u,s} = \rho_u q^f_s$, and where $\pi_{u,s}$ denotes the $(v, v^0)$-independent expected immediate overtaking probability of vehicles of class $u$ overtaking a vehicle of class $s$.

4. Macroscopic multiclass traffic flow model

We establish in this section the macroscopic multiclass traffic flow equations on the basis of the gas-kinetic equations presented in the Section 3. The derivation approach is similar to the approaches followed by among others, Prigogine and Herman (1971), Leutzbach (1988) and Helbing (1996). To establish the dynamics of the class-specific density, velocity and velocity variance the method of moments is applied, resulting in a non-closed set of equations of the moments of the velocity distribution. Subsequently, additional modifications to the model are proposed resulting in the following closed-form approximate system of partial differential equations, describing the dynamics of the class-specific density $r_u$, expected velocity $V_u$ and velocity variance $\Theta_u$:

$$\frac{\partial r_u}{\partial t} + \frac{\partial (r_u V_u)}{\partial x} = 0,$$

(25)

$$\frac{\partial V_u}{\partial t} + V_u \frac{\partial V_u}{\partial x} = \frac{V^c_u - V_u}{\tau_u} - \frac{1}{r_u} \frac{\partial}{\partial x} \left( r_u \Theta_u - \eta_u \frac{\partial V}{\partial x} \right),$$

(26)

$$\frac{\partial \Theta_u}{\partial t} + V_u \frac{\partial \Theta_u}{\partial x} = \frac{\Theta^c_u - \Theta_u}{\tau_u} - 2 \Theta_u \frac{\partial V_u}{\partial x} + \frac{1}{r_u} \frac{\partial}{\partial x} \left( \kappa_u \frac{\partial \Theta_u}{\partial x} \right),$$

(27)

where the equilibrium velocity $V^c_u$ and the equilibrium variance $\Theta^c_u$ respectively equal:

$$V^c_u \triangleq V^0_u - (1 - \pi_u) \tau_u \sum_{s \in \mathbb{V}} \mathcal{R}^{(1)}_{u,s},$$

(28)

and

$$\Theta^c_u \triangleq C_u - (1 - \pi_u) \frac{\tau_u}{2} \sum_{s \in \mathbb{V}} \left( \mathcal{R}^{(2)}_{u,s} - 2 V_u \mathcal{R}^{(1)}_{u,s} \right),$$

(29)

where $\mathcal{R}^{(k)}_{u,s}$ reflects the influence of interacting vehicles of classes $u$ and $s$ which follows from its definition (Eq. (38)) and the dynamics of the $k$th velocity moment (Eq. (39)).
In the remainder, we will derive these partial differential equations. However, we will first derive the main dependent variables (e.g. density $r_u$, velocity $V_u$, velocity variance $H_u$, desired velocity $V_0$, and velocity covariance $C_u$) from the MUC-PSD.

**Velocity moments:** In the following the method of moments is applied to density functions describing the distribution of the velocities, with the aim to establish equations describing the dynamics of the velocity moments. We show that the class-specific densities, velocities and velocity variances are completely determined by these velocity moments. Consequently, establishing the dynamics of the velocity moments also reveals the dynamics of the densities, velocities and velocity variances.

Let us first define the so-called mean-operator $\langle \cdot \rangle_u$ of class $u$ applied to any function $a$ of $(v, v^0)$:

$$
\langle a(v, v^0) \rangle_u \triangleq \int_v \int_v a(v, v^0) \phi_u(v, v^0 | x, t) \, dv \, dv^0,
$$

where $\phi_u$ denotes the joint probability density function of the velocity and the desired velocity. Then, the joint $k$-th velocity and $l$-th desired velocity moment $M^{(k,l)}_u$ is defined by:

$$
M^{(k,l)}_u \triangleq \langle v^k \cdot (v^0)^l \rangle_u.
$$

We will first consider the velocity moments for $k = 0, 1, 2$ and 3 for $l = 0$. Let $M^{(k)}_u \triangleq M^{(k,0)}_u$. For $k = 0$ we have the identity relation $M^{(0)}_u = 1$. For $k = 1$ we find that the first velocity moment equals the expected velocity of class $u$, i.e. $M^{(1)}_u = V_u$. For $k = 2$, we find:

$$
M^{(2)}_u = V^2_u + \Theta_u.
$$

That is, the velocity variance $\Theta_u = \Theta_u(x, t)$ of user-class $u$ can be determined from the second velocity moment and the mean velocity of user-class $u$. Finally, the third velocity moment equals:

$$
M^{(3)}_u = V^3_u + 3V_u \Theta_u + \Gamma_u,
$$

where $\Gamma_u = \Gamma_u(x, t)$ denotes the skewness of the velocity distribution.

Let us now consider some specific moments for $l = 1$. Considering $k = 1$, the zeroth-covariance moment equals the expected desired velocity $V^0_u$ of class $u$:

$$
M^{(0,1)}_u \triangleq \langle v^0 \rangle_u = V^0_u.
$$

The covariance $C_u = C_u(x, t)$ between the actual and the desired velocity is determined from the first covariance moment $k = l = 1$, the expected actual velocity and the expected desired velocity. That is:

$$
M^{(1,1)}_u \triangleq \langle v \cdot v^0 \rangle_u = V^0_u V_u + C_u.
$$

Finally, let us define the interaction moments $J^{(k)}_{u,s}$ and $X^{(k)}_{u,s}$ respectively by:

$$
 r_u J^{(k)}_{u,s} \triangleq (1 - \theta_s) r_u \left( v^k \int_{w < v} (v - w) \bar{p}_s(w) \, dw \right)_u,
$$

and

$$
 r_u X^{(k)}_{u,s} \triangleq (1 - \theta_s) r_s \left( v^k \int_{w > v} (v - w) \bar{p}_u(w) \, dw \right)_s
$$
where the reduced MUC-PSD $\tilde{p}_x$ is defined by $\tilde{p}_x(x, v, t) \triangleq \int_0^t p_x(x, v, t^0, t) dt^0$. Finally, let us define:

$$\mathcal{G}^{(k)}_{u,s} \triangleq \mathcal{G}^{(k)}_{u,s} + \mathcal{A}^{(k)}_{u,s},$$  \tag{38}$$

The multiclass velocity moments dynamics: In applying the method of moments, the special MUC continuity equations are multiplied with $v^k$, and subsequently integrated with respect to $v$ and $v^0$. By doing so, the following equations describing the dynamics of the velocity moments $M_u^{(k)} = M_u^{(k)}(x, t)$ are established (for details, we refer to Hoogendoorn and Bovy, 1998a):

$$\frac{\partial (r_u M_u^{(k)})}{\partial t} + \frac{\partial (r_u M_u^{(k+1)})}{\partial x} = r_u \frac{M_u^{(k+1)} - M_u^{(k)}}{\tau_u} - \frac{(1 - \pi_u)}{s \in \mathcal{U}} \sum_{s \in \mathcal{U}} \mathcal{G}^{(k)}_{u,s},$$  \tag{39}$$

where $\pi_u$ is the expected immediate overtaking probability, and $\tau_u = \tau_u(x, t)$ denotes the expected acceleration time. The dynamic equation shows how the $k$th velocity moment $M_u^{(k)}$ changes due to convection (term $C$), acceleration (term $A$), and deceleration (term $D$). The influence of deceleration is the result of a term $\sum_{s \in \mathcal{U}} \mathcal{G}^{(k)}_{u,s}$ reflecting changes over time in $M_u^{(k)}$ due to vehicles of class $u$ interacting with vehicles of class $s$ on the one hand, and the probability on immediate overtaking $(1 - \pi_u)$ of vehicles of class $u$ on the other hand.

Derivation of the multiclass macroscopic equations: In this section we establish the macroscopic MUC equations for the density, the velocity and the velocity variance. To this end, Eq. (39) is assessed for $k = 0, 1, 2$ respectively, yielding the conservation of vehicle equation, the velocity dynamics and the velocity variance dynamics.

The conservation of vehicles equation: For $k = 0$, and using $\mathcal{G}^{(0)}_{u,s} + \mathcal{A}^{(0)}_{u,s} = 0$ we find Eq. (25) as the expected result. This expression can be considered as the multiclass generalization of the traditional conservation-of-vehicle equation. It states that the density changes according to the balance between the inflow and outflow of vehicles of user-class $u$.

The velocity dynamics equation: Next, Eq. (39) is assessed for $k = 1$. By using of the conservation-of-vehicle equation (25) in the expansion of the left-hand side of (39) for $k = 1$ we find:

$$r_u \frac{\partial V_u}{\partial t} + r_u V_u \frac{\partial V_u}{\partial x} + \frac{\partial P_u}{\partial x} = r_u \frac{V_u^0 - V_u}{\tau_u} - (1 - \pi_u) \sum_{s \in \mathcal{U}} \mathcal{G}^{(1)}_{u,s},$$  \tag{40}$$

where the so-called traffic pressure $P_u$ for class $u$ is defined by $P_u = r_u \Theta_u$. The following dynamic equation thus holds for the velocity $V_u$:

$$\frac{dV_u}{dt} = \frac{\partial V_u}{\partial t} + V_u \frac{\partial V_u}{\partial x} = \frac{V_u^0 - V_u}{\tau_u} - \frac{\partial P_u}{r_u \partial x},$$  \tag{41}$$

where the equilibrium velocity $V_u^e$ is defined by:

$$V_u^e \triangleq V_u^0 - (1 - \pi_u) \sum_{s \in \mathcal{U}} \mathcal{G}^{(1)}_{u,s},$$  \tag{42}$$

where $V_u^0$ denotes the average desired velocity of user-class $u$. 


Eq. (41) shows the changes in the traffic conditions experienced by an observer moving along with the vehicles of class \( u \). These changes are caused by a term considered as an *anticipation* term \( A \) on the one hand, and a relaxation term \( R \) on the other hand. Term \( A \) shows how the total time derivative of the velocity \( V_u \) changes due to spatial changes in the traffic pressure \( P \).

Eq. (42) specifies the *equilibrium velocity* in accordance with the gas-kinetic multiclass flow equations (5). It can be interpreted as a theoretical result concerning the dependence of the equilibrium velocity on the microscopic processes of traffic flow. According to (42), the equilibrium velocity is given by the expected class-dependent desired velocity \( V_0^u \) reduced by the term arising from forced instantaneous deceleration due to within-class and between-class interactions.

The *velocity variance equation*: Derivation of the equations describing the dynamics of the velocity variance \( H_u \) involves the evaluation of Eq. (39) for \( k = 2 \). Similar to the dynamics of the class-specific expected velocity \( V_u \), we can determine the following dynamic equations:

\[
\begin{align*}
\frac{d\Theta_u}{dt} & = \frac{\partial \Theta_u}{\partial t} + V_u \frac{\partial \Theta_u}{\partial x} = 2 \frac{\Theta_u^0 - \Theta_u}{\tau_u} - 2 \Theta_u \frac{\partial V_u}{\partial x} - \frac{1}{r_u} \frac{\partial J_u}{\partial x}, \\
\text{where the flux of velocity variance } J_u \text{ is defined by } J_u &= r_u \Gamma_u = \langle (v - V_u)^3 \rangle_u, \text{ and the equilibrium velocity variance is defined by:} \\
\Theta_u^e & = C_u - (1 - \eta_u) \frac{\tau_u}{2} \sum_{s \in \#} \left( \mathcal{R}_{u,s}^{(2)} - 2V_u \mathcal{R}_{u,s}^{(1)} \right). \\
\end{align*}
\]

Expression (44) reflects the velocity variance due to relaxation of the velocity variance towards the covariance \( C_u = \langle vv^0 \rangle_u - V_u^0 V_u \) of class \( u \) on the one hand, and the changes in the variance due to interactions on the other hand.

*Drivers anticipation and traffic viscosity*: In fluidic or gas-flows, viscosity expresses the influence of friction between the particles in the fluid or gas. The effect of the internal friction is expressed by a second order term with respect to the velocity present in the momentum dynamics. However, several authors (e.g. Payne, 1971, 1979; Kerner and Konhäuser, 1995; Kerner et al., 1996; Helbing, 1996) have argued that traffic viscosity cannot be plausibly interpreted from gas-kinetic theory, since due to the spatial one-dimensionality of the traffic flow equations, the viscosity term cannot be shear viscosity (originating from friction between e.g. the boundaries of the flow-region).

However, viscosity can be interpreted from the viewpoint of transitions in the driver’s attitude towards prevailing traffic conditions: it describes the higher-order anticipation-behavior of drivers. In other words, if drivers travel in a region of spatial acceleration \( \partial^2 V / \partial x^2 > 0 \) they will anticipate, and accelerate (i.e. \( dV_u / dt > 0 \)) (cf. Kerner et al., 1996). In, among others, Payne (1971) and Kerner et al. (1996) the spatial derivative of the traffic pressure (in the multiclass case \( \partial P_u / \partial x \)) in Eq. (41) has been related to driver’s anticipation. The authors argue that when the pressure increases spatially, driver’s anticipate on these worsening downstream conditions and *decelerate* (for the multiclass case, i.e. \( dP_u / dt < 0 \)). This justifies modeling this anticipation-behavior by adding a diffusive term to the traffic pressure, implying \( P_u := P_u - \eta_u \partial V / \partial x \), yielding Eq. (26). In
this expression, \( V = \sum s(r_s V_s) / \sum s(r_s) \) is the expected aggregate-class velocity, where \( \eta_u \) is the bulk-viscosity coefficient.

**Critique on the low-order driver’s anticipation:** We can easily show that the interpretation of term \( A \) in (41) for the perspective of driver’s anticipation (see among others Payne, 1971; Payne, 1979; Kerner and Konhäuser, 1995; Kerner et al., 1996; Helbing, 1996) is not justified, since term \( A \) shows that drivers only anticipate on changes in the pressure of their own class \( u \). For instance, if the pressure \( P_u \) of class \( u \) decreases spatially (\( \partial P_u / \partial x < 0 \)), only drivers of this class \( u \) will accelerate, even if the aggregate-class pressure \( P = \sum P_s \) does not (\( \partial P / \partial x = 0 \)).

For a macroscopic multiclass (multilane) traffic flow this implies that the interpretation of the term \( \partial P_u / \partial x \) from the perspective of drivers’ anticipations is incorrect. Rather, the term stems from the convection of groups of vehicles with different variations in their velocities (see Hoogendoorn, 1999). However, if we extend this conception, the classical interpretation of the \( \partial P_u / \partial x \) is equally incorrect for traditional Payne-type macroscopic flow models.

Thus, describing the traffic process as a mixed stream of different user-classes rather than considering each class separately leads to the misinterpretation of the pressure term. In terms of gaining correct insights into the mechanisms of traffic flow operations, a multiclass distinction is hence invaluable.

**Closure of the model equations:** The system of partial differential equations (25), (41) and (43) is not closed, in the sense that the dynamics of the velocity variance contains three unspecified moments, namely the expected desired velocity \( V_0^u \), the covariance \( C_u \), and the flux of velocity variance \( J_u \). For the single-class case, Helbing (1997) proposes dynamic equations for the aggregate-class velocity–covariance equations. However, for the multiclass case, we assume that the dynamic variations in the expected desired velocity \( V_0^u \) and the covariance \( C_u \) can be neglected, and that both can be described by functions of the densities \( r \) of the respective classes \( r = \{ r_u \} \):

\[
V_0^u = V_0^u(r) \quad \text{and} \quad C_u = C_u(r). \tag{45}
\]

Due to the increasing fraction of platooning vehicles, these functions are monotonically decreasing with respect to the class-specific densities (e.g. \( \partial V_0^u / \partial r_s \leq 0 \)). In Hoogendoorn (1999) the specification of both terms is discussed in more detail.

If we assume that even in dynamic situations the velocity distributions are Gaussian distributed random variates with mean \( \tilde{V}_u(x, t) \) and variance \( \Theta_u(x, t) \), then by definition, the flux of velocity variance equals \( J_u(x, t) = r_u(\langle v - V_u \rangle^2 + ) = 0 \). In Helbing (1996) it is shown that since drivers need a finite amount of time to adapt their velocities caused by their finite reaction and braking times (reflected by the adaptation of the velocity distribution to the local equilibrium distribution). This causes a non-zero skewness of the velocity distribution. As a consequence, additional transport terms result, that can be expressed by the term:

\[
J_u = -\kappa_u \frac{\partial \Theta_u}{\partial x}. \tag{46}
\]

Using this expression, the velocity variance dynamics equation (43) yields Eq. (27). See Helbing (1996) for details.

Let us remark that the resulting multiclass model is similar to the tentative multiclass model of Helbing (1997). The main differences between the models stem from the improved multiclass acceleration term, and the vehicular interaction incorporating the correlation between the pla-
tooning vehicles and their free-flowing platoon-leaders of the multiclass model presented in the paper.

**Finite space requirements:** The analogy between a fluidic flow and a vehicular flow is based on the presumption that vehicles can be adequately modeled as infinitesimal particles. However, this assumption is not realistic and causes both theoretical as well as practical problems. To resolve such problems, we propose incorporation of the space needed by each of the vehicle due to both its user class dependent physical length and the additional velocity dependent safety margin. To this end, let \( l_u(v) \) be the average roadway space needed by a vehicle of user-class \( u \) to safely traverse the roadway at speed \( v \). This amount of roadway space can be determined by considering the car-following behavior of drivers. For recent empirical findings on car-following behavior, we refer to Dijker et al. (1998). We will use the model of Jepsen (1998) to describe the space requirements \( l_u(v) \) of a vehicle of class \( u \) driving with velocity \( v \):

\[
l_u(v) = L_u + d_u + vT_u + v^2F_u,
\]

where \( L_u \) denotes the length of the vehicle, \( d_u \) the minimal gap between two vehicles, \( T_u \) the reaction time of drivers, and finally \( F_u \) the speed-risk factor following from drivers aiming to minimize the potential damage or injuries. If we assume that these parameters are equal for all vehicles in a class, then we can define the dimensionless space-requirement correction factor \( \delta = \delta(x,t) \in [1, \infty) \) by (see Hoogendoorn, 1999):

\[
\delta \triangleq \frac{1}{1 - \sum_{s \in y} r_s(L_s + d_s + V_sT_s + (V_s^2 + \Theta_s)F_s)}.
\]

Using this correction factor, we introduce the modified traffic flow variables. For instance, we can defined the modified density \( \hat{\rho}_u \) of class \( u \) by:

\[
\hat{\rho}_u \triangleq \delta \cdot \rho_u.
\]

The modified density equals the expected number of vehicles of class \( u \) per unit non-occupied roadway space. If we assume that the changes in the traffic conditions experienced by an observer moving along with vehicles of class \( u \) change very slowly (the so-called adiabatic elimination; cf. Haken, 1983), we find the modified multiclass conservation-of-vehicle equation:

\[
\frac{\partial \hat{\rho}_u}{\partial t} + \frac{\partial (\hat{\rho}_u V_u)}{\partial x} = 0.
\]

Similarly, we can include the vehicle spacing requirements in the dynamics of the velocity and the velocity variance. Let the ‘hat’ indicate multiplication of the respective variable or parameter with the correction factor \( \delta(x,t) \). We need not include the resulting model equations here, since the structure of the equations does not change due to incorporation of the finite-space requirements. Rather, the modified density \( \hat{\rho}_u \), pressure \( \hat{P}_u \), viscosity coefficient \( \hat{\eta}_u \), kinematic coefficient \( \hat{\kappa}_u \), and finally the interaction term \( \hat{\mathcal{M}}_{u,s}^{(k)} \) replace their regular counterparts in Eqs. (26) and (27), respectively. We can consequently show that when the densities \( \rho_u \) increase, the available space decreases, and the correction factor \( \delta \) increases accordingly. As a consequence, the influence of vehicle-interaction expressed by the modified interaction moments \( \hat{\mathcal{M}}_{u,s}^{(k)} \triangleq \delta \cdot \mathcal{M}_{u,s}^{(k)} \) increases (the
introduction of the space-requirements yields an increase of the interaction rates). This increase results in a larger decrease in the modified equilibrium velocity.  

This concludes the derivation of the multiclass traffic flow model. That is, we have now shown how the macroscopic multiclass equations can be determined from the mesoscopic gas-kinetic equations using the method of moments. Moreover, we have briefly discussed the closure of the model, the introduction of viscosity, and the finite-space-requirements. For details, see Hoogendoorn (1999), Hoogendoorn and Bovy (1998b) and Helbing (1996).

**The equilibrium relations and vehicular interactions**: Now that we have established the macroscopic multiclass flow model, let us discuss some of the model properties. That is, in this section we will discuss the interactions of vehicles in the same user-class, and the asymmetric interactions among vehicles from different user-classes.

**Within-class interactions (Interactions)**: Let us illustrate the mechanisms in the macroscopic model that reflect the influence of acceleration and deceleration due to vehicle interactions. Let us first consider the case where only one user-class $u$ is present. In this case, we can show that:

$$R_{u,u}^{(1)} \triangleq (1 - \theta_u) \left( v^h \int (v - w) \tilde{\rho}_u(w) \, dw \right)_u = (1 - \theta_u) P_u$$

yielding:

$$V_u^e = V_u^0 - (1 - \pi_u) (1 - \theta_u) \tau_u P_u.$$  

In other words, the equilibrium velocity $V_u^e$ is determined by the mean desired velocity $V_u^0$ of the user-class, decreased by the mean rate at which vehicles of the same user-class actively interact with each other $(1 - \theta_u) P_u$, without being able to overtake these vehicles (with probability $(1 - \pi_u)$), multiplied by the acceleration time $\tau_u$. This mean interaction rate is relative to the traffic pressure. From (52) we observe that if the acceleration time $\tau_u$ is small, the influence of interactions is reduced due to vehicles being able to increase their velocity rapidly after interacting. Let us finally remark that expression (52) is similar to the expression describing the equilibrium velocity derived by Helbing (1996), albeit the latter does not consider the effect of relaxing the vehicular chaos assumption.

If we consider the equilibrium velocity variance in the single-class case, then Hoogendoorn and Bovy (1998a) show that $\Theta_u^e$ equals:

$$\Theta_u^e = C_u^e - \frac{1}{2} (1 - \pi_u) (1 - \theta_u) \tau_u J_u,$$

which is again similar to the single-class equation describing the equilibrium variance derived by Helbing (1996).

**Between-class interactions**: Next, let us consider the influence of interactions between vehicles of different classes. In contrast to the velocity dynamics derived in Helbing (1996), interaction between user-classes is present in the equations, both directly in the equilibrium velocity, and indirectly in the traffic pressure. This novelty of our macroscopic MUC traffic flow model is reflected in the way in which the distinguished classes interact. That is, the model does not only

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5 For notational convenience, the hat is dropped from notation for the remainder of this article.
consider influences on the velocity and on the velocity variance caused by within-class interactions; also between-class interactions are considered. These influences are directly reflected in the equilibrium velocity and the equilibrium velocity variance.

Changes in the equilibrium velocity (28) of class $u$ that are caused by interactions with vehicles from class $s$ are reflected by the term $(1 - \pi_u)\tau_u \mathcal{R}_{u,s}^{(1)}$ for $u \neq s$. Let us now illustrate class interaction by assuming that the supports of classes $u$ and $s$ are disjoint. Let us first assume that $V_u < V_s$. In other words, each vehicle of class $u$ is driving slower than any vehicle of class $s$. In this case we have (see Hoogendoorn and Bovy, 1998a):

$$\mathcal{R}_{u,s}^{(1)} = 0$$

and thus

$$V_e^u = V^0_u - (1 - \pi_u)(1 - \theta_u)\tau_u P_u.$$  

We can observe that the equilibrium velocity (28) of vehicles from class $u$ does not change due to interaction with vehicles in user-class $s$. That is, since each vehicle in user-class $s$ is driving faster than any vehicle in user-class $u$, no interaction exists. This holds equally for the equilibrium velocity variance of user-class $u$.

However, from the viewpoint of the other class $s$, we can show:

$$\mathcal{R}_{s,u}^{(1)} = (1 - \theta_u)r_u (\Theta_u + \Theta_s + (V_u - V_s)^2)$$

that is

$$V_e^s = V^0_s - \cdots - (1 - \pi_s)\tau_s(1 - \theta_u)r_u (\Theta_u + \Theta_s + (V_u - V_s)^2).$$

In this case, interaction between vehicles is present due to velocity variances of both classes, and the squared difference in mean speed between the user-classes, causing a reduction in the equilibrium velocity of user-class $s$. This holds equally for the equilibrium velocity variance of user-class $s$.

**Competing processes:** In Helbing (1996) and Kerner et al. (1996) competing processes in single user-class traffic flow operations are observed and interpreted. The equilibrium velocity $V_e^r$ and the equilibrium variance $\Theta_e^r$ reflect these processes. That is, they describe how drivers accelerate to traverse along the roadway at their desired velocity on the one hand, while they may need to decelerate due to impeding, slower vehicles on the other hand.

Kerner et al. (1996) approximate the variance $\Theta$ by a constant $\Theta_0$ and assume that the equilibrium velocity $V_e^r$ is an explicit function of the traffic density $r$:

$$V_e^r = V^0 - \tau(1 - \pi)r\Theta_0,$$

where $\pi = \pi(r)$ is the immediate overtaking probability. When the equilibrium velocity is determined explicitly, the probability of immediate overtaking can be determined from $V_e^r(r)$:

$$\pi(r) = 1 - \frac{V^0 - V^e(r)}{r\tau\Theta^0}.$$  

Considering these competing processes for single user-class traffic flow, the relaxation term:

$$\frac{V_e - V}{\tau} = \frac{V^0 - V}{\tau} + (1 - \pi)\Theta^0$$
describes the dynamic changes in the velocity $V$ due to relaxation towards the equilibrium velocity $V^e$. Here, the term $(V^0 - V)/\tau$ reflects the acceleration or active process caused by drivers aiming to traverse the roadway at their desired velocity. Conversely, the term $(1 - \pi)\Theta^0$ reflects a deceleration or damping due to vehicles interacting.

In Kerner and Konhäußer (1995) a stability analysis is performed by considering spatial perturbations in the mean density. It is shown that, if the amplitude of the fluctuation in the density exceeds a certain threshold value, the density perturbation causes a change in the mean velocity which itself leads to an increase in the traffic density. This ‘avalanche’ occurs when the spatial fluctuations are large enough to overcome the influence of traffic viscosity and relaxation.

We can also identify these competing processes for the multiclass case. Consider Eq. (26) describing dynamic changes in the velocity. In this equation, the term:

$$\frac{V^e - V}{\tau_u} = \frac{A}{\tau_u} - \frac{D}{\tau_u} - (1 - \pi_u)\sum_{s \in \mathcal{W}} \mathcal{R}^{(1)}_{u,s}$$

represents the relaxation process towards $V^e_u$. Also for the MUC case, the processes comparable to the single user-class case can be identified. On the one hand, drivers aim to traverse along the roadway at the class-specific desired velocity $V^0_u$. These acceleration processes are reflected by the term $A$. On the other hand, the term $D$ results from both within and between-class interactions. This term reflects a deceleration process due to vehicles interacting. As shown in the previous sections, main difference between the single-class case and the multiclass case is impact of vehicle interaction between the classes. We remark that in the velocity variance equation (27), comparable processes can be identified.

**Equivalence with other macroscopic models:** In Hoogendoorn and Bovy (1998a) it is shown that our generalized multiclass model collapses to well-known macroscopic models in case of a single user-class. For instance, if we consider only one user-class $u$, and choose $\theta = 0$, the system of equations (25), (26), and (27) yield Helbing’s improved single user-class macroscopic traffic flow model presented in Helbing (1996). In the Payne-model (cf. Payne, 1971), the equilibrium velocity $V^e = V^e(r)$ is specified a priori. Assuming a constant velocity variance $\Theta = c_0^2$, assuming $\kappa = 0, \eta = 0$, and finally choosing $\pi(r)$:

$$\pi(r) = \frac{c_0^2 \tau r - (V^0 - V^e(r))}{c_0^2\tau r}$$

reduces our multiclass model reduces to the Payne model.

5. Fundamental issues and macroscopic simulation

It is the purpose of this section to investigate whether the issues raised by among others, Daganzo (1995) and Helbing (1996), are remedied by the developed MUC model. Some of these issues can be checked straightaway by considering the model equations. Others, such as the anisotropy and the unaffected slow vehicles, are investigated using macroscopic simulations.

**Approach to macroscopic simulation:** To perform macroscopic simulations using the multiclass traffic flow model presented in this article, we need to numerically approximate solutions to the
system of partial differential equations describing the dynamics of the user-class specific density, velocity and velocity variance. To this end, several numerical approaches have been developed. These are based on recasting the model equations using appropriate variables (the so-called conservative variables). The following results have been computed by applying Roe’s Approximate Riemann solver to the multiclass flow equations. This scheme is based on exact solutions of the inviscid multiclass model (e.g. $\eta_u = \kappa_u = 0$), operating under equilibrium conditions. Approximations at timestep $t_k$ are represented as piecewise constant functions in $x$. Locally, piecewise constant initial conditions yield the so-called Riemann-problem, which can be solved analytically for the inviscid multiclass equations (assuming equilibrium conditions). These solutions are then used to approximate the (partial) solution at the next timestep. The influence of relaxation and deceleration are subsequently determined. Finally the influence of the higherorder terms are effectuated using central finite-differences. These results are combined to determine the approximation at the next timestep $t_{k+1}$. For details, we refer to Hoogendoorn and Bovy (1998a).

Description of the test-cases (Specification of the relations and parameters): In this section we will illustrate macroscopic multiclass simulation using a hypothetical test case (Fig. 2). The case describes a two-lane ringroad of 30 km length. We identify two classes, namely person-cars and trucks. These differ with respect to their mean lengths (4 and 11 m, respectively), their (constant) desired velocities (31 and 23 m/s), and relations for the covariance $C_u(r)$.

The overtaking probabilities are specified by considering the gap-distribution (see Hoogendoorn, 1999): having assumed that these gap-distributions are a function of the total density, the probability $p_u$ that a vehicle is able to immediately change lanes equals the probability that a gap is available that is larger than the gap needed by the interacting vehicle. The gap that is needed equals 2.5 times the length of the vehicle increased by a safety margin, which is dependent on the

![Fig. 2. Initial distribution of person-cars and trucks on the two-lane 30 km ringroad.](image-url)
reaction time of the drivers and their velocity (cf. Jepsen, 1998). Since the overtaking probability
distribution function is a decreasing function of the gap needed, the overtaking probability of a
truck is in general smaller than the overtaking probability of person cars.

The reaction time $T_r$ equals 1.6 and 2.6 s for person-cars and trucks, respectively. For the speed-
risk factor, and the minimal distance headway, we respectively choose $F_u = 0.022 \text{ m}^2/\text{s}^2$ and $d_u = 1$ m (see Jepsen, 1998). The acceleration time $\tau_u$ is assumed constant and equals 10 s for both classes. For the kinematic viscosity coefficient and the kinematic coefficient we have used $\eta_u = 150 \, r_u$ and $\kappa_u = 150 \, r_u$.

Case I (Multiclass traffic operations in homogeneous regions): The first case is characterized by
the following initial conditions: at time $t = 0$, the traffic conditions are characterized by three regions of different person-car densities, respectively equal to 10, 20 and 50 veh/km. In the regions, the truck densities are equal, namely 4 veh/km. The regions are respectively located at $X_1 = [3 \text{ km}, 6 \text{ km})$, $X_2 = [9 \text{ km}, 12 \text{ km})$ and $X_3 = [15 \text{ km}, 18 \text{ km})$. We assume that initially, the traffic is in equilibrium (in Hoogendoorn, 1999 the author presents an algorithm to determine the initial equilibrium conditions). As a result, in each region the velocity of the vehicles is different (in $X_3$ the vehicles are nearly stationary).

Fig. 3 depicts the density contour plots and the trajectories of person-cars and trucks on the
ringroad during a 15-minute period. In the two moderate density regions $X_1$ and $X_2$ the velocities
of the persons-cars and the trucks are high, while in the high density region $X_3$ the velocities are
nearly equal to zero (congested traffic conditions). Clearly, in neither the free-flow, nor the
congested regions, vehicles flow back into the empty upstream regions. This strengthens our belief
that the discrete MUC-model satisfies the anisotropy condition, stating that drivers mainly react
to stimuli in front of them. We can also see how the individual drivers are affected by the traffic
conditions if we consider the vehicle trajectories. Note that the mean velocity of the person-cars is
reduced significantly in each density region, while this reduction is more profound in the higher-
density regions. Only in the high-density region $X_3$ the velocities of the trucks are reduced as well.

That is, from the trajectories we can see that the trucks-velocities are virtually unaffected by the
person-cars present in the regions $X_1$ and $X_2$. In the high-density region $X_3$ however, the number of
interactions with slow person-cars is such that the truck-velocity is affected significantly. More-
over, in this region the velocity of person-cars and trucks in nearly identical. Concluding, similar
to traffic operations in real-life traffic, the slow users (trucks) remain virtually unaffected by fast
users (person-cars), at least in non-congested traffic conditions. This implies that the unaffected slow-users critique of Daganzo (1995) in remedied in our multiclass model. In Hoogendoorn
(1999) it is discussed how the multiclass modelling of traffic loosens the invariant personality issue

Case II (Formation of localized structures in multiclass traffic flow): Let us now consider the
following case, again using person-cars and trucks. In this scenario, a small perturbation is applied
to the initial density distribution of the person-cars. Fig. 4 shows the results from the
macroscopic simulation: The figure density contour plots and the trajectories of person-cars and
trucks, respectively. On the one hand, Fig. 4 shows that initially, both the truck density, as well as
the truck velocity are initially unaffected by the passenger-car density perturbation. After some
time, the increased number of person-car interactions in the higher-density region yields the
formation of a localized structure, resulting in traffic breakdown. Since the person-car velocities in
this region are low, passenger-car/truck interactions also increase substantially. Consequently, eventually the truck velocities are affected by the person–car disturbances ($t = 6\text{ min at } x = 5–7\text{ km}$). This example shows that our model is able to describe the seemingly spontaneous formation of traffic jams.

**Macroscopic modeling issues:** The test-cases illustrate the models aptness with respect to the issues raised by Daganzo (1995) (anisotropy, unaffected slow vehicles, and driver’s personality). Furthermore, the multiclass model also considers the issues of Lyrintzis et al. (1994) and Kerner et al. (1996) (deviation from the speed–density relation, traffic hysteresis, formation of localized structures; see Hoogendoorn, 1999 for details).

Finally, let us note that the remarks of Helbing (1996) have explicitly been taken into account: the velocity variance is a model variable, the finite space requirements have been explicitly introduced using the modified density, while the finite braking and reaction times have been introduced by the inclusion of the flux of velocity variance.

Fig. 3. Density contour plots and trajectories of person-cars and trucks for test case I.
6. Conclusions

In this article, we have presented a multiple user-class generalization of a macroscopic single-class traffic flow model. The model, being derived from mesoscopic foundations, features class-dependent competing processes: on the one hand, drivers aim to traverse the roadway at a class-dependent desired velocity, while on the other hand, drivers are slowed down by slower vehicles from their own and other classes. Depending on the prevailing traffic conditions, these competing processes may result in the formation of localized structures.

Other model features are: incorporation of dynamic equations describing class-specific velocity variance, vehicle interactions within and between the distinguished classes, and incorporation of class-specific parameters, such as acceleration time, reaction time, desired velocity, and vehicle lengths. Additionally, critical issues raised by among others, Daganzo (1995), Lyrintzis et al. (1994) and Helbing (1996), do not hold for our model.
From a model-application perspective, the usefulness of the macroscopic model stems among other from multiclass traffic control. For one, the developed model may provide insight into the multiclass response–behavior of heterogenous traffic. Moreover, it can be applied in among others model-based traffic control approaches of multiclass control options, automated multiclass incident-detection, data-checking and data-completion. Finally, a by-product of the model are (equilibrium) multiclass travel-time functions, that can be used in multiclass dynamic assignment.

The paper has presented a new multiclass model formulated in so-called primitive variables, namely the density, the velocity, and the velocity variance. It appears that the model advantageously can be reformulated in the so-called conservatives (see Hoogendoorn, 1999). The advantages of this approach are among others the simplified and elegant model derivation, and the availability of improved numerical solution approaches. Its main disadvantage is the lack of interpretation of the resulting model equations from the driver's perspective.

Acknowledgements

We would like to express our gratitude to the anonymous referees whose comments have contributed to improve both the clarity and contents of this paper.

References