Optimization of the transportation expense of a firm with contractual supplies

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Abstract

We consider nonlinear nonconvex capacitated transportation problems where the nonlinearity occurs only in the last row of the transportation tableau. This transportation model can be successfully applied to economics representing the nonlinearity caused by penalties for unsatisfied contractual quantities or by changing in price. An algorithm for local optimization, based on the algorithms for solving linear transportation problems, is suggested. It consists of three phases – initial, linear and nonlinear. The nonlinear phase uses auxiliary linear transportation problems. The algorithm is illustrated by proper examples. Sufficient conditions for satisfying contractual supplies are given. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Transportation; Nonlinear problem; Local minimum

1. Introduction

The increasing interest in nonlinear transportation problems is based on nonlinear relationships between quantities and the real cost for their transportation. In this paper we consider nonlinear capacitated transportation problems where the nonlinearity is presented only in the last row of the transportation tableau. To explain the economic model and the difficulties in solving such nonconvex problems we consider the following example:

Let a firm supply two warehouses with a particular product with the transportation costs being $18 and $13 per unit. The firm has produced 6 units. The first warehouse wants to receive all the 6 units and it is willing to pay a premium of $6 each for any received units over and above the first 2 units received. With the second warehouse 2 units of the product have been contracted and the
contract requires $6 compensation for every unit less than contractual. The warehouse can receive maximum 4 units. The firm wants to minimize the total cost – transportation cost plus penalties for unsatisfied contractual supplies minus benefits of increasing the price. The total cost for the first warehouse is $18 if the received units $x$ are in the interval $[0,2]$ and $36 + 12(x - 2)$ if $x \in [2,6]$. This function can be written as $15x - 3|x - 2| + 6$. The cost function for the second warehouse could be presented as $7y + 12$ if $y \in [0,2]$ and $13y$ if $y \in [2,6]$ or $10y + 3|y - 2| + 6$. The mathematical problem is

$$\min F = 15x + 10y - 3|x - 2| + 3|y - 2| + 12$$

subject to:

$$x + y = 6, \quad 0 \leq x \leq 6, \quad 0 \leq y \leq 4.$$ 

By presenting $y$ as $6 - x$ we receive the cost function

$$F = 5x - 3|x - 2| + 3|x - 4| + 72$$

or tabulated:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
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<tr>
<td>$F$</td>
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<td>87</td>
<td>86</td>
<td>91</td>
<td>96</td>
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Even this simple nonconvex problem is difficult to be solved by linear methods. Function $F$ is not convex in the interval $[0,6]$. For the considered example $x$ belongs to the interval $[2,6]$ (using the restriction for $y$: $0 \leq 6 - x \leq 4$ we receive $x \geq 2$), where the cost function $F$ is convex. A linearization would change the cost function to convex (eventually to $3.5x + 1.5|x - 4| + 72$) with minimum at point 2, whereas function $F$ has its minimum at point 4. This explains the difficulties in solving partially linear nonconvex problems.

Until now, few nonlinear transportation models have been considered. In many books (for example, in Kovalev, 1977; Murty, 1976; Wagner, 1975) nonlinear problems are approached by means of linearization. Transportation problems with uncertain demands and a penalty for short supply are presented in Dantzing (1955) and Ferguson and Dantzing (1956). A method for solving the minimum cost flow problem when the costs over arcs are piecewise-linear convex functions is given in Ahuja and Batra (1984). Some problems for maximizing the flow in a network with the possibility of increasing the capacities of arcs (Bansal and Jacobson, 1975; Fulkerson, 1959; Hu, 1969; Price, 1967) could be considered as partially linear problems. Methods for solving some nonlinear transportation problems are also given in Dangalchev (1991, 1992, 1996, 1997).

The problem considered here can be solved (if it is convex) as a linear problem. In the nonconvex case it can be solved by the method in Dangalchev (1997) by spending substantially more time than the algorithms for linear transportation problems. The algorithm presented in this paper solves nonlinear problem (1) as a transportation problem by taking advantage of the special structure of the problem, without increasing its dimensions. For finding a better solution of the problem we have to start from different initial points. The algorithm consists of 3 phases: initial, linear and nonlinear (partially linear).

Sufficient conditions for satisfying different conditions, arising in the economic model, are given. The algorithm is illustrated by proper examples.
2. Formulation of the problem

Let there be one firm with \(m-1\) sections which produce a particular product. Each of the sections has a given production capacity \(A_i, \ i \in M = \{1, \ldots, m-1\}\). Suppose there are \(n\) warehouses with given requirements \(L_j, \ j \in N = \{1, \ldots, n\}\) of the product. The firm and every warehouse have contracted for the quantities \(L_j\). Every section \(i\) is connected with every warehouse \(j\) and the communication line between \(i\) and \(j\) has a capacity limit \(U_{ij} \geq 0\). If \(X_{ij}\) units of the product are sent from section \(i\) to the warehouse \(j\) then the transportation cost is \(C_{ij}X_{ij}\).

Let, for some reason (shortage of raw materials or markets for example), the produced and contractual quantities be out of balance. If warehouse \(j\) receives \(Q_j\) units of the product \(\left( Q_j = \sum_{i=1}^{m-1} X_{ij} \right)\) which is less than contractual \(L_j\) then the firm will have to pay penalties, \(P_j(L_j - Q_j)\). The warehouse \(j\) may receive quantity \(Q_j\) which is more than contractual \(L_j\), but then the firm will have to make a reduction in price \(R_j\), i.e., the penalties are \(R_j(Q_j - L_j)\). In the both these cases the penalties \(P_j\) and \(R_j\) are greater than zero. Let the manager of the \(j\)th warehouse wants to receive more units of the product and is willing to pay a premium for units received over and above the contract quantity. In this case the reduction \(R_j\) in price is negative, i.e., there is an increase in price. The situation is analogous if the manager of the warehouse wants to receive less than contractual quantity. In this case \(P_j\) is negative. In general if the changes in supplies are caused by the producer the penalties are positive. On the contrary if a warehouse changes its demand then the penalties are negative. There is an upper limit \(B_j\) of the quantity which can be received by the warehouse \(j\). The whole produced quantity of the product is less than or equal to the sum of all upper limits \(B_j\).

The objective is to select a plan which minimizes the transportation costs and penalties of the firm. The mathematical model of the problem is

\[
\begin{align*}
\min \quad & F(X) = \sum_{i \in M} \sum_{j \in N} C_{ij}X_{ij} + \sum_{j \in N} F_j \left( \sum_{i \in M} X_{ij} \right) \\
\text{s.t.} \quad & \sum_{j \in N} X_{ij} = A_i, \quad i \in M, \\
& \sum_{i \in M} X_{ij} \leq B_j, \quad j \in N, \\
& 0 \leq X_{ij} \leq U_{ij}, \quad i \in M, \quad j \in N,
\end{align*}
\]

where

\[
F_j(X) = \begin{cases} 
P_j(L_j - X), & X \leq L_j, \\
R_j(X - L_j), & X > L_j.
\end{cases}
\]

If \(P_j \geq 0\) and \(R_j \geq 0\), which ensure the convexity of the cost function \(F\), the problem can be solved by linear methods. If \(P_j\) or \(R_j\) are negative the problem could be nonconvex and can not be solved as a linear one.

There are some questions to be answered:

- How to solve problem (1)?
- What are sufficient conditions for fulfilling the contractual requirement \(L_j\) of the \(j\)th warehouse?
- What conditions for \(P_j\), \(R_j\) and \(C_{ij}\) make the product received by \(j\)th warehouse equals to 0 or \(B_j\)?

These questions can be represented: When the total supply \(Q_j\) is equal to 0, \(L_j\) or \(B_j\)?
3. Initial plan

We make a classic transportation tableau for the problem (1) with dimensions $m - 1$ by $n$. We add to the tableau a $m$th row corresponding to a fictitious section of the firm which produces quantities equal to the difference between the total maximal demand and the whole produced quantities, i.e., $A_m = \sum_{j \in N} B_j - \sum_{i \in M} A_i$. In this last $m$th row we shall put the quantities of the product which must be added to the total received supplies $Q_j$ to reach the upper limit $B_j$, i.e., $X_{mj} = B_j - Q_j$. The sum of all variables in the column $j$ must be equal to $B_j$. In every cell of the new row we put $R_j$ in the left upper corner and $P_j$ in the right upper corner. Between them we put the difference $B_j - L_j$. If cell $j$ of the last row is filled with the quantity $X_{mj}$ which is equal to the middle number above $(B_j - L_j)$ then the penalty is equal to zero. If $X_{mj}$ is less than $B_j - L_j$ then the penalty is $R_j(B_j - L_j - X_{mj})$. If $X_{mj}$ is greater than $B_j - L_j$ then the penalty is $P_j(X_{mj} + L_j - B_j)$.

We use the Greedy algorithm to determine the total initial quantities received by the warehouses, i.e., to fulfill the last row. Let (for example) there be a shortage in the production. There are 3 groups of warehouses (one of these groups may be the empty set). The first group contains the warehouses with the greatest penalties $P_j$. They all receive quantities equal to the corresponding $L_j$ (the last $m$ row is fulfilled with $B_j - L_j$). The second group contains warehouses with the smallest penalties and they receive the quantities equal to zero ($B_j$ in the last row). The third group contains one (or zero) warehouse with the penalty smaller than the penalties of the first group and bigger than the penalties of the second group. It receives quantity between 0 and $L_j$ so the sum of the quantities of the last row equals $A_m$. Meanwhile, for every warehouse we ensure that the received quantity can be transported, i.e., the sum of all upper limits $U_{ij}$ is greater than $L_j$ – otherwise we fix the maximal possible quantity. On the contrary, if there is a shortage in requirements we start from the greatest reduction $R_j$ to fix the quantities (the first group warehouses will receive $L_j$, the second – $B_j$).

The initial variables $X_{ij}$, $i = 1, \ldots, m - 1$, $j = 1, \ldots, n$ can be received by the Northwest corner rule or Vogel’s approximation method. As there are upper limits $U_{ij}$ for finding an initial plan of problem (1) we use the standard technique for capacitated problems or we may follow Dangalchev (1997), i.e., we have to solve the auxiliary nonlinear problem with the cost function sum of all functions $f_{ij}(X_{ij})$, $i = 1, m - 1$, $j = 1, n$, where

$$f_{ij}(X_{ij}) = \begin{cases} 0 & \text{if } 0 \leq X_{ij} \leq U_{ij}; \ X_{ij} - U_{ij} & \text{if } X_{ij} > U_{ij}. \end{cases}$$

If the cost function of the auxiliary problem is equal to zero then we have found an initial plan, otherwise there is not a feasible solution for the chosen total quantities $(Q_j = B_j - X_{mj})$ and we have to solve an auxiliary problem with all the $m$ rows.

The following example illustrates the procedure for finding an initial plan.

**Example 3.1.** Let us consider the problem: $m = 3$, $n = 3$; $C_{11} = 13$, $C_{12} = 10$, $C_{13} = 16$, $C_{21} = 10$, $C_{22} = 10$, $C_{23} = 15$; $A_1 = 20$, $A_2 = 20$, $L_1 = 5$, $B_1 = 15$, $L_2 = 10$, $B_2 = 20$, $L_3 = 10$, $B_3 = 20$; $U_{ij} = 20$, $i \in 1, 2$, $j \in 1, 2, 3$; $P_1 = -1$, $R_1 = 2$, $P_2 = 2$, $R_2 = 3$, $P_3 = 1$, $R_3 = 1$.

We follow the Greedy algorithm: As $L_1 + L_2 + L_3 < A_1 + A_2$ we start to fix the quantities from the maximal $R_1$ ($R_2 > R_1 > R_3$). As $B_1 + L_2 + B_3 > A_1 + A_2$ we set the quantity, received by warehouse 2 equal to $L_2$, i.e., $X_{32} = B_2 - L_2 = 20 - 10 = 10$. As $10 + L_1 + B_3 < A_1 + A_2$ we put the
quantity of warehouse 1 to \( A_1 + A_2 - 10 - B_3 = 10 \), i.e., \( X_{11} = B_1 + B_3 + 10 - A_1 - A_2 = 5 \). As \( 10 + 10 + B_3 = A_1 + A_2 \) the quantity for warehouse 3 is equal to the upper limit \( B_3 = 20 \), i.e., \( X_{33} = 0 \). The variables of the first two rows can be obtained by the Northwest corner method: \( X_{11} = 10, X_{12} = 10, X_{23} = 20 \).

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We evaluate the cost function adding the penalties \( 2 \cdot (10 - 5) + 3 \cdot (10 - 10) + 1 \cdot (10 - 0) = 20 \) to the transportation cost \( 13 \cdot 10 + 10 \cdot 10 + 15 \cdot 20 = 530 \) and receive total cost \( F = 550 \).

4. Linear phase

We solve the transportation problem with dimensions \( m \times n \) (except for the last row) by standard methods. If there are intermediate balances between the quantities of sections and warehouses we change the basis. The essence of this technique is: whenever we find a nonbasic cell \((i, j)\), which provides a direction decreasing the cost function with a step in this direction equal to zero (because of a basic cell \((k, p)\)), we change the basis, including the cell \((i, j)\) into the place of \((k, p)\).

For finding directions, decreasing the cost function we use the modified distribution method. We determined an index \( u_i \) for each row \( i = 1, \ldots, m - 1 \) and an index \( v_j \) for each column \( j \) such that \( u_i + v_j = C_{ij} \) for every basic cell \((i, j)\). Then we calculate the improvement potentials \( d_{ij} \) for every nonbasic cell, \( d_{ij} = u_i + v_j - C_{ij} \). The increment of the cost function \( \Delta F \) is presenting by the formula \( \Delta F = - \sum d_{ij} \Delta X_{ij} \), where the sum is on all nonbasic cells \( \Delta X_{ij} \) is the increment of variable \( X_{ij} \). A direction provides decreasing of the cost function if \( X_{ij} = 0 \) and \( d_{ij} > 0; X_{ij} = U_{ij} \) and \( d_{ij} < 0 \); \( 0 < X_{ij} < U_{ij} \) and \( d_{ij} \neq 0 \).

To determine the maximal step possible in this direction we use the denotations \( Z_{kp} \) – the set of all nonbasic cells \((i, j)\) which form the cycle of basic cells, including basic cell \((k, p)\); \( Z^+_{kp} \) – a subset of \( Z_{kp} \) for which \( X_{ij} \) increases with the increasing of \( X_{kp} \) in their common cycle; \( Z^-_{kp} = Z_{kp} \setminus Z^+_{kp} \). Now we can evaluate the maximal possible step \( S_{ij} \) in direction with increasing nonbasic variable \( X_{ij} \):

\[
S_{ij} = \min \{ U_{ij} - X_{ij}; U_{kp} - X_{kp}, (i, j) \in Z^+_{kp}, X_{kp}, (i, j) \in Z^-_{kp} \},
\]

where the minimum is taken on all basic cells \((k, p)\) for which \((i, j) \in Z_{kp} \). In the direction with decreasing nonbasic variable \( X_{ij} \) the maximal possible step \( S_{ij} \) is

\[
S_{ij} = \min \{ X_{ij}; X_{kp}, (i, j) \in Z^+_{kp}; U_{kp} - X_{kp}, (i, j) \in Z^-_{kp} \}.
\]
If the maximal possible step is greater than zero \( S_{ij} > 0 \) we make this step and decrease the cost function. Let the process of solving problem (1) give a basis with basic cells \((k,p)\) for which \( X_{kp} = 0 \) or \( X_{kp} = U_{kp} \). If moving in the direction of decreasing is impossible because of the basic variable \( X_{kp} \), then we change the basis, placing the cell \((i,j)\) into the basis in the place of the cell \((k,p)\). In this way we prevent cycling of the algorithm.

The linear phase ends after reaching the optimal solution of the linear transportation problem with dimension \( m - 1 \) by \( n \).

We illustrate the linear phase by the following example:

**Example 4.1.** Let us continue to solve the transportation problem of Example 3.1. Let \{\((1,1), (1,2), (2,2), (2,3)\)\} be the basis with indexes: \( u_1 = 0, u_2 = 0; v_1 = 13, v_2 = 10, v_3 = 15 \). For every nonbasic cell we evaluate the possibility of decreasing the cost function, i.e., the improvement potentials for the nonbasic cells (1,3) and (2,1): \( d_{13} = u_1 + v_3 - C_{13} = 0 + 15 - 16 = -1 \), \( d_{21} = u_2 + v_1 - C_{21} = 0 + 13 - 10 = 3 \). The cycle with cell (1,3) can not decrease the cost function. The cycle with cell (2,1) can decrease the cost function, but the maximal possible step in this direction is \( S_{21} = \min\{X_{11}, X_{22}, U_{12} - X_{12}, U_{21} - X_{21}\} = \min\{10, 0, 10, 20\} = 0 \).

We change the basis by putting the cell (2,1) into the place of (2,2) (because \( X_{22} = 0 \)), i.e., the new basis is \{\((1,1), (1,2), (2,1), (2,3)\)\}. We evaluate the new indexes, \( u_1 = 3, u_2 = 0; v_1 = 10, v_2 = 7, v_3 = 15 \) and the new improvement potentials, \( d_{13} = 2, d_{22} = -3 \). The maximal possible step with increasing \( X_{13} \) is \( S_{13} = \min\{X_{11}, X_{23}, U_{13} - X_{13}, U_{21} - X_{21}\} = \min\{10, 20, 20 - 0, 20 - 0\} = 10 \).

We make this step and receive the plan:

```
   13 10 16 10
   10 10 15 10
  2 10 1 3 10 2 1 10 1
   5 10 0
```

The new basis is \{\((1,2), (1,3), (2,1), (2,3)\)\} and the corresponding characteristics of the plan are \( u_1 = 1, u_2 = 0; v_1 = 10, v_2 = 9, v_3 = 15 \); \( d_{11} = -2, d_{22} = -1 \), which means that the plan satisfies the criteria for optimality. The transportation cost decreases to \( F = 550 + (-2) \cdot 10 = 530 \).

### 5. Nonlinear phase

After outlining the initial and linear steps of the algorithm we present the nonlinear part. In the nonlinear phase we have to solve a transportation problem with dimensions \( m \) by \( n \), where the nonlinearity may occur only in the last row. We can follow Dangalchev (1997), but the algorithm
there is more complex, because it can solve problems with nonlinearity in every cell. Here we present the nonlinear phase of an algorithm, constructed especially for this problem.

We want to estimate the possibility of decreasing the cost function with a cycle, including the cells of the last row. For this purpose we have to know the contribution of these cells to the decrement. The nonlinear component of the decrement of the cost function $E_j^-$, caused by column $j$ (warehouse $j$) in case of decreasing $X_{mj}$ can be evaluated by the formula

$$E_j^- = \{R_j \text{ if } X_{mj} \leq B_j - L_j; \quad -P_j \text{ if } X_{mj} > B_j - L_j\}.$$  \hfill (2)

In the case of increasing $X_{mj}$ we obtain for the nonlinear component

$$E_j^+ = \{-R_j \text{ if } X_{mj} < B_j - L_j; \quad P_j \text{ if } X_{mj} \geq B_j - L_j\}. \quad \hfill (3)$$

Our objective is to decrease the cost function by changing the plan. To find an appropriate direction we have to find one which decreases the cost function and, at the same time, is feasible (with possible step $S > 0$).

Let the direction decreasing the cost function contain 4 cells of the last row

$$(m,p), (m,q), (i,q), \ldots, (j,r), (m,r), (m,s), (k,s), \ldots (t,p)$$

and $X_{mp}$ increase. From the condition for decreasing the cost function we receive

$$E_p^+ + E_q^- + C_{iq} - \cdots - C_{jr} + E_r^+ + E_s^- + C_{ks} - \cdots - C_{tp} < 0.$$  \hfill (5)

We may divide this cycle into two subcycles:

$$(m,p), (m,s), (k,s), \ldots, (t,p),$$

$$(m,q), (i,q), \ldots, (j,r), (m,r).$$

At least one of the following inequalities is satisfied:

$$E_p^+ + E_q^- + C_{ks} - \cdots - C_{tp} < 0,$$

$$E_q^- + C_{iq} - \cdots - C_{jr} + E_r^+ < 0,$$

which means that at least one of these subcycles is a cycle decreasing the cost function.

Cycles with 6 or more cells of the last $m$ row we divide into 3 or more subcycles containing 2 cells of row $m$ and at least one of them is a cycle, decreasing the cost function. Hence we have proved.

**Theorem 5.1.** If there exists a cycle, decreasing the cost function, which contains more than 2 cells of the last (nonlinear) row then there exists a cycle of decreasing with only 2 cells of the last row.

Further, we shall consider only cycles with two cells in the last row.

6. Auxiliary problems

To find a cycle decreasing the cost function we solve auxiliary linear problems. Let us consider columns $p$ and $q$ for which $X_{mp}$ can increase and $X_{mq}$ can decrease. We may set the prices: $C_{mp}$ to
\( E^+_p; C_{mq} \) to \(-E^-_q\). Then we make one iteration of the auxiliary transportation problem with dimensions \( m \) by \( n \), these new prices, and a taboo for using other cells of the last \( m \)th row (the prices of the other \( m - 1 \) rows are the original). If we find a direction of decreasing for this auxiliary problem we can use this direction to decrease the cost function of problem (1).

Another way to find a direction of decreasing is to evaluate the improvement potentials without constructing auxiliary problems. In the linear phase we have solved the system: \( u_i + v_j = C_{ij} \), for all basic cells \((i, j)\). Let the basic cell for the last row index be \((m, p)\) which value can increase:

\[ u_m + v_p = E^+_p. \]

The improvement potentials for the other cells of the last row we receive by the formulae,

\[ d_{mq} = u_m + v_q - C_{mq} = u_m + v_q + E^-_q, \]

or finally \( d_{mq} = E^+_p + E^-_q - v_p + v_q \).

The increment of the cost function is \( \Delta F = -d_{mq} \Delta X_{mq} \) and \( \Delta X_{mq} \) decreases. Condition \( \Delta F \geq 0 \) can be presented:

\[ E^+_p + E^-_q - v_p + v_q \geq 0. \tag{4} \]

If \( X_{mp} \) can only increase (for example if \( X_{mp} = 0 \) or \( X_{ip} = U_{ip} \) for all \( i = 1, m - 1 \)) and condition (4) is satisfied then there is no cycle decreasing the cost function because the linear phase has been performed and there are only two cells \((m, p)\) and \((m, q)\) in the last row, i.e., this condition is sufficient for optimality of the auxiliary problem. In case of more than two cells in the last row one sufficient condition for local optimality is satisfying conditions (4) for all possible pairs \((p, q)\).

When \( X_{mp} < B_p - L_p \) we use \( E^+_p = -R_p \) to receive from (4):

\[ E^-_q + v_q \geq R_p + v_p. \tag{5} \]

In case of \( X_{mp} \geq B_p - L_p \), using \( E^+_p = P_p \) condition (4) is presented:

\[ E^-_q + v_q \geq -P_p + v_p. \tag{6} \]

We continue to solve Example 4.1 and to show how the nonlinear phase of the algorithm can be performed.

**Example 6.1.** In the last transportation tableau of Example 4.1 with basis \( \{(1,2), (1,3), (2,1), (2,3)\} \) the indexes are: \( u_1 = 1, u_2 = 0, v_1 = 10, v_2 = 9, v_3 = 15 \). Using column 3 as a basis, conditions (5) for columns 1 and 2 become

\[ E^+_1 + v_1 \geq R_3 + v_3 \quad \text{or} \quad 2 + 10 \geq 1 + 15, \]

\[ E^+_2 + v_2 \geq R_3 + v_3 \quad \text{or} \quad 3 + 9 \geq 1 + 15. \]

Both columns 1 and 2 supply directions of decreasing. The same directions can be received if we consider the auxiliary linear transportation problem:

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Using column 2 we evaluate the maximal possible step $S = 10$ and receive the new plan:

\[
\begin{array}{ccc|c|c|c}
13 & 10 & 20 & 16 & 0 \\
10 & 10 & 15 & 10 \\
2 & 10 & -1 & 3 & 10 & 2 & 1 & 10 & 1 \\
5 & 0 & 10 \\
\end{array}
\]

with cost function $F = 530 - 4 \cdot 10 = 490$.

Using the same basis and indexes, from condition (6) for column 1 we receive

\[E_1 + v_1 \geq -P_3 + v_3 \quad \text{or} \quad 2 + 10 \geq -1 + 15.
\]

Decreasing $X_{31}$ and increasing $X_{33}$ with maximal possible step 5 we receive the plan

\[
\begin{array}{ccc|c|c|c}
13 & 10 & 20 & 16 \\
10 & 15 & 10 & 5 \\
2 & 10 & -1 & 3 & 10 & 2 & 1 & 10 & 1 \\
0 & 0 & 15 \\
\end{array}
\]

with cost function $F = 490 - 2 \cdot 5 = 480$.

When $X_{mp}$ can decrease we change the places of $p$ and $q$ in condition (4). Using $E_p = R_p$ when $X_{mp} \leq B_p - L_p$ and $E^p = -P_p$ when $X_{mp} > B_p - L_p$ we receive

\[
E^+_q - v_q \geq -R_p - v_p, \quad (7)
\]

\[
E^+_q - v_q \geq P_p - v_p. \quad (8)
\]

When $X_{mp} = 0$ the variable $X_{mp}$ can only increase. A sufficient condition for optimality (in case of only two cells in the last row) is (5). When $X_{mp} = B_p$ the variable $X_{mp}$ can only decrease – condition (8) is sufficient for optimality.

When $X_{mp} = B_p - L_p$ the variable $X_{mp}$ can increase as well as decrease. Combining (6) and (7) we receive a sufficient condition for local optimality (in case of only two cells in the last row):

\[
R_p + E^+_q \geq -v_p + v_q \geq -P_p - E^-_q, \quad X_{mp} = B_p - L_p. \quad (9)
\]
When \( 0 < X_{mp} < B_p - L_p \) or \( B_p - L_p < X_{mp} < B_p \) the variable \( X_{mp} \) can increase and decrease. Combining (5) with (7) or (6) with (8) we receive the sufficient conditions:

\[
\begin{align*}
E_q^- & \geq R_p + v_p - v_q \geq -E_q^+, \quad 0 < X_{mp} < B_p - L_p, \\
E_q^- & \geq v_p - v_q - P_p \geq -E_q^+, \quad B_p - L_p < X_{mp} < B_p.
\end{align*}
\]

(10) (11)

Theorem 5.1 gives us the opportunity to consider cycles with only two cells in the last row, i.e., we receive:

**Lemma 6.1.** A sufficient condition for local optimality (after the linear phase) is for every pair \((p, q)\) one of conditions 5, 8, 9, 10 or 11 (corresponding to the value of \(X_{mp}\)) to be satisfied.

These conditions are not necessary conditions – some of them can be violated but restrictions for \(X_{ij}\) (e.g., \(X_{ij} \geq 0\) or \(X_{ij} \leq B_i\)) can prevent the cost function from decreasing.

The last iteration of example 6.1 can also be performed using conditions (9) for cells (3,3), (3,1). Example 6.2 is illustrating the auxiliary problems, constructed by the help of condition (9). It is a very interesting example from economic point of view: the problem is balanced, the initial solution is optimal for the linear phase but not for the nonlinear problem.

**Example 6.2.** Let us consider a transportation problem with upper limits \(U_{ij} = 20\) and a tableau after the linear part of the algorithm:

\[
\begin{array}{|c|c|c|c|}
\hline
5 & 10 & 3 & 20 \\
7 & 5 & & 4 \\
-1 & 5 & 2 & 1 \\
& & 5 & 10 \\
& & & 4 \\
\hline
\end{array}
\]

The cost function of the problem is \( F = 5 \cdot 10 + 3 \cdot 20 + 4 \cdot 20 + 5 \cdot 20 = 290\).

From the initial basis \(\{(1,1), (1,2), (1,3), (2,2)\}\) we find a direction of decreasing (cycle (1,2), (1,3), (2,3), (2,2) with possible step equal to zero) and move to the basis \(\{(1,1), (1,3), (2,2), (2,3)\}\). Using indexes \(u_1 = 4, u_2 = 4, v_1 = 1, v_2 = 1, v_3 = 0\) and condition (9) for basic cell (3,1) we receive

\[
\begin{align*}
-1 + E_2^+ & \geq -1 + v_2 \geq -2 - E_2^- \quad \text{or} \quad -1 + 0 \geq -1 + 1 \geq -2 - 1, \\
-1 + E_3^+ & \geq -1 + v_3 \geq -2 - E_3^- \quad \text{or} \quad -1 + 4 \geq -1 + 0 \geq -2 - 3.
\end{align*}
\]

As the condition for column 2 is violated \((-1 \geq 0\) we consider an auxiliary transportation problem with prices \(C_{31} = -E_1^- = 1, C_{32} = E_2^- = 0\). The use of cell (3,3) is forbidden:
The cycle (3,1),(3,2),(2,2),(2,3),(1,3),(1,1) is a cycle of decreasing:
$$E_3^1 + E_2^2 - C_{22} + C_{23} - C_{13} + C_{11} = -1 + 0 - 5 + 4 - 4 + 5 = -1.$$  
We make a step $$S = 5$$ and receive a new plan

\[
\begin{array}{cccc}
5 & 15 & 3 & 20 \\
7 & 5 & 4 & 15 \\
1 & 20 & 0 & X \\
5 & 5 & X & X
\end{array}
\]

with cost function $$F = 285$$.

Using condition (4) we will construct another sufficient condition for local optimality. We divide all cells of the last $$m$$th row into 5 classes:

- $$O = \{k \in N: X_{mk} = 0\}$$ – the variables are equal to zero.
- $$U = \{k \in N: X_{mk} = B_k\}$$ – the variables are on their upper bounds, i.e., the warehouses do not receive the product.
- $$L^- = \{k \in N: 0 < X_{mk} < B_k - L_{mk}\}$$ – the variables are on the linear part of the cost function. The warehouses receive more than their contractual quantities.
- $$L^+ = \{k \in N: B_k - L_{mk} < X_{mk} < B_k\}$$ – the variables are on the linear part of the cost function and the warehouses receive less than their contractual quantities.
- $$Q = \{k \in N: X_{mk} = B_k - L_k\}$$ – the warehouses receive their contractual quantities and these variables can cause nonlinearity.

Some of these sets may be empty.

We have to check conditions (4) for all possible pairs $$(i,j)$$. Let us consider the following condition:

$$\min_{i \in N}(v_i + E_i^-) \geq \max_{j \in N}(v_j - E_j^+).$$

This condition is stronger than considering (4) for all possible pairs as it considers even pairs with two equal elements – $$(i,i)$$. Sets $$U, L^+, L^-, Q$$ can decrease their values: $$E_i^-$$ is equal to $$-P_i$$ if $$i \in (U \cup L^+)$$ and $$R_i$$ if $$i \in (Q \cup L^-)$$; sets $$O, L^-, L^+, Q$$ can increase their values: $$E_j^+$$ is equal to $$-R_j$$ if $$j \in$$...
\((O \cup L^-)\) and \(P_j\) if \(j \in (Q \cup L^+)\). Replacing \(E^-_i\) and \(E^+_j\) in the upper condition we receive the following theorem.

**Theorem 6.2.** One sufficient condition for local optimality of problem 1 (after the linear phase) is satisfying

\[
\min \left\{ \min_{i \in U, L^+} (v_i - P_i), \min_{i \in Q, L^-} (v_i + R_i) \right\} \geq \max \left\{ \max_{j \in O, L^+} \left( v_j + R_j \right), \max_{j \in Q, L^-} \left( v_j - P_j \right) \right\}. \tag{12}
\]

Condition (12) is easier to be used but stronger (it considers pairs \((i, i)\)) than the condition of Lemma 6.1. It is not a necessary condition – some pairs can violate it but the restrictions of problem (1) can prevent the cost function from decreasing.

### 7. Algorithm

As the problem (1) may not be convex we have to find different local optimal solutions and choose the best of them. For this purpose we have to start from different initial quantities received by the warehouses (the nonlinearity may occur only in the last row). To assign the initial quantities we can use the Greedy algorithm described in Section 3 (it gives a good starting point for problem (1)) or generate these quantities at random.

1. **Initial phase:** We assign the initial quantities received by the warehouses (by the Greedy algorithm or generated at random). Then we continue with the Northwest Corner method or Vogel’s Approximation method to obtain an initial solution.

2. **Linear phase:** We apply the linear phase (following Section 4) after the initial phase to receive the optimal linear solution.

The cells of the sets \(O, U, L^-, L^+\) are considered as a part of the linear phase (if in the process of solving the linear problem some of the cells become a part of set \(Q\) they will be considered in the nonlinear phase). The prices \(C_{mk}\) are \(-R_k\) for \(k \in (O \cup L^-)\) or \(P_k\) for \(k \in (U \cup L^+)\).

3. **Nonlinear phase:** To implement the nonlinear part we have to consider all possible cycles with two cells in the last row (according to Theorem 5.1). As the nonlinearity can be caused only by cells of \(Q\) we can consider all pairs of cells of the last row with at least one cell of \(Q\) – after the linear phase we can use condition (9). Implementing the nonlinear phase some cells of \(Q\) can become parts of other sets and they still can provide a direction of decreasing – we have to use conditions (5), (8), (10) and (11). To decrease the number of cases we use the sufficient condition for optimality (12). After finding a pair \((i, j)\) \((i \neq j)\) violating condition (12) we construct an auxiliary linear problem. The prices \(C_{mk}\) for the cells of set \(Q\) depend on changing the variables: \(-R_k\) if \(X_{mk}\) decreases or \(P_k\) if \(X_{mk}\) increases. Actually we have only to evaluate the maximal possible step \(S\) in the selected direction – if \(S > 0\) we make this nonlinear step.

Then we again look for a pair violating (12) (now the sets may be changed) and we continue with the implementation of the nonlinear phase. We keep the list of considered auxiliary problems to avoid repeating these considerations.
The algorithm stops when all possibilities are considered – either all pairs satisfy condition (12) or the maximal possible steps into directions, defined by pairs violating (12), are equal to 0.

**Example 7.1.** To finish solving this example we compose an auxiliary problem with sets \( O \) and \( L^+ \) and prices \(-R_1, -R_2\) and \( P_3\).

\[
\begin{array}{ccc}
13 & 10 & 16 \\
10 & 20 & 15 \\
15 & 10 & 5 \\
-2 & -3 & 1 \\
0 & 0 & 15
\end{array}
\]

The solution of this linear transportation problem is optimal. Using condition (12) we can receive this result without constructing auxiliary problem:

\[
\min (v_3 - P_3) \geq \max (v_1 + R_1, v_2 + R_2, v_3 - P_3) \text{ or }
15 - 1 \geq \max (10 + 2, 9 + 3, 15 - 1).
\]

Hence, the minimal cost function for Example 7.1 is \( F = 480 \).

---

**8. Sufficient characterizing conditions**

Now we can give sufficient conditions for the characterization of the optimal plan.

1. Let the optimal plan fulfill: \( Q_p = 0 \), i.e., \( X_{mp} = B_p \). From condition (8) we receive

   \[
   v_p - P_p \geq v_q - E_q^+.
   \]

   In the case of a cycle with only 4 cells we receive

   \[
   C_{kp} - P_p \geq C_{kq} - E_q^+.
   \]

   This condition must be satisfied for every row \( k = 1, \ldots, m - 1 \) and for every column \( q \). One sufficient condition is the prices of the \( p \)th column must be substantially greater than the other prices. This condition is quite natural: the warehouse with greater transportation costs will receive less quantity of the product. The same economic result will be received if the penalty \( P_p \) is a big negative number, i.e., warehouse \( p \) pays to receive less quantity of the product.

   Let the cycle have more than 4 cells: \((m, p), (m, q), (k, q), (k, s), \ldots, (j, t), (j, p)\). Using equations, \( u_k + v_q = C_{kq}, u_k + v_s = C_{ks}, \ldots, u_j + v_t = C_{jt}, u_j + v_p = C_{jp} \), we receive

   \[
   v_q - v_p = C_{kq} - C_{jp} + (-C_{ks} + \cdots + C_{jt}) \text{ or }
   C_{jp} - P_p \geq C_{kq} - E_q^+ + (-C_{ks} + \cdots + C_{jt}).
   \]

   From this inequality the same sufficient conditions must be delivered.
2. Let us consider the situation: \( Q_p = B_p \). As \( X_{mp} \) can only increase from condition (5) in case of a cycle with only 4 cells we receive
\[
C_{kp} + R_p \leq C_{kq} + E_q^-.
\]
This condition must be satisfied for every row \( k = 1, \ldots, m - 1 \) and for every column \( q \). It can be expressed in words as the prices of \( p \)th column must be substantially less than the other prices. This condition is also natural – the warehouse with the minimal transportation costs will receive more quantity of the product. Another sufficient condition is the manager of warehouse \( p \) is willing to pay more for the additional quantity, i.e., the reduction \( R_p \) is a large negative number. The same sufficient conditions could be received in case of more than 4 cells per cycle.

3. For the condition \( Q_p = L_p \), i.e., \( X_{mp} = B_p - L_p \) from (9) (or from (6) and (7)) we receive the inequalities
\[
C_{kp} + R_p \geq C_{kq} - E_q^+,
\]
\[
C_{kp} - P_p \leq C_{kq} + E_q^-.
\]
In this case one sufficient condition is the penalties \( P_p \) and \( R_p \) are substantially greater than the other penalties. With large penalties \( P_p \) and \( R_p \) the satisfaction of contractual supplies is ensured.

4. Let in the optimal plan there exist two warehouses \( p \) and \( q \) with \( Q_p > L_p \) and \( Q_q < L_q \), i.e., \( X_{mp} < B_p - L_p \) and \( X_{mq} > B_q - L_q \). We consider a direction for which \( X_{mp} \) increases and \( X_{mq} \) decreases, i.e., \( E_p^+ = -R_p \) and \( E_q^- = -P_q \). From condition (5) we obtain
\[
-R_p - P_q \geq C_{kp} - C_{kq}.
\]
The upper condition must be satisfied for every row \( k \). One sufficient condition for this is the prices of \( q \)th column are substantially greater than the prices of \( p \)th column. If the transportation costs of a warehouse substantially exceeds the transportation costs of the other one then the first warehouse may receive less than its contractual supplies and the second – more than its contractual supplies.

Another condition is \( R_p \) and \( P_q \) are negative, i.e., the first warehouse pays more for the additional quantity, whereas the second pays to receive less quantity.

9. Concluding remarks

In this paper a partially linear transportation model is proposed. As the suggested method is a transportation one (in its nature) the time for solving such partially linear transportation problems (starting from a given initial plan) is comparable to the time needed to solve ordinary transportation problems (a number of computational experiments showed that the time for nonlinear problems (1) does not exceed more than 50% the time for linear ones with the same dimensions). The total time for solving problem (1) strongly depends on the number of used initial plans.
References


